Stable near-rational sunspot equilibria*

George W. Evans
University of Oregon
University of St. Andrews

Bruce McGough
University of Oregon

February 2, 2018

Abstract

A new class of near-rational sunspot equilibria is identified in economies expressed as non-linear forward-looking models. The new equilibria are natural extensions of the usual sunspot equilibria associated with the linearized version of the economy, and are near-rational in that agents use the optimal linear forecasting model when forming expectations. Generic existence results are established. Stability under learning is also examined: the near-rational sunspot equilibria are found to be E-stable provided that the corresponding linearized model's minimal state variable solution is E-stable.

1 Introduction

Dynamic macroeconomic models that include forward-looking agents may exhibit equilibrium multiplicity: there may exist rational expectations equilibria (REE) that depend upon extrinsic stochastic processes, that is, a sequence of shocks that influences the economy only because agents condition expectations on these shocks. Importantly, this dependency is self-fulfilling: it exists only because agents think it exists. Equilibria that depend upon such extrinsic shocks are called sunspot equilibria, with the shocks themselves referred to as the sunspots.

The possibility that competitive rational expectations models could have self-fulfilling solutions driven by extraneous stochastic processes was demonstrated by various authors, notably through the work of Shell (1977), Azariadis (1981), Cass and Shell (1983), Azariadis

*This paper has benefited from comments received in talks at the University of Oregon, the June 2012 INEXC conference, Paris, Aix-Marseille University, Simon-Fraser University, UC Riverside, UC Irvine, CEF 2017 NYC, and LAEF, UC Santa Barbara. Financial support from National Science Foundation Grant No. SES-1559209 is gratefully acknowledged.

and Guesnerie (1986) and Guesnerie (1986).[1] These existence results were originally obtained in simple stylized models, such as the overlapping generations model of money, and the sunspot drivers were typically taken to be a finite-state Markov process; but generic results providing criteria for local equilibrium uniqueness have also been established. Blanchard and Kahn (1980) present a practical technique for determining whether a linear model has a unique equilibrium, and Woodford (1986) shows that local equilibrium uniqueness in a non-linear model is implied by uniqueness in the linearized model.

The method of Blanchard and Kahn may also be used to establish the existence of sunspot equilibria in linear models. Importantly, this existence result is constructive: the equilibria present in an easily analyzed VAR form; and, the extrinsic processes – the sunspots – characterizing the sunspot equilibria in these linear models have (or, at least, can have) continuous support, and are thus more general than the finite-state equilibria examined in the earlier literature. This observation has been exploited by many authors toward a number of ends. For example, Farmer and coauthors have developed an entire research program devoted to explaining business cycle co-movements through the incorporation of non-convexities into competitive DSGE models and through the analysis of the sunspot equilibria associated with the linearized versions of these models: see, for example, Farmer and Guo (1994) and Benhabib and Farmer (1994). Separately, a large literature has emerged warning of the dangers of sunspot equilibria resulting from poorly designed policy in DSGE models with price frictions. This literature too relies on the examination of sunspot equilibrium existence in the linearizations of the associated models.

Results establishing the existence of sunspot equilibria in non-linear models are available: Woodford (1986) showed that equilibrium multiplicity in the linearized model implies local equilibrium multiplicity in the non-linear model. Unlike their linearized counterparts, however, the sunspot equilibria associated with the non-linear models are not easily analyzed: the existence result relies on an implicit function theorem and is not constructive in nature; indeed, given a non-linear model, there is no general technique for establishing a closed-form representation, or even a numerical approximation of an equilibrium associated to a sunspot with continuous support.

The existence of sunspot equilibria raises the question of equilibrium selection. The simple presence of exotic equilibria does not justify their importance: why, after all, would we as modelers anticipate that agents would (choose to) coordinate their expectations and actions on some extrinsic process that has no inherent economic immediacy? Woodford (1990) used adaptive learning to provide an answer to this question. Woodford showed, in a non-linear overlapping generations model, that if agents thought certain finite state Markov sunspot processes *might* be relevant for forecasting, these agents would learn that the sunspots *are* relevant: Woodford showed that the economy converged, in an appropriate sense, to the associated sunspot equilibrium.

Subsequent research on the stability under learning of constructible sunspot equilibria

[1] See the extensive survey in Guesnerie and Woodford (1992).

associated with linearized models has been less definitive. While certain linear(ized) models are known to have stable sunspot equilibria, Evans and Honkapohja (2001) showed that the sunspot equilibria associated with the model examined by Farmer and Guo (1994), at least for the particular calibration used, were not stable under learning. Evans and McGough (2005a) and Duffy and Xiao (2007) extended this instability result to a host of non-convex RBC-type models.

Stability also depends upon the stochastic properties of the sunspot process associated with the equilibrium. For example, in a model previously thought to have no stable sunspot equilibria, Evans and McGough (2005c) found that the equilibria may be stable provided that the associated sunspot process exhibited the appropriate serial correlation, known as the "resonance frequency." Using this insight, Evans and McGough (2005b) established the existence of stable sunspot equilibria in a variety of New Keynesian specifications.

The research on sunspot equilibria and their stability under learning has raised a number of concerns, a few of which we catalog here.

1. **No non-linear equilibrium recursions.** The challenge of constructing and analyzing continuous-support sunspot equilibria in non-linear models is problematic not only for the modeler, but also (indeed, even more so) for the model's agents. If we, as theoretical economists, are unable to recursively represent a particular equilibrium and thereby capture the conditional distributions of the endogenous variables, how then do we imagine agents making optimal forecasts? And even if we wish to adopt a learning perspective, what forecasting model do we provide our agents?

2. **The knife-edge of resonance.** The discovery of resonance frequency sunspots has greatly expanded the literature's catalog of models exhibiting stable sunspot equilibria; however, some researchers have questioned reliance on the existence of extrinsic processes meeting the knife-edge resonance frequency condition.

3. **No general stability results.** Woodford's stability result has been extended to the general univariate, forward-looking case by Evans and Honkapohja (2003), provided that the sunspots are finite state. No stability results are available for equilibria in non-linear models associated with sunspots that have continuous support. In particular, it is not known whether sunspot stability in a linearized model is, in general, even related to stability of sunspot equilibria in the non-linear model.

In this paper, we develop a new equilibrium concept designed to simultaneously address the above questions and concerns. We take our cue from the literature on bounded rationality and embrace the possibility that our agents have insufficient information and/or cognitive capacity to uncover the economy's endogenous distributions. Instead, we assume agents use simple, linear forecasting models when forming expectations. If the linear forecasting model used by agents is optimal among all similarly specified linear models then the economy is in a near-rational equilibrium. If the linear model includes a conditional dependency upon a sunspot process then the economy is in a near-rational sunspot equilibrium (NRSE).

We establish a generic existence result: if the linearized model is indeterminate then near-rational sunspot equilibria (NRSE) exist. Importantly, while the existence result itself relies on a bifurcation argument and is thus not constructive in nature, NRSE are identified as fixed points of finite dimensional functions and thus easily computed; furthermore the associated equilibrium process has a VAR structure and so is amenable to detailed analysis. This addresses point one.

The sunspot processes associated with NRSE are found to be natural generalizations of the linearized model's resonance frequency sunspots: the processes are serially correlated, with the required correlation converging to the associated resonance frequency as the model's curvature (non-linearity) vanishes. However, for given curvature there is an open set of serial correlations corresponding to NRSEs. We conclude that the knife-edge resonance frequency condition is an artifact of the linearization, and point two is addressed.

The linear structure of an NRSE makes it amenable to stability analysis: simply provide agents with a linear perceived law of motion that precisely includes the conditioning variables in the NRSE. We find that if the linearized model is indeterminate and the minimal state variable (MSV) solution is stable under learning, then the NRSE are stable under learning. In Evans and McGough (2011) we showed that, in this linearized model, indeterminacy together with stability of the MSV solution is equivalent to the existence of stable sunspot equilibria. This provides a link between the linear and nonlinear models: stable sunspot equilibria in the linear model imply stable NRSE in the non-linear model. This addresses point three.

After applying our univariate results to a simple over-lapping generations (OLG) model, we turn to extensions. It is not our intention to develop our results within the broadest possible framework in part because of the tediousness of the exercise and also in part because it not clear what the most useful framework is for applied work, at least that remains tractable analytically. Instead, we extend our results along different dimensions separately, thus providing an architecture for future extensions should they become needed. In particular, we provide results establishing the existence of NRSE when the endogenous variable is implicitly defined, when the model has fundamental stochasticity, and when the model is multivariate – each of these results is demonstrated using the same proof strategy as the non-stochastic, univariate case, but each also holds its own special nuances. The remaining natural extension – the inclusion of a lagged endogenous variable – involves a significant technical barrier, so we consider this case only numerically.

The paper is organized as follows. In Section 2, we develop with care our existence and stability results within the context of a univariate, non-stochastic model, and we apply our results to an OLG framework; Section 3 develops and concisely presents the results in a variety of generalized modeling environments; Section 4 concludes, and all formal proofs are in the Appendix.

