Learning and Expectations in Macroeconomics

Problems for Chapter 3

1. Consider the variation of the cobweb model in which p_t depends on the observable exogenous variable w_t rather than w_{t-1}:

$$\begin{aligned} p_t &= \mu + \alpha p_t^e + \delta w_t + \eta_t, \text{ where } \alpha \neq 1, \\ w_t &= k + \lambda w_{t-1} + \varepsilon_t, \text{ where } |\lambda| < 1, \end{aligned}$$

where ε_t and η_t are independent white noise processes. This model was considered earlier in a Chapter 2 problem. Suppose now that agents forecast according to

$$p_t^e = a_{t-1} \text{ where } a_t = a_{t-1} + t^{-1}(p_t - a_{t-1}),$$

i.e. they ignore the dependence of p_t on the w_t variable. Using the stochastic approximation technique, show that $a_t \to \hat{a}$ (for an appropriate sense of convergence) for some $\hat{a}$, provided a stability condition holds. What can you say if the stability condition fails?

2. Consider the model

$$y_t = \alpha + \beta_0 E_{t-1}^* y_t + \beta_1 E_{t-1}^* y_{t+1} + v_t + \kappa w_{t-1},$$

where v_t is a white noise disturbance and w_t is an observable *iid* random variable with mean $E(w_t) = \mu$. Consider an adaptive learning rule in which agents neglect the influence of w_{t-1} and use a PLM of the form

$$y_t = a + v_t.$$

Assume they estimate a by

$$a_t = a_{t-1} + t^{-1}(y_t - a_{t-1}),$$

and form their forecasts as $E_{t-1}^* y_t = E_{t-1}^* y_{t+1} = a_{t-1}$. Using the stochastic approximation approach, write this as an SRA (stochastic recursive algorithm) and obtain the corresponding ODE (ordinary differential equation). Find the equilibrium point of the ODE, a corresponding stability condition, and state a result concerning convergence of a_t.

3. Consider the nonstochastic model with a lag

$$y_t = \delta y_{t-1} + \beta E_{t-1}^* y_{t+1}.$$

(a) Show that there are RE solutions of the form

$$y_t = \lambda y_{t-1},$$

where λ must satisfy the quadratic equation

$$\lambda = \delta + \beta \lambda^2 \equiv T(\lambda).$$

(b) Suppose agents have a PLM of the same form, so that the ALM is

$$y_t = T(\lambda_{t-1})y_{t-1}$$

and they use the constant gain algorithm

$$\lambda_t = \lambda_{t-1} + \gamma[(y_t/y_{t-1}) - \lambda_{t-1}].$$

Obtain the convergence condition for the RE solutions.

4. Suppose that in a coordination game the utility of any agent i depends on his own action x_i and the median action $M(x)$, where $x = (x_1, ..., x_n)$, so that the best response of any agent is quadratic in the median

$$b(M) = \omega M(1 - M).$$

Assume $\omega > 1$. Consider the adaptive learning dynamics

$$\begin{aligned} M_t &= b(M_t^e) \\ M_t^e &= M_{t-1}^e + \gamma_{t-1}(M_{t-1} - M_{t-1}^e). \end{aligned}$$

(a) Show that there are two Nash equilibria to this game and that one of them is zero and the other one is interior.
(b) Show that this learning dynamics converges to the interior Nash equilibrium.

5. Consider the multivariate Muth model

$$\begin{aligned} y_t &= \mu + Ay_t^e + Cw_t \\ w_t &= Bw_{t-1} + v_t, \end{aligned}$$

where y_t is an $m \times 1$ vector, w_t is a $p \times 1$ vector. y_t^e denotes the average expectation of y_t, formed at time $t - 1$. The eigenvalues of B are inside the unit cirle, so that w_t is a stationary process. Let

$$z_t = \begin{pmatrix} 1 \\ w_t \end{pmatrix}, \varphi' = \begin{pmatrix} a & b \end{pmatrix},$$

where a is $m \times 1$ and b is $m \times p$.
(a) Show that the unique REE is

$$y_t = \bar{\varphi}' z_{t-1} + \eta_t,$$

where

$$\begin{aligned} \bar{\varphi}' &= \begin{pmatrix} (I - A)^{-1}\mu & (I - A)^{-1}CB \end{pmatrix} \\ \eta_t &= Cv_t. \end{aligned}$$

(b) Suppose outside REE the expectations can be heterogenous, so that

$$y_t^e = N^{-1}\sum_{i=1}^{N} y_{i,t}^e,$$

where each group $i = 1, ..., N$ forms expectations by least squares learning as follows. Agent i forecasts according to

$$y_t^e = \varphi'_{i,t-1} z_{t-1}$$

and the estimates $\varphi'_{i,t-1}$ are formed by least squares regressions based on different amounts of past data, so that

$$\begin{aligned} \varphi_{i,t} &= \varphi_{i,t-1} + (t+T_i)^{-1} R_{i,t}^{-1} z_{t-1}(y_t - \varphi'_{i,t-1} z_{t-1}) \\ R_{i,t} &= R_{i,t-1} + (t+T_i)^{-1}(z_{t-1} z'_{t-1} - R_{i,t-1}). \end{aligned}$$

Derive the SRA and the associated ODE.

(c) Show that the asymptotic behavior of the associated ODE is governed by the "small" differential equation

$$d\Phi/d\tau = \left\{ N^{-1} \begin{bmatrix} A & \cdots & A \\ \vdots & \vdots & \vdots \\ A & \cdots & A \end{bmatrix} - I \right\} (\Phi - \bar{\Phi}),$$

where $\Phi = (\varphi'_1, \ldots, \varphi'_N)$ and $\bar{\Phi} = (\bar{\varphi}'_1, \ldots, \bar{\varphi}'_N)$. Prove that this differential equation is globally stable if and only if all the eigenvalues of A have real parts less than one.

6. Consider the Cagan model

$$y_t = \alpha + \beta E_t^* y_{t+1} + \delta' w_t + v_t$$

and assume that agents use the forecast rule

$$E_t^* y_{t+1} = a_{t-1} + b'_{t-1} w_t = \phi'_{t-1} z_t,$$

where $\phi_{t-1} = (a_{t-1}, b'_{t-1})$ and $z'_t = (1, w'_t)$. The REE values of the parameters are $a = (1-\beta)^{-1}\alpha$ and $b = (1-\beta)^{-1}\delta$.

(a) Parameters are updated by the stochastic gradient algorithm

$$\phi_t = \phi_{t-1} + \gamma_t z_{t-1}(y_t - \phi'_{t-1} z_t).$$

Show that this learning rule can be formally represented as an SRA and derive the associated ODE.

(b) Prove that the convergence condition is given by the E-stability condition of the REE of interest.