Learning and Expectations in Macroeconomics

Problems for Chapter 4

1. In appendix 1 to Chapter 10 it is shown that a general multivariate framework of the form
$$f(Y_t) + E_t g(Y_{t+1}) = 0,$$
where $Y_t = (Y_{1t}, ..., Y_{nt})$ is an $n-$vector can be log-linearized as follows. Let $\bar{Y}$ be a steady state of the nonstochastic structural equation. Introduce the notation $f_i(\bar{Y}) = \frac{\partial f}{\partial Y_i}(\bar{Y}), g_i(\bar{Y}) = \frac{\partial g}{\partial Y_i}(\bar{Y})$, define $y_{it} = \ln\left(\frac{Y_{it}}{\bar{Y}_i}\right)$. and utilize the approximation $\exp(y_i) \approx y_i + 1$. The log-linearization can be shown to be
$$\sum_{i=1}^{n} \bar{Y}_i f_i(\bar{Y}) y_{it} + \sum_{i+1}^{n} \bar{Y}_i g_i(\bar{Y}) E_t y_{i,t+1} = 0.$$
Use this methodology to obtain the linearization to the Ramsey model in the appendix 4.8.1 of the book.

2. Background: For linear second order difference equations
$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2},$$
the equilibrium point is asymptotically stable if the following conditions hold
$$\phi_1 + \phi_2 < 1, \phi_2 < 1 + \phi_1, \phi_2 > -1.$$
Consider the Diamond growth model for learning as in appendix 4.8.2.
(i) Verify that the dynamics of the capital stock under learning is given by the formula
$$K_{t+1} = (1 + a_k + \gamma(a_r b_k - 1))K_t + (\gamma - 1)a_k K_{t-1}$$
as shown in the book.
(ii) Verify that for $\gamma > 0$ sufficiently small the stability conditions under learning are $a_k < 1$ and $a_k + a_r b_k < 1$ and that these imply
$$0 < \frac{dK_{t+1}}{dK_t}(\bar{K}) < 1.$$
Moreover, show that the two sets of conditions are equivalent, provided $a_r < 0$, i.e. savings depend negatively on the interest rate.

3. In the overlapping generations model, Section 4.2 of the book, suppose that agents are trying to learn the steady state and are able to use contemporaneous data. The learning rule (4.2) is then modified to
$$q^e_{t+1} = q^e_t + \gamma_t(q_t - q^e_t),$$
where $q_t = \mathcal{F}(q^e_{t+1})$ and γ_t is a decreasing sequence of gains with $\gamma_1 < 1, \lim_{t-\infty} \gamma_t = 0$ and $\sum \gamma_t = \infty$. Using a linearization show that the conditions for local convergence and non-convergence are still given by the E-stability and E-instability of a steady state. Consider also what happens in learning with contemporaneous data and a small constant gain.

4. Suppose that the model

$$y_t = E_t F(y_{t+1})$$

has a solution that is a perfect foresight 2-cycle,

$$\begin{aligned} y_t &= \bar{y}_1 \text{ if } t \text{ is odd,} \\ y_t &= \bar{y}_2 \text{ if } t \text{ is even,} \end{aligned}$$

where $\bar{y}_1 \neq \bar{y}_2$. Then clearly we must have $\bar{y}_1 = F(\bar{y}_2)$ and $\bar{y}_2 = F(\bar{y}_1)$. Draw a figure showing a 2-cycle. Using the equations defining a 2-state Markov stationary sunspot equilibrium (SSE), illustrate the existence of an SSE (y_1^*, y_2^*) near $(\bar{y}_1, \bar{y}_2)$. Show that for (y_1^*, y_2^*) near $(\bar{y}_1, \bar{y}_2)$ the transition probabilities π_{11} and π_{22} are near 0.

5. For the model

$$y_t = \alpha + \beta_0 E_{t-1} y_t + \beta_1 E_{t-1} y_{t+1} + v_t,$$

where v_t is white noise, use the method of undetermined coefficients to find the AR(1) solutions, i.e. RE solutions of the form

$$y_t = a + b y_{t-1} + v_t.$$

Compute the implied $E_{t-1} y_t$ and $E_{t-1} y_{t+1}$ and substitute in to obtain the implied restrictions on a and b. Give the possible solutions for (a, b).

6. Consider the model of social increasing returns. Assume that agents use a constant gain learning algorithm

$$\begin{aligned} n_t &= \mathcal{F}(n_{t+1}^e), \\ n_{t+1}^e &= n_t^e + \gamma(n_{t-1} - n_t^e), \; 0 < \gamma < 1. \end{aligned}$$

Derive the local stability condition $1 - 2/\gamma < \mathcal{F}'(n^*) < 1$ for a steady state n^*. For the case of $\mathcal{F}' > 0$ and three interior steady states, as illustrated in Figure 4.5, derive the basins of attraction for n_L and n_H. Are the basins of attraction different with decreasing gain?

7. Consider the model

$$y_t = E_t F(y_{t+1}),$$

where F is nonlinear. (i) Give the equations for a 2-state Markov SSE (stationary sunspot equilibrium). (ii) Draw a sketch in which F has 3 steady states and show on the sketch a 2-state Markov SSE in which y_t takes values near the high and low steady states, $\bar{y}_H$ and $\bar{y}_L$. (iii) Using the equations in (i) solve for π_{11} and π_{22}, the probabilities of staying in states 1 and 2, respectively. What happens to π_{11} and π_{22} as the values of y_t in the SSE become close to $(\bar{y}_L, \bar{y}_H)$? (iv) Draw a sketch in which F has a single irregular steady state and show on the sketch a 2-state Markov SSE.