2 Stable NRSE: the univariate case

We examine the univariate case with considerable care in order to provide intuition for our results. Throughout we take as primitive a complete probability space (Ω, μ). Let $F : \mathbb{R} \to \mathbb{R}$ be C^k with $F(0) = 0$ and $F'(0) = \beta \neq 0$. Unless otherwise noted, $|\beta| > 1$. The abstract economy is taken as characterized by the following sequence of reduced-form equations:

$$y_t = E_t^* F(y_{t+1}). \tag{1}$$

Here E_t^* denotes the representative agent's subjective expectation based on their time t forecasting model of y_{t+1}. Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (1) satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

When agents satisfy the rational expectations hypothesis a rational expectations equilibrium (REE) of the model is any appropriately bounded stochastic process y_t satisfying (1) when $E_t^* = E_t$, where E_t denotes the true time t conditional expectation. By assumption, $y_t = 0$ is an REE, often referred to as the minimal state variable (MSV) solution. Because we have assumed $|\beta| > 1$, we know from Woodford (1984) that the model is locally indeterminate: given any open neighborhood V of the origin, there is a non-MSV equilibrium (a sunspot equilibrium) with support in V; however, as noted in Section 1, these sunspot equilibria are, in general, difficult to characterize or even numerically approximate.

2.1 NRSE: existence

Our construction of near-rational sunspot processes for the nonlinear model (1) is motivated by the corresponding sunspots in the rational linear model. The linearized model associated to (1) is given by

$$y_t = \beta E_t y_{t+1}. \tag{2}$$

We define an REE of this model to be any stationary process y_t satisfying (2). Now let ε_t be a zero-mean iid process, and with $\lambda = \beta^{-1}$, set $\eta_t = \lambda \eta_{t-1} + \varepsilon_t$. Then η_t is stationary provided that $|\beta| > 1$. Further, if $\hat{y}_t = \eta_t$ then $E_t \hat{y}_{t+1} = \lambda \eta_t$, so that $\hat{y}_t$ is a solution to (2). The stochastic process η_t is usually referred to as a "sunspot" and the solution $\hat{y}_t = \eta_t$ as a sunspot equilibrium; $\hat{y}_t$ is an REE associated with the serially correlated sunspot process η_t. We conclude with the well-known result that if $|\beta| > 1$ then sunspot equilibria exist.

We now define a new equilibrium notion couched in the language and paradigms of bounded rationality. Similar to rational sunspot equilibria, the equilibrium processes we identify will also depend upon extrinsic noise in a self-fulfilling manner: the dependence exists only if agents believe it exists. Unlike sunspot equilibria, however, the new equilibria are easily characterized, and amenable to both numerical and analytical examination.

To develop the notion of NRSE, we embrace bounded rationality: we assume agents form expectations using linear forecasting models; and to impart discipline, we require in an NRSE that the agent's forecasting model is optimal among similarly specified linear models.

Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be an iid process in $L^\infty(\Omega)$ with zero mean and $\sigma_\varepsilon^2 > 0$. We further assume that, as a random variable, ε_t has compact support. Assume $\xi \in \mathbb{R}$ is such that $\lambda(\xi) = \beta^{-1} + \xi \in (-1, 1)$. It follows that

$$\eta_t^\xi = \sum_{k\geq 0} \lambda(\xi)^k \varepsilon_{t-k} \in L^\infty(\Omega).$$

The agents' Perceived Law of Motion (PLM), that is, the linear forecasting model used to form expectations, is given as

$$\begin{aligned} y_t &= a + b\eta_t^\xi & (3) \\ \eta_t^\xi &= \lambda(\xi)\eta_{t-1}^\xi + \varepsilon_t. & (4) \end{aligned}$$

Observe that since $\eta_t^\xi \in L^\infty(\Omega)$, we have that for any continuous $f : \mathbb{R} \to \mathbb{R}$, $f \circ \eta_t^\xi(\cdot) \in L^\infty(\Omega)$; further, since η_t^ξ is stationary, it follows that for any s and t,

$$\int_\Omega f \circ \eta_t^\xi(\omega) d\mu(\omega) = \int_\Omega f \circ \eta_s^\xi(\omega) d\mu(\omega) = \int_\Omega f\left(\eta^\xi(\omega)\right) d\mu(\omega),$$

where

$$\eta^\xi(\omega) = \sum_{m\geq 0} \lambda(\xi)^m \varepsilon_m(\omega),$$

which exploits the time-invariant nature of the distribution against which the integral is taken. We will use this and similar observations repeatedly in the computations below, without further comment.

The PLM specifies E^*, yielding the following Actual Law of Motion (ALM):

$$y_t = \int_\Omega F(a + b\lambda(\xi)\eta_t^\xi + b\varepsilon_{t+1}(\omega)) d\mu(\omega) \equiv \hat{F}(a, b, \xi, \eta_t^\xi).$$

We need to be able to differentiate $\hat{F}$ (and many other functions like it). For this, we require a simple generalization of Leibniz's rule. While surely well known, for completeness, we present a proof of this Lemma in the Appendix.

Lemma 1 *Let $U \subset \mathbb{R}^n$ be open and $h : U \times \Omega \to \mathbb{R}$ have the following properties:*

1. *For all $x \in U$, $h(x, \cdot) \in L^\infty(\Omega)$*
2. *For almost all $\omega \in \Omega$, $h(\cdot, \omega) \in C^k(U)$*
3. *There exists $G \in L^1(\Omega)$ so that for all $x \in U$, $|D_{x_i} h(x, \omega)| \leq G(\omega)$ for almost all $\omega \in \Omega$.*

If $H : U \to \mathbb{R}$ is given by $H(x) = \int_\Omega h(x,\omega)d\mu(\omega)$ then $H \in C^k(U)$ and

$$D_{x_i} H(x) = \int_\Omega D_{x_i} h(x,\omega)d\mu(\omega).$$

We outline the simple argument for the application of Lemma 1 to $\hat{F}$ here. Since F is continuous, it follows that $\hat{F}(a,b,\xi,\eta_t^\xi(\cdot)) \in L^\infty(\Omega)$ for all t. Further, the analysis below will be local to the steady state $(0,0,0)'$, thus we may assume the existence of an open neighborhood $U \subset \mathbb{R}^3$ of the steady state, with compact closure, so that $\hat{F} : U \times \Omega \to \mathbb{R}$; and since $\hat{F}(\cdot, \eta_t^\xi(\omega))$ is $C^4(U)$, the compact closure of U provides the uniform bounds on the various partials needed to apply Lemma 1. Below, for expedience, we apply Lemma 1 without further comment.

We may define the T-map $T(\cdot,\cdot,\xi) : \mathbb{R}^2 \to \mathbb{R}^2$ as the Least Squares projection of the ALM onto the span of $\{1, \eta_t^\xi\}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{T=T(a,b,\xi)} \begin{pmatrix} \int_\Omega \hat{F}(a,b,\xi,\eta^\xi(\omega))d\mu(\omega) \\ \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}(a,b,\xi,\eta^\xi(\omega))d\mu(\omega) \end{pmatrix} \equiv \begin{pmatrix} T^a(a,b,\xi) \\ T^b(a,b,\xi) \end{pmatrix}, \tag{5}$$

where

$$\sigma^2_{\eta^\xi} = \int_\Omega \left(\eta^\xi(\omega)\right)^2 d\mu(\omega).$$

Observe that $T(0,0,\xi) = (0,0)'$.

Definition. A non-trivial fixed point of the T-map is a *near-rational sunspot equilibrium.*

2.1.1 The simple cubic

To establish existence of NRSE, we must study the fixed points of T, which requires a somewhat tedious two-dimensional bifurcation analysis. Before tackling the general specification of F, we first restrict attention to the case that F is cubic and symmetric about the origin:

$$F(y) = \beta y + \phi_3 y^3.$$

We also assume here that $\mu_3^\varepsilon = \mu_3^{\eta^\xi} = 0$. All of these assumptions will be relaxed in the general case. The assumed symmetry implies (abusing notation) that $T(0,b) = (0,T(b))$. This reduces the dimension of the problem to one, greatly simplifying the analysis.

Recall $\lambda(\xi) = \beta^{-1} + \xi$. Then, emphasizing the dependence on ξ, we easily compute

$$\hat{F}(b,\xi,\eta^\xi) = \beta b\lambda(\xi)\eta^\xi + \phi_3 b^3 \left(\lambda(\xi)^3(\eta^\xi)^3 + 3\lambda(\xi)\eta^\xi\sigma_\varepsilon^2\right),$$

$$T(b,\xi) = \beta b\lambda(\xi) + \phi_3\theta(\xi)b^3, \tag{6}$$

where

$$\theta(\xi) = \frac{\lambda(\xi)^3 \mu_4^{\eta^\xi}}{\sigma^2_{\eta^\xi}} + 3\lambda(\xi)\sigma^2_\varepsilon,$$

and here and in the sequel, the n^{th}-moment of a random variable x is given by μ_n^x. Given ξ, the function T is a cubic in b and so has either one or three fixed points, and in case there are three fixed points, it follows that NRSE exist.

The solutions to $T(b,\xi) = b$ are given by $b = 0$ and $b = \pm\left(\frac{-\beta\xi}{\phi_3\theta(\xi)}\right)^{\frac{1}{2}}$. Since, for ξ near zero, $\theta(\xi)$ is always positive it follows that NRSE exist when $\phi_3 < 0$ and $\xi > 0$, or when $\phi_3 > 0$ and $\xi < 0$. Importantly, there is an open set of "resonance frequencies" near β^{-1} for which NRSE exist: the "knife-edge of resonance" is indeed an artifact of the linearization. Of course our work allows us to conclude much more. We know exactly what the associated sunspots look like, and given the map F, we know how to compute the NRSE.

To prepare ourselves for the work of the next section, it is helpful to revisit the existence question using bifurcation theory: for standard results and a reference on nonlinear dynamics, see Wiggins (1990). To this end, we interpret the T-map as identifying a dynamic system with rest points corresponding to NRSE. While we could envision this interpretation quite naturally as a discrete time system, for reasons that will become apparent later it will be helpful to work in continuous time. Thus we consider the dynamic differential equation system

$$\dot{b} = H(b,\xi) \equiv T(b,\xi) - b, \tag{7}$$

and note that $b \neq 0$ corresponds to an NRSE provided that $H(b,\xi) = 0$.

Observe that $H(0,\xi) = 0$ and that $H_b(0,0) = 0$, indicating that there is a steady state at $b = 0$ and that the system bifurcates at $\xi = 0$. Also, $H_\xi(0,0) = H_{bb}(0,0) = 0$, and

$$H_{b\xi}(0,0) = \beta \neq 0, \text{ and } H_{bbb}(0,0) = 6\phi_3\theta(0) \neq 0, \tag{8}$$

which identifies the occurrence of a pitchfork bifurcation.

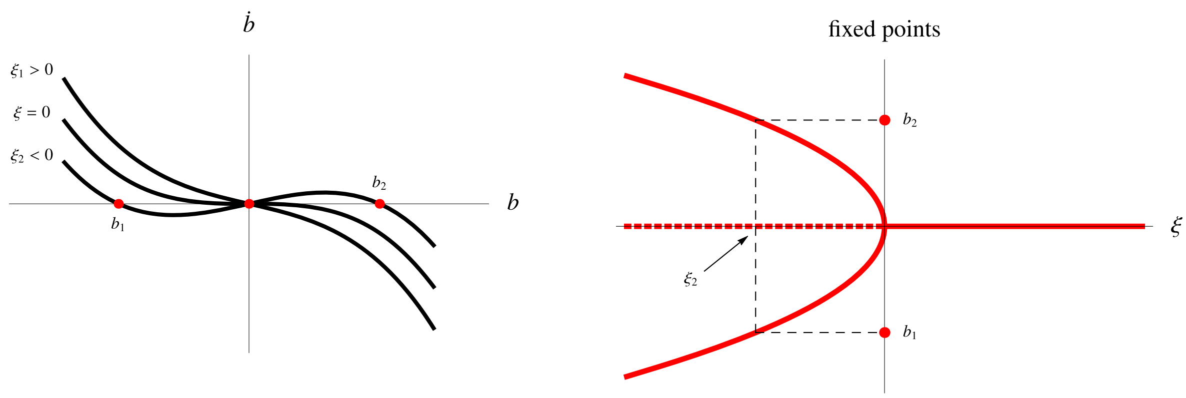

Figure 1: Supercritical pitchfork bifurcation, $H_{b\xi} < 0$, $H_{bbb} < 0$.

As we have noted, $b = 0$ is always a rest point of (7). A pitchfork bifurcation is characterized by the emergence of two additional rest points as ξ crosses the origin from the appropriate direction. Geometrically, as ξ crosses zero, the graph of the cubic H morphs to intersect the horizontal axis at two additional points. This phenomenon is witnessed in Figure 1. Here, we have chosen $H_{bbb} < 0$ and $H_{b\xi} < 0$. The left panel shows the graph of $H(b, \xi)$ for ξ greater than, equal to, and less than zero, and the right panel identifies the fixed points of (7) for given ξ. Notice that for $\xi \geq 0$ there is a unique fixed point, and for $\xi < 0$ there are three fixed points, indicating the existence of NRSE.

Stability information, which will be useful in the sequel, can also be gleaned from Figure 1. A rest point of the ordinary differential equation (ode) (7) is Lyapunov stable if H_b is negative. By observing the left panel, we see that as ξ crosses the origin from above, the zero steady state destabilizes and the two emergent steady states – the NRSE – are stable. This example illustrates a more general phenomenon: at a pitchfork bifurcation the stability of the origin flips and the stability of the non-trivial fixed points are opposite to the stability of the origin. The exact pattern of emergence and stability depend on the relative signs of $H_{b\xi}$ and H_{bbb}, as indicated in Figure 2. For the cubic case, using (8), these signs are easily translated into conditions on λ and ϕ_3; however, and importantly, we note that Figure 2 is general: it holds for any univariate system $\dot{b} = H(b, \xi)$ that undergoes a pitchfork bifurcation at the origin. We will use this fact in Section 2.2 when studying the stability of NRSE under learning.

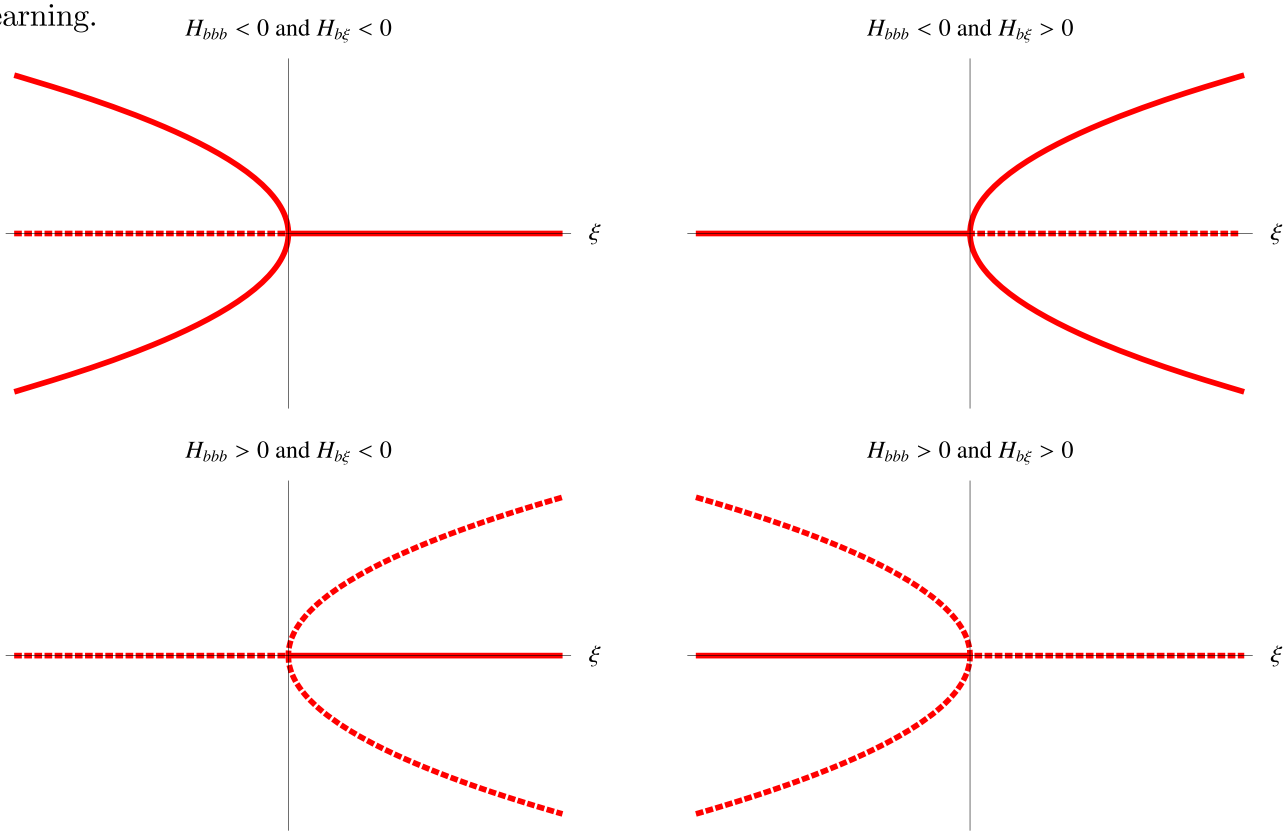

Figure 2: Pitchfork bifurcations. Dashed curves indicate unstable fixed points.

2.1.2 The general case

In this section, we allow F to be general and the odd moments of the sunspot to be non-zero. The essential difference is that we can no longer rely on the existence of NRSE with $a = 0$: the T-map is generically two dimensional. Establishing existence of NRSE proceeds as above, but the bifurcation analysis is more tedious because a center manifold reduction must first be performed. We have the following result, which is stated to emphasize the open set of resonance frequencies consistent with NRSE.

Theorem 1 (*existence*) *Assume that $|\beta| > 1$ and that either of the following two regularity conditions is met:*

1. *$F''(0) \neq 0$ and $\mu_3^\varepsilon \neq 0$;*
2. *$F'''(0)\left(\frac{3\sigma_\varepsilon^2}{\beta} + \frac{\mu_4^{\eta^\xi}}{\beta^3\sigma_{\eta^\xi}^2}\right) + \left(\frac{3(F''(0))^2}{(1-\beta)\beta}\right)\sigma_{\eta^\xi}^2 \neq 0.$*

Then NRSE exist. Specifically, there exists a neighborhood V of β^{-1} so that given any open set $W \subset V$ containing β^{-1} there is a $\lambda(\xi) \in W$ and a point $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$with $T(a, b, \xi) = (a, b)'$.

It may appear surprising that the generic existence conditions turn on the third moment of the sunspot's conditional shock, but brief reflection provides the intuition: if $\mu_3^\varepsilon \neq 0$ and $F''(0) \neq 0$ then H^b is $\mathcal{O}\left(\|(a,b)\|^2\right)$ so that the associated bifurcation is transcritical; in the case $\mu_3^\varepsilon = 0$ or $F''(0) = 0$, and condition 2 is met, it follows that H^b is $\mathcal{O}\left(\|(a,b)\|^3\right)$ so that the associated bifurcation is pitchfork. The nature of the bifurcation does not impinge on existence; however, there is an interesting implication for stability: see discussion following Theorem 2.

The proofs of all theorems are in the Appendix. While the details of the proof of Theorem 1 are somewhat tedious, a discussion of the argument is useful. Letting $\gamma = (a, b)'$, write $H(\gamma, \xi) = T(\gamma, \xi) - \gamma$. Direct computation allows for the following decomposition:

$$H(\gamma, \xi) = \begin{pmatrix} \beta - 1 & 0 \\ 0 & \beta\xi \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} f(a, b, \xi) \\ g(a, b, \xi) \end{pmatrix}, \tag{9}$$

where f and g are $\mathcal{O}\left(\|(a, b, \xi)\|^2\right)$.

It is evident that a bifurcation of the system $\dot{\gamma} = H(\gamma, \xi)$ occurs at $\xi = 0$. To assess the nature of this bifurcation, we appeal to the center manifold theorem. This theorem guarantees the existence of a sufficiently smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ characterizing an invariant, parameter-dependent manifold, that is, a differentiable subset $W_c(\xi)$ of $\mathbb{R}^2$, tangent to the b-axis, so that

- For ξ and b near zero, $W_c(\xi)$ is the graph of $a = h(b, \xi)$.
- $W_c(\xi)$ is invariant under the action of H, that is, the trajectory of γ implied by $\dot{\gamma} = H(\gamma, \xi)$ remains in $W_c(\xi)$ if it is initialized in $W_c(\xi)$.

The invariance of the center manifold may be used to specify a functional equation characterizing h. Specifically, by definition, $\dot{a} = (\beta - 1)a + f(a, b, \xi)$; and, on $W_c(\xi)$, $a = h(b, \xi)$, so that

$$\dot{a} = h_b(b, \xi)\dot{b} = h_b(b, \xi)\left(\beta\xi b + g(a, b, \xi)\right).$$

We conclude that h must satisfy the functional equation

$$(\beta - 1)a + f(a, b, \xi) = h_b(b, \xi)\left(\beta\xi b + g(a, b, \xi)\right).$$

Using this equation together with the implicit function theorem allows for the computation of the Taylor expansion of h to arbitrary order.

The importance of the manifold $W_c(\xi)$ follows from a corollary to the center manifold theorem which states that the dynamic behavior of the two-dimensional system $\dot{\gamma} = H(\gamma)$ is locally equivalent in a natural sense to its behavior on $W_c(\xi)$; and, using h, this behavior is captured by the univariate system

$$\dot{b} = \beta\xi b + g(h(b, \xi), b, \xi).$$

Finally, because this system is univariate, bifurcation analysis proceeds just as in the cubic case. The proof in the Appendix simply involves fleshing out the details of this analysis.

Theorem 1 generically addresses the first two concerns raised in the introduction and identified as motivating this effort. We now know when NRSE exist and what they look like. Further, we know that the resonance frequency restriction is an artifact of the linearization procedure: in fact, the sunspot's serial correlation acts a bifurcation parameter in the general case. Finally, and perhaps most interestingly, existence of NRSE obtains *if and only if* rational sunspot equilibria exist. This observation is particularly important from a practical perspective: assessing whether a given model may exhibit NRSE requires no new analytic tools.

2.2 NRSE: stability

Having established the generic existence of NRSE in the case $|\beta| > 1$, we now turn to the question of stability under learning. As is standard in the literature and natural given our assumptions regarding the forecasting behavior of agents, we have agents update their beliefs over time using recursive least squares: see Marcet and Sargent (1989) and Evans and Honkapohja (2001). Let $\gamma_t = (a_t, b_t)'$ represent agents' beliefs conditional on information dated t and earlier. These beliefs evolve according to the following recursions:

$$\begin{aligned} \gamma_t &= \gamma_{t-1} + \frac{1}{t} R_t^{-1} \begin{pmatrix} 1 \\ \eta_t^{\xi} \end{pmatrix} \left(\hat{F}\left(a_{t-1}, b_{t-1}, \xi, \eta_t^{\xi}\right) - \gamma_{t-1}' \begin{pmatrix} 1 \\ \eta_t^{\xi} \end{pmatrix} \right) \\ R_t &= R_{t-1} + \frac{1}{t} \left(\begin{pmatrix} 1 \\ \eta_t^{\xi} \end{pmatrix} \begin{pmatrix} 1 & \eta_t^{\xi} \end{pmatrix} - R_{t-1} \right), \end{aligned} \tag{10}$$

where R_t captures the sample second-moments matrix. The asymptotic behavior of this system may be analyzed by considering the differential equation system

$$\begin{aligned} \dot{\gamma} &= R^{-1} \int_{\Omega} \left(\begin{pmatrix} 1 \\ \eta^{\xi}(\omega) \end{pmatrix} \hat{F}\left(a, b, \xi, \eta^{\xi}(\omega)\right) \right) d\mu(\omega) - R^{-1} M \gamma \\ \dot{R} &= M - R, \end{aligned}$$

where

$$M = \int_{\Omega} \left(\begin{pmatrix} 1 \\ \eta^{\xi}(\omega) \end{pmatrix} \begin{pmatrix} 1 & \eta^{\xi}(\omega) \end{pmatrix} \right) d\mu(\omega)$$

is the a.e. limit of R_t by the law of large numbers. The stability of this system at a given rest point (γ^*, M) is determined by the stability of

$$\dot{\gamma} = T(\gamma) - \gamma \tag{11}$$

at γ^*. Since γ^* corresponds to a fixed point of the T-map, it identifies an NRSE. The theory of stochastic recursive algorithms tells us that if this fixed point is a Lyapunov stable rest point of (11), then an appropriately modified version of (10) will converge to it:[2] the associated NRSE is stable under learning. We note that the ordinary differential equation (ode) given by (11) corresponds to the usual E-stability differential equation, and thus in the sequel, we will rely on E-stability when assessing the stability NRSE under learning.

If the model is linear then, as noted above, NRSE correspond to resonance frequency sunspot equilibria: $\lambda = \beta^{-1}$. Assuming agents know λ, it follows that $E_t y_{t+1} = a + b\lambda\eta_t$, so that the actual law of motion is given by

$$y_t = \beta a + b\eta_t.$$

We find that $T(a, b) = (\beta a, b)'$, so that the eigenvalues of DT are β and 1. We conclude that for the linear model sunspot stability obtains provided that $\beta < -1$.[3]

[2] To guarantee almost sure convergence, learning algorithms may require a projection facility: see Evans and Honkapohja (2001) for details.

[3] It is standard, in the stability analysis of sunspot equilibria associated to linear models, for the T-map to have at least one unit eigenvalue. This neutral stability reflects the (artificial) fact that, in a linear environment, any scalar multiple of a sunspot is again a sunspot. For a discussion, see Evans and McGough (2005a).

2.2.1 The simple cubic

It is again revealing to begin by obtaining the stability condition for the case of the simple cubic, and we again restrict attention to PLM's of the form $y_t = b\eta_t$, which is consistent with NRSE in the cubic case. The T-map is given by (6), so the E-stability ode may thus be written

$$\dot{b} = \beta\xi b + \phi_3\theta(\xi)b^3. \tag{12}$$

We learned above that as ξ crosses zero from the appropriate side, a pitchfork bifurcation indicates the emergence of three fixed points. Importantly, the bifurcation also switches the stability of the original fixed point, and the new fixed points inherit the stability previously afforded the original fixed point. Thus if before the bifurcation the origin was stable then after the bifurcation the origin is unstable and the two new fixed points are stable. These stability transfers are evident in Figure 1, where we remember that a sufficient condition for Lyapunov stability in the univariate case is that the derivative be negative.

Applying these observations to the system (12), we conclude that if $\text{sign}(\beta) \neq \text{sign}(\phi_3)$ then the the cubic NRSE is stable under learning mechanisms consistent with (12). We note that the cubic assumption is not innocuous: as we will see in the next section, requiring that agents include a constant in their regression imparts additional restrictions.

2.2.2 The general case

Returning to the general case the relevant ode is given by (11). The analysis here again proceeds as it did with the cubic, and again, the principal distinction and difficulty is the center manifold analysis. Fortunately, we can rely on all of the hard work already done in the existence proof. We have the following result:

Theorem 2 (*stability*) *Assume that $\beta < -1$ and that either of the following two regularity conditions is met:*

1. $F''(0) \neq 0$ *and* $\mu_3^{\varepsilon} \neq 0$*;*
2. $F'''(0)\left(\frac{3\sigma_\varepsilon^2}{\beta} + \frac{\mu_4^{\eta^\xi}}{\beta^3\sigma_{\eta^\xi}^2}\right) + \left(\frac{3(F''(0))^2}{(1-\beta)\beta}\right)\sigma_{\eta^\xi}^2 < 0.$

Then there exist NRSE that are stable under adaptive learning.

When $\beta < -1$, the coefficients of $\sigma_{\eta^\xi}^2$ and $F'''(0)$, in the regularity inequality identified in item 2 of Theorem 2, are negative. This observation leads to the following corollary:

Corollary 1 *If $\beta < -1$ and either $F''(0) \neq 0$ or $F'''(0) > 0$ then stable NRSE exist.*

This result should be understood to mean that if the conditions of the corollary are met then stable NRSE exist for suitable choices of sunspot processes. More specifically, if $F'''(0) > 0$ then Condition 2 is met for any sunspot process with ξ near and on the appropriate side of β^{-1}, while if $F''(0) \neq 0$ then Condition 1 will be met for sunspots with ξ near and on the appropriate side of β^{-1} and $\mu_3^{\varepsilon} \neq 0$.

Returning now to the case of the simple cubic in Section 2.2.1, recall the associated stability condition $\text{sign}(\beta) = -\text{sign}(\phi_3)$, where $F'''(0) = 6\phi_3$. Note that no specific restriction on β was required: the PLM did not include an intercept because we assumed the agents understood the symmetry in F. If instead agents' PLM included a constant term, as in (3), then stability would require that $\beta < -1$ and $\phi_3 > 0$, in accordance with Corollary 1.

Theorems 1 and 2 provide vindication for resonance frequency sunspot equilibria: the knife-edge requirement needed in linear models is an artifact of the linearization and the tendency of resonance frequency sunspot equilibria to inherit the stability of the MSV solutions prevails in the non-linear world. Put differently, by Theorem 2, E-stability of resonance frequency sunspot equilibria in the linear model guarantees the existence of stable NRSE in the non-linear model (provided $F''(0) \neq 0$), which is a striking demonstration of the deep and broad reach of the E-stability principle.

2.3 NRSE and REE

While near-rational sunspot equilibria comprise a stand-alone equilibrium concept, it is natural to wonder about their connection to rational expectations equilibria. Establishing a formal connection requires taking a stand on the metric used for comparison, and is further complicated by the concepts' inherent multiplicities: even with a selected metric, which NRSE should be compared to which REE?

To make progress, we first characterize, to the extent possible, the REE local to the (indeterminate) steady state $y^* = 0$ of our model (1). Fix a martingale difference sequence (mds) $\hat{\varepsilon}_t$ with small support, and interpret it as the following rational forecast error: $\hat{\varepsilon}_t = F(y_t) - E_{t-1}F(y_t)$. It follows that the associated REE y_t must satisfy $F(y_t) = y_{t-1} + \hat{\varepsilon}_t$. Since $\hat{\varepsilon}_t$ has small support and $F'(0) \neq 0$, provided that $|y_{t-1}|$ is small, there is an open neighborhood U of the origin in $\mathbb{R}^2$, and a function $h : U \to \mathbb{R}$ so that $y_t = h(y_{t-1}, \hat{\varepsilon}_t)$. Furthermore, expanding h, we have that

$$y_t = \beta^{-1} y_{t-1} + \beta^{-1} \hat{\varepsilon}_t + \mathcal{O}\left(\| (y_{t-1}, \hat{\varepsilon}_t) \|^2 \right),$$

which, by indeterminacy (i.e. $|\beta| > 1$) guarantees that $|y_{t-1}|$ will remain small if initialized near the origin. We conclude that the function h characterizes the REE associated to the mds $\hat{\varepsilon}_t$.[4] Conversely, all REE local to the steady state can be represented in this fashion: simply

[4]Note that, provided F is sufficiently smooth, h can be approximated to arbitrarily high order by expanding each side of $F(h(y_{t-1}, \hat{\varepsilon}_t)) = y_{t-1} + \hat{\varepsilon}_t$ around $(0, 0)$ and equating coefficients.

note that if y_t is an REE local to the steady state then, by setting $\hat{\varepsilon}_t = F(y_t) - E_{t-1}F(y_t)$. we may construct a function h so that $y_t = h(y_{t-1}, \hat{\varepsilon}_t)$.

The characterization of REE by the function h provides the connection between REE and NRSE. In particular, note that, to first order, *any* mds $\hat{\varepsilon}_t$ induces an REE with serial correlation given by β^{-1}, and the serial correlation of *any* NRSE is a perturbation of this same value β^{-1}. Thus, to-first-order/up-to-perturbation, the correlograms of all REE and all NRSE are the same.

2.4 An overlapping generations model

Here we develop a simple economy that fits the hypotheses of Theorems 1 and 2. Consider an OLG environment in which there is a continuum of agents born at each time t indexed by $\omega_t \in \Omega$. Each agent lives two periods, works when young and consumes when old. The population is constant at unit mass. Each agent owns a production technology that is linear in labor and produces a common, perishable consumption good. The agent can sell his produced good in a competitive market for a quantity of fiat currency, anticipating that he will be able to use this currency when old to purchase goods for consumption.

Let $\omega_t \in \Omega$ be the index of a representative agent born in time t. This agent's problem is given by

$$\max_{c_{t+1}(\omega_t),\ n_t(\omega_t),\ M_t(\omega_t)} \quad E^*(\omega_t)\left(u(c_{t+1}(\omega_t)) - \nu(n_t(\omega_t))\right) \tag{13}$$

$$\text{subject to} \quad n_t(\omega_t) = q_t M_t(\omega_t) \ \text{ and } \ c_{t+1}(\omega_t) = q_{t+1}M_t(\omega_t).$$

Here, $n_t(\omega_t)$ is the agent's labor supply when young and $n_t(\omega_t)$ is his output. Also, q_t is the time t goods price of money and $c_{t+1}(\omega_t)$ is the agent's planned consumption when old. The expectations operator $E^*(\omega_t)(\cdot)$ denotes the expectation of agent ω_t at time t, taken with respect to his subjective beliefs conditional on the information available to him. This information includes $n_t(\omega_t)$, $M_t(\omega_t)$ and current and lagged values of q_t.

The first order condition is given by

$$\nu'(n_t(\omega_t)) = E^*(\omega_t)\left(\frac{q_{t+1}}{q_t}u'(c_{t+1}(\omega_t))\right), \tag{14}$$

and to make our model particularly tractable, we assume that $\nu' = 1$ and $u(c) = \frac{1}{1-\sigma}\left(c^{1-\sigma} - 1\right)$. With simplification, we obtain agent ω_t's decision rules:

$$\begin{aligned} n_t(\omega_t) &= \left(q_t^{\sigma-1}E^*(\omega_t)\left(q_{t+1}^{1-\sigma}\right)\right)^{\frac{1}{\sigma}} \\ M_t(\omega_t) &= \left(\frac{1}{q_t}E^*(\omega_t)\left(q_{t+1}^{1-\sigma}\right)\right)^{\frac{1}{\sigma}}; \end{aligned}$$

and we note that, as is natural, the quantity of money demanded by agent ω_t at time t, depends on, among other things, the price at time t.

Assuming a constant (unit) supply of money, we obtain the market-clearing condition

$$\int_\Omega M_t(\omega_t)d\omega_t = 1,$$

which yields

$$q_t = \left(\int_\Omega \left(E^*(\omega_t)\left(q_{t+1}^{1-\sigma}\right)\right)^{\frac{1}{\sigma}} d\omega_t\right)^\sigma. \tag{15}$$

Equation (15) characterizes the equilibrium price path.

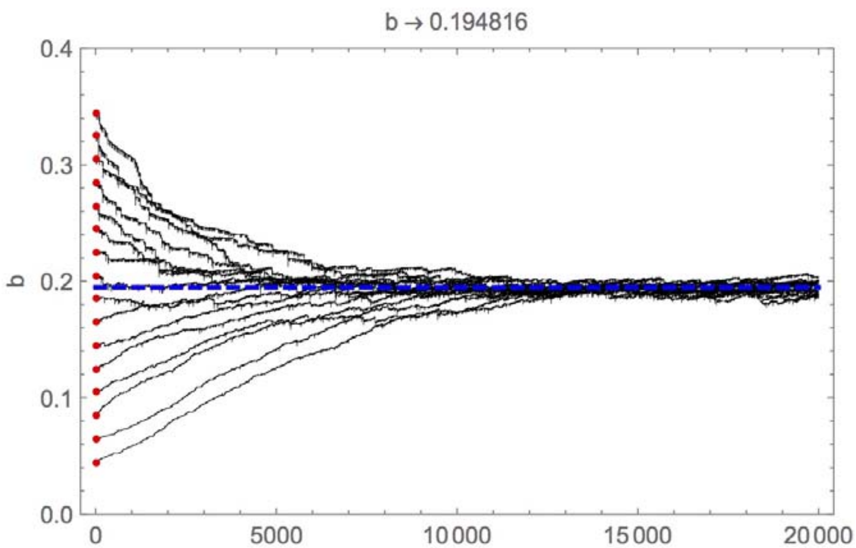

Figure 3: Learning dynamics

If agents are homogeneous, the model reduces to $q_t = E_t^* q_{t+1}^{1-\sigma}$, which is consistent with the framework considered in Section 2. If all agents have rational expectations then $q = 1$ is the unique, non-autarky, perfect-foresight steady state. The system may be log-linearized around this steady state to yield $\log q_t = (1-\sigma)E_t \log q_{t+1}$. The steady state is indeterminate if $\sigma > 2$: in this case the expectational feedback parameter is negative and sunspot equilibria exist in both the linearized and non-linear models.

To apply our theorems, let $F(y) = (y+1)^{1-\sigma} - 1$, so that the model becomes $y_t = E_t F(y_{t+1})$, with $y = q - 1$. We compute $F'(0) = 1 - \sigma$ and $F''(0) = \sigma(\sigma - 1)$, so that stable NRSE exist provided $\sigma > 2$.

To assess this claim numerically, we calibrate the model by setting $\sigma = 2.5$, and, since $F''(0) > 0$, we select a negative perturbation ($\xi < 0$), so that the NRSE is stable. Then, choosing an asymmetric iid martingale difference sequence ε_t, we simulate the real-time learning dynamics corresponding to a variety of initial conditions: see Figure 3, which plots the dynamics of b_t, the time t-value of the sunspot coefficient in the agent's forecasting model.[5] We observe convergence to the estimated NRSE value of $b^* = .195$.[6]

3 Stable NRSE: extensions

The reduced form model (1) served as a platform to discuss and provide intuition for our main existence and stability results; however, most applied macro models do not present so simply. Ideally, the theory of NRSE should be developed against a sequence of reduced-form equations of the form

$$E_t^* F(y_t, y_{t+1}, y_{t-1}, v_t) = 0, \tag{16}$$

where $y_t \in \mathbb{R}^n$ is endogenous, $v_t \in \mathbb{R}^m$ is a stationary exogenous process, and $F : \mathbb{R}^{3n} \oplus \mathbb{R}^m \to \mathbb{R}^n$ is C^k; however, results for models at this level of generality are not yet available. To make some progress, and to show how modifications of our underlying framework and arguments apply in our general settings, in this Section we consider, separately, a variety of extensions suggested by the model (16). Because the development and argument structure are similar to the work done in Section 2, our discussions here will be considerably more brief.

3.1 Stable NRSE: the implicit case

In many modeling environments, the time t endogenous variable is defined only implicitly in terms of expectations of future variables. To consider this case, let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^k $(k \geq 4)$, with $F(0,0) = 0$, $F_1(0,0) \neq 0$, and $\beta = -F_2(0,0)/F_1(0,0)$, where, in this section, F_i is the partial of F with respect to the i-th variable.

The sequence of reduced-form equations is given by

$$E_t^* F(y_t, y_{t+1}) = 0. \tag{17}$$

Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (17) satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

As in the previous section, let $\{\varepsilon_t\} \subset L^\infty(\Omega)$ be a zero-mean iid process with compact support, and assume $\xi \in \mathbb{R}$ is such that $\lambda(\xi) = \beta^{-1} + \xi \in (-1, 1)$. The agents' PLM is given as

$$\begin{aligned} y_t &= a + b\eta_t^\xi \\ \eta_t^\xi &= \lambda(\xi)\eta_{t-1}^\xi + \varepsilon_t, \end{aligned}$$

[5] For this Figure we use the following specification for the sunspot process: $\varepsilon_t \in \{-.475, .025\}$ is iid with $\Pr(\varepsilon_t = .025) = .95$, and $\xi = -.0175$. Since $\beta = -1.5$ this gives $\lambda(\xi) = -.684$.

[6] It can be shown that if $2 < \sigma < \frac{1}{4}\left(5 + \sqrt{17}\right) \approx 2.28$ then the sunspot's stochastic driver ε_t can be taken as symmetric.

which, by specifying E^*, gives the following implicitly defined ALM:

$$\tilde{F}\left(y_t, a, b, \xi, \eta_t^\xi\right) \equiv \int_\Omega F\left(y_t, a + b\lambda(\xi)\eta_t^\xi + b\varepsilon(\omega)\right) d\mu(\omega) = 0.$$

Noting that $\tilde{F}(0,0,0,\xi,\eta_t^\xi) = 0$ and that, evaluated at $y_t = a = b = 0$, we have $\tilde{F}_y = F_1(0,0) \neq 0$, the implicit function theorem implies that locally the ALM may be written $y_t = \hat{F}\left(a, b, \xi, \eta_t^\xi\right)$, where we are now assuming that the support of ε_t is such that η^ξ remains in the domain of $\hat{F}$ for small ξ.

With $\hat{F}$ so defined, we may proceed just as in the previous case by defining the T-map $T(\cdot,\cdot,\xi) : \mathbb{R}^2 \to \mathbb{R}^2$ as the projection of the ALM onto the span of $\{1, \eta_t^\xi\}$. This again yields the formula in equation (5). Again observe that $T(0,0,\xi) = (0,0)'$. An NRSE of this model is a non-trivial fixed point of this T-map.

The following result provides conditions for existence and stability. The complicated expression corresponding to $\mathcal{IC}$ is given by equation (17f) in the Appendix. All derivatives are evaluated at zero.

Theorem 3 *Assume that* $|\beta| > 1$.

- ***Existence.*** *Assume that any one of the following two regularity conditions is met:*

  1. $(2\beta F_{12} + F_{22})\,\mu_3^\varepsilon \neq \beta F_{112}\mu_4^{\eta^\xi}$;
  2. $\mathcal{IC} \neq 0$.

  Then NRSE exist. Specifically, there exists a neighborhood V *of* β^{-1} *such that given any open set* $W \subset V$ *containing* β^{-1} *there is a* $\lambda(\xi) \in W$ *and a point* $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *with* $T(a,b,\xi) = (a,b)'$.

- ***Stability.*** *Assume that* $\beta < -1$ *and that any of the following two regularity conditions is met:*

  1. $(2\beta F_{12} + F_{22})\,\mu_3^\varepsilon \neq \beta F_{112}\mu_4^{\eta^\xi}$;
  2. $\mathcal{IC} < 0$.

  Then the NRSE are stable under adaptive learning.

We observe that existence, and stability in case $\beta < -1$, are generic in the sense that they obtain for appropriate ε_t if $F_{112} \neq 0$. We note also that setting $F_1 = -1$ and $F_{1*} = 0$ corresponds to the previous case in which $y_t = E_t^* \breve{F}(y_{t+1})$ (for appropriate $\breve{F}$); and, the conditions we obtain here reduce to the conditions found in Theorems 1 and 2.

3.2 Stable NRSE: the stochastic case

In this section we allow for the presence of a stationary stochastic driver. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^k ($k \geq 4$), with $F(0,0) = 0$ and $\beta = DF_y(0,0)$. Let $\zeta_t \in L^\infty(\Omega)$ be an iid process, let $0 < \rho < 1$, and let

$$v_t^\sigma = \rho v_{t-1}^\sigma + \sigma\zeta_t = \sigma \sum_{k\geq 0} \rho^k \zeta_{t-k} \in L^\infty(\Omega),$$

for $\sigma \in \mathbb{R}$. The model is given by $y_t = E_t^* F(y_{t+1}, v_{t+1}^\sigma)$. Given the specification of E^*, we are interested in solutions $\{y_t\}$ satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

We first consider the concept and existence of near rational equilibria that depend only on v_t^σ and not also on a sunspot process. Henceforth, for notational simplicity, we will suppress the dependence of v_t on σ. Following the literature, we refer to equilibria of this type as *restricted perceptions equilibria* (RPE).

We assume agents use a PLM of the form $y_t = a + bv_t$, which yields the following ALM:

$$y_t = \int_\Omega F(a + b\rho v_t + b\sigma\zeta_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \equiv \tilde{F}(a, b, \sigma, v_t). \tag{18}$$

The corresponding T-map is given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{T=T(a,b,\sigma)} \begin{pmatrix} \int_\Omega \tilde{F}(a,b,\sigma,v(\omega))d\mu(\omega) \\ (\sigma_v^2(\sigma))^{-1} \int_\Omega v(\omega)\tilde{F}(a,b,\sigma,v(\omega))d\mu(\omega) \end{pmatrix} \equiv \begin{pmatrix} T^a(a,b,\sigma) \\ T^b(a,b,\sigma) \end{pmatrix},$$

where

$$\sigma_v^2(\sigma) = \int_\Omega (v(\omega))^2 \, d\mu(\omega).$$

A fixed point of the T-map provides an RPE of this non-linear model. The next lemma provides the existence result.

Lemma 2 *If $DF_v(0,0) \neq 0$ and $\beta \neq 1$ or ρ^{-1} then, for $|\sigma|$ sufficiently small, an RPE exists, that is, there exists $(a^*(\sigma), b^*(\sigma))' \in \mathbb{R}^2$, with $b^*(\sigma) \neq 0$, such that $T(a^*(\sigma), b^*(\sigma), \sigma) = (a^*(\sigma), b^*(\sigma))'$.*

We now turn to existence and stability of NRSE associated to this RPE, taking the form $y_t = a + bv_t + c\eta_t^\xi$. In what follows, unless otherwise specified, derivatives are evaluated at

$$(a, b, c, \xi) = (a^*, b^*, 0, 0).$$

As argued in the proof of Theorem 4, by choosing $|\sigma|$ small we may assume that $DF_\star(y, v) \approx DF_\star(0,0)$ for $\star = y, v, yy$, etc. Thus, we may assume, for the remainder of this section, that $|\beta| > 1$, whence we may choose σ small enough that $|DF_y| > 1$.

Turning first to expectations, the PLM is given by

$$\begin{aligned} y_t &= a + bv_t + c\eta_t^\xi \\ v_t &= \rho v_{t-1} + \sigma\zeta_t \\ \eta_t^\xi &= \lambda(\xi)\eta_{t-1}^\xi + \varepsilon_t, \end{aligned}$$

where $\lambda(\xi) = \beta^{-1} + \xi \in (-1, 1)$. We further assume that $\zeta_t \perp \varepsilon_s$ for all t, s. For fixed small σ, the ALM is given by $y_t = \hat{F}\left(a, b, c, \xi, v_t, \eta_t^\xi\right)$ where

$$\hat{F} = \int_\Omega F(a + b\rho v_t + b\sigma\zeta_{t+1}(\omega) + c\lambda(\xi)\eta_t^\xi + c\varepsilon_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega).$$

We note that the function $\hat{F}$ here is different from the function used in the previous proof. Exploiting independence, the T-map is given by

$$\begin{aligned} a &\xrightarrow{T^a(a,b,c,\xi)} \int_\Omega \hat{F}\left(a, b, c, \xi, v(\omega), \eta^\xi(\omega)\right) d\mu(\omega) \\ b &\xrightarrow{T^b(a,b,c,\xi)} \frac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}\left(a, b, c, \xi, v(\omega), \eta^\xi(\omega)\right) d\mu(\omega) \\ c &\xrightarrow{T^c(a,b,c,\xi)} \frac{1}{\sigma_{\eta^\xi}^2}\int_\Omega \eta^\xi(\omega)\hat{F}\left(a, b, c, \xi, v(\omega), \eta^\xi(\omega)\right) d\mu(\omega) \end{aligned}$$

Finally, let $(a^*(\sigma), b^*(\sigma)) = (a^*, b^*)$ be an RPE corresponding to σ, and note, using (18), that

$$T(a^*, b^*, 0, \xi) = (a^*, b^*, 0)'.$$

A non-trivial (i.e. $c \neq 0$) fixed point of the T-map is an NRSE.

Theorem 4 *Assume $DF_v(0,0) \neq 0$, $|\beta| > 1$, $\beta\rho \neq 1$, and that $|\sigma|$ is sufficiently small.*

- ***Existence.*** *Assume that either of the following two regularity conditions is met:*

  1. *$DF_{yy} \neq 0$ and $\mu_3^\varepsilon \neq 0$;*
  2. *$DF_{yyy}\left(\frac{3\sigma_\xi^2}{\beta} + \frac{\mu_4^{\eta^\xi}}{\beta^3\sigma_{\eta^\xi}^2}\right) + \frac{3(DF_{yy})^2\left(\sigma_{\eta^\xi}^2 + \beta^2\sigma_\xi^2\right)}{(1-\beta)\beta^3} \neq 0.$*

  Then NRSE exist. Specifically, there exists a neighborhood V of β^{-1} so that given any open set $W \subset V$ containing β^{-1} there is a $\lambda(\xi) \in W$ and a point $(a, b, c) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ with $T(a, b, c, \xi) = (a, b, c)'$.

- ***Stability.*** *Assume further that $\beta < -1$ and that either of the following two regularity conditions is met:*

1. $DF_{yy} \neq 0$ *and* $\mu_3^{\xi} \neq 0$*;*

2. $DF_{yyy}\left(\frac{3\sigma_{\xi}^2}{\beta} + \frac{\mu_4^{\eta^\xi}}{\beta^3\sigma^2_{\eta^\xi}}\right) + \frac{3(DF_{yy})^2\left(\sigma^2_{\eta^\xi}+\beta^2\sigma^2_{\xi}\right)}{(1-\beta)\beta^3} < 0.$

Then the NRSE are stable under adaptive learning.

3.3 Stable NRSE: the multivariate case

In principle, there is no difficulty conducting the above analysis in higher dimensions, though in practice the work is somewhat more tedious; and, two distinct cases arise, depending on the nature of the model's roots. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be C^k with $F(0) = 0$ and assume $DF(0) \in \mathbb{R}^{n\times n}$ is diagonalizable. Write $DF(0) = S \cdot \oplus_{i=1}^n \beta_i \cdot S^{-1}$, with $\beta_i \in \mathbb{C}$ the eigenvalues of $DF(0)$. The economic model is given by

$$y_t = E_t^* F(y_{t+1}). \tag{19}$$

Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (19) satisfying $y_{it} \in L^\infty(\Omega)$, and $\sup_t \|y_{it}\|_\infty < \infty$.

So that the model is indeterminate, we assume at least one root, which we label as β_n, lies outside the unit circle. We make the further assumption that $\beta_n \in \mathbb{R}$. This is for simplicity, as the analysis is considerably more involved if all roots that lie outside S^1 are complex: the sunspot is necessarily a two-dimensional VAR(1) process, and co-dimension-2 bifurcation analysis is required.[7]

Working as before, assume $\xi \in \mathbb{R}$ is such that $\lambda(\xi) = \beta_n^{-1} + \xi \in (-1, 1)$. The agents' PLM is given as

$$\begin{aligned} y_t &= a + b\eta_t^\xi \\ \eta_t^\xi &= \lambda(\xi)\eta_{t-1}^\xi + \varepsilon_t, \end{aligned}$$

with $a, b \in \mathbb{R}^n$. Writing $F = (F^1, \ldots, F^n)'$, the ALM is given by

$$y_{it} = \int_\Omega F^i\left(a + b\lambda(\xi)\eta_t^\xi + b\varepsilon_{t+1}(\omega)\right) d\mu(\omega) \equiv \hat{F}^i\left(a, b, \xi, \eta_t^\xi\right).$$

The T-map is given by

$$\begin{aligned} a_i &\to \int_\Omega \hat{F}^i\left(a, b, \xi, \eta^\xi(\omega)\right) d\mu(\omega) \\ b_i &\to \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \hat{F}^i\left(a, b, \xi, \eta^\xi(\omega)\right) d\mu(\omega). \end{aligned}$$

[7]Preliminary results indicate that, in the complex case, appropriate perturbation of the sunspot process's covariance matrix results in a Bogdanov-Takens bifurcation, from which a stable NRSE emerges. We also note that if $DF(0)$ has $k \leq n$ eigenvalues lying outside the unit circle, then sunspot processes up to dimension k may exist. We are developing these results in current work.

It is immediate that $(a, b) = (0, 0) \in \mathbb{R}^n \oplus \mathbb{R}^n$ is a fixed point of the T-map. A fixed point with non-zero b is an NRSE. The next theorem establishes existence and stability of NRSE in the simpler, transcritical case, which occurs when $\mu_3^\varepsilon \neq 0$. As notation, let S_n be the n^{th}-column of S (i.e. an eigenvector associated to β_n), $S^{-1} = (S^{ij})$ and D^2F^i the Hessian of F^i evaluated at zero.

Theorem 5 *Let $DF(0) = S \cdot \oplus_{i=1}^n \beta_i \cdot S^{-1}$ with $\beta_n \in \mathbb{R}$ and $|\beta_n| > 1$.*

- ***Existence.*** *Assume the following regularity conditions hold:*

  1. $\mu_3^\varepsilon \neq 0$
  2. $\sum_{i=1}^n S^{ni} \left(S_n' \cdot D^2F^i \cdot S_n\right) \neq 0.$

  Then NRSE exist. Specifically, there exists a neighborhood V of β_n^{-1} so that given any open set $W \subset V$ containing β_n^{-1} there is a $\lambda(\xi) \in W$ and a point $(a, b) \in \mathbb{R}^n \oplus \mathbb{R}^n \setminus \{(0, 0)\}$ with $T(a, b, \xi) = (a, b)'$.

- ***Stability.*** *Assume further that $\mathrm{Re}\,(\beta_i) < 1$ for all $i = 1, \ldots, n$ and $\frac{\mathrm{Re}(\beta_i)}{\beta_n} < 1$ for all $i = 1, \ldots, n-1$. Then the NRSE are E-stable.*

We remark that the second regularity condition for existence (above) can be viewed as generic in the following sense: S is invertible (and thus the S_{ij} and S^{ij} are not all zero) and S is a first-order term whereas the D^2F^i are second-order.

3.4 Stable NRSE: the case with lags

As a final extension, we consider a univariate reduced-form model with an endogenous lag. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^k $(k \geq 4)$, with $F(0, 0) = 0$, $F_1(0, 0) = \beta \neq 0$, and $F_2(0, 0) = \delta \neq 0$, where, in this section, F_i is the partial of F with respect to the i-th variable.

The sequence of reduced-form equations is given by

$$y_t = E_t^* F(y_{t+1}, y_{t-1}). \tag{20}$$

Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (20) satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

The linearized, RE version of this model is given by

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1}. \tag{21}$$

This model is indeterminate and has real roots provided $|\beta + \delta| > 1$, $|\delta| < |\beta|$ and $\beta\delta < \frac{1}{4}$, and we assume these conditions hold throughout this section.

As in previous sections, let $\{\varepsilon_t\} \subset L^\infty(\Omega)$ be a zero-mean iid process with compact support. Evans and McGough (2005c) showed that the process given by

$$\begin{aligned} y_t &= \varphi_i y_{t-1} + \eta(j)_t \\ \eta(j)_t &= \varphi_j \eta(j)_{t-1} + \varepsilon_t \end{aligned}$$

where $i, j \in \{1, 2\}$ with $i \neq j$, is a sunspot REE of (21), and

$$\varphi_1 = \frac{1 - \sqrt{1 - 4\beta\delta}}{2\beta} \text{ and } \varphi_2 = \frac{1 + \sqrt{1 - 4\beta\delta}}{2\beta}$$

are the roots of the model; and further, if $\beta < 0$ then this REE is stable under adaptive learning.

As we did in Section 2, we use perturbations of sunspots in the linear model to generate NRSE in the non-linear model. Let $\xi \in \mathbb{R}$ be such that $\lambda_i(\xi) = \varphi_i + \xi \in (-1, 1)$. The agents' PLM is given as

$$\begin{aligned} y_t &= a + by_{t-1} + c\eta(i)_t^\xi \\ \eta(i)_t^\xi &= \lambda_i(\xi)\eta(i)_{t-1}^\xi + \varepsilon_t, \end{aligned}$$

which, by specifying E^*, gives the following implicitly defined ALM:

$$y_t = \int_\Omega F\left((1+b)a + b^2 y_{t-1} + c\left(\lambda_i(\xi) + b\right)\eta(i)_t^\xi + c\varepsilon(\omega), y_{t-1}\right) d\mu(\omega).$$

Here we are assuming, as in common in the literature, that when agents form expectations their information set includes y_{t-1} and η_t, but not y_t. The next step in the analysis would normally be to define the T-map, but this requires knowledge of the asymptotic distribution of the regressors for fixed beliefs (a, b, c). Unfortunately, given the presence of y_{t-1}, this distribution is endogenous to beliefs, which appears to be a formidable technical impediment. Based on our work thus far, the following conjecture seems reasonable:

Conjecture 1 *Assume that* $|\beta + \delta| > 1$, $|\delta| < |\beta|$ *and* $\beta\delta < \frac{1}{4}$.

- ***Existence.*** *NRSE generically exist.*
- ***Stability.*** *If, in addition,* $\beta < 0$, *then E-stable NRSE generically exist.*

To provide support for this conjecture, we present numerical results. First, observe that a "sample-version" of a T-map may be defined. Specifically, for fixed beliefs (a, b, c), we may draw a sequence of N shocks $\left\{\eta(i)_t^\xi\right\}_{t=0}^N$, and using quadrature to evaluate $\hat{F}^i$, compute the associated endogenous realizations $\{y_t\}_{t=0}^N$, where y_0 and $\eta(i)_0^\xi$ are taken as given. The sample T-map is given by simply using these data to regress y_t on y_{t-1}, $\eta(i)_t^\xi$ and a constant. If the sample size N is large enough (and if the associated asymptotic distributions exist, etc.) then the sample T-map should well-approximate the true T-map, which means a fixed point of the sample T-map should well-approximate an NRSE. Finally, if the NRSE is E-stable, it is expected that iteration of the sample T-map, possibly modified to include a damping factor, should converge to a fixed point.

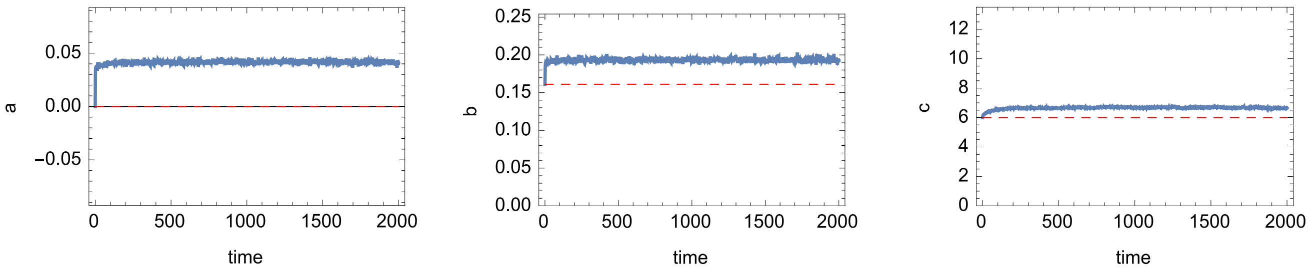

Figure 4: NRSE with lags, T-map

Precisely this experiment is carried out in Figure 4. The map F used to construct this figure has linear terms $\beta = -1.5$ and $\delta = .2$ and an ad-hoc quadratic form to capture the non-linearity. Thus the linear model is indeterminate, and, according to the conjecture, we expect stable NRSE to exist. Sample size is set at $3,000$, and the shock ε_t is uniformly distributed on $[-.1, .1]$. The initial conditions for beliefs, as indicated by the red, dashed lines, correspond to the linear REE values, with c set arbitrarily at 6. The sample T-map is then iterated, and the "time-plot" is provided in the Figure. We see convincing evidence of rapid convergence to non-REE values, suggesting the presence of a stable NRSE.[8]

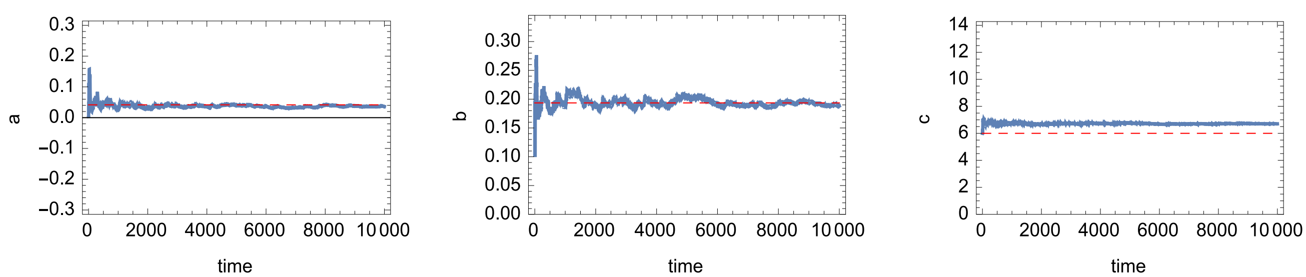

Figure 5: NRSE with lags, RTL

We may also conduct the analogous real-time learning simulation – See Figure 5. In this case, as new data become available, beliefs are updated over time using recursive least squares. As above, the dynamics are initialized at the linear REE values, and a decreasing gain algorithm is used. The red, dashed lines in the first two panels identify the fixed point of the sample T-map, and in the last panel, the red, dashed line corresponds to the initial

[8] That the sample T-map never settles down to a fixed point is a reflection of the finite sample properties of the map.

condition for beliefs c. We note that convergence appears to obtain to the fixed point of the sample T-map identified in Figure 4, thus supporting our conjecture.

4 Conclusion

According to Blanchard, " ...the world economy is pregnant with multiple equilibria – self-fulfilling outcomes of pessimism or optimism, with major macroeconomic implications."[9] This conclusion, and others like it, makes imperative understanding when and how sunspot equilibria, which represent and characterize the class of stationary multiple equilibria, are consistent with the dynamic stochastic general equilibrium modeling paradigm of the macroeconomic literature.

Investigations of sunspot equilibria in mainstream models have met with a variety of obstacles. Most notably, and as indicated in the Introduction, sunspot equilibria in non-linear models have complicated stochastic structure, making them difficult for researchers and economic agents to model, and thus rendering stability analysis impossible.

Our embrace of a linear-forecasting framework allows us to circumvent this obstacle while preserving natural, agent-level behavior. We establish the existence of (near-rational) sunspot equilibria that have simple recursive stochastic structure. By providing agents an understanding of this structure, we are then able to assess stability under adaptive learning, and indeed establish generic stability results.

It is important to emphasize the link between the existence and stability of NRSE, and the existence and stability of sunspot equilibria in the corresponding rational model. We find that if sunspot equilibria exist in the rational (linearized, and thus the non-linear) model then NRSE exist; and if sunspot equilibria are stable in the linearized model then the NRSE are stable. In fact, an even deeper connection prevails: the extension of an observed phenomenon, which we call *The MSV Principle,* to a non-linear environment. The MSV Principle states that in a linear(ized) model, if the steady state is indeterminate and the MSV solution is stable under learning then there exist stable sunspot equilibria. While we have not formally established this in a completely general setting, in all of our work we know of no counterexample. The power of this principle lies in its computational simplicity: it is often quite easy to identify and analyze the stability of the MSV solution to a linear model. Our work here generalizes the principle as follows: If the steady state is indeterminate and the MSV solution to the linearized model is stable under learning then there exist stable NRSE in the associated non-linear model.

[9] IMF blog, http://blog-imfdirect.imf.org/2011/12/21/2011-in-review-four-hard-truths/

References

Azariadis, C. (1981): "Self-Fulfilling Prophecies," *Journal of Economic Theory*, 25, 380–396.

Azariadis, C., and R. Guesnerie (1986): "Sunspots and Cycles," *Review of Economic Studies*, 53, 725–737.

Benhabib, J., and R. A. Farmer (1994): "Indeterminacy and Increasing Returns," *Journal of Economic Theory*, 63, 19–41.

Blanchard, O., and C. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48, 1305–1311.

Cass, D., and K. Shell (1983): "Do Sunspots Matter?," *Journal of Political Economy*, 91, 193–227.

Duffy, J., and W. Xiao (2007): "Instability of Sunspot Equilibria in Real Business Cycle Models under Adaptive Learning," *Journal of Monetary Economics*, 54, 879–903.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.

——— (2003): "Existence of Adaptively Stable Sunspot Equilibria near an Indeterminate Steady State," *Journal of Economic Theory*, 111, 125–134.

Evans, G. W., and B. McGough (2005a): "Indeterminacy and the Stability Puzzle in Non-Convex Economies," *The B.E. Journal of Macroeconomics* (Contributions), 5, Iss. 1, Article 8.

——— (2005b): "Monetary Policy, Indeterminacy and Learning," *Journal of Economic Dynamics and Control*, 29, 1809–1840.

——— (2005c): "Stable Sunspot Solutions in Models with Predetermined Variables," *Journal of Economic Dynamics and Control*, 29, 601–625.

——— (2011): "Representations and Sunspot Stability," *Macroeconomic Dynamics*, 15, 80–92.

Farmer, R. E., and J.-T. Guo (1994): "Real Business Cycles and the Animal Spirits Hypothesis," *The Journal of Economic Theory*, 63, 42–72.

Guesnerie, R. (1986): "Stationary Sunspot Equilibria in an N-commodity World," *Journal of Economic Theory*, 40, 103–128.

Guesnerie, R., and M. Woodford (1992): "Endogenous Fluctuations," in Laffont (1992), chap. 6, pp. 289–412.

LAFFONT, J.-J. (ed.) (1992): *Advances in Economic Theory: Sixth World Congress. Volume 2.* Cambridge University Press, Cambridge, UK.

MARCET, A., AND T. J. SARGENT (1989): "Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

SHELL, K. (1977): "Monnaie et Allocation Intertemporelle," Working paper, CNRS Seminaire de E.Malinvaud, Paris.

WIGGINS, S. (1990): *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer-Verlag, Berlin.

WOODFORD, M. (1986): "Stationary Sunspot Equilibria: The Case of Small Fluctuations around a Deterministic Steady State," Manuscript, University of Chicago and New York University.

——— (1990): "Learning to Believe in Sunspots," *Econometrica*, 58, 277–307.

Appendix

Proof of Lemma 1. Fix $x \in U$, $i \in \{1,\dots,n\}$, and let Δ_m be a real sequence converging to zero such that

$$x(\Delta_m) = (x_1,\dots,x_{i-1},x_i+\Delta_m,x_{i+1},\dots,x_n) \in U.$$

Define $h_n(x,\omega) = \Delta_m^{-1}(h(x(\Delta_m) - h(x))$. Since $L^\infty(\Omega) \subset L^1(\Omega)$ it follows that $h_n(x,\cdot) \in L^1(\Omega)$ and $h_n(x,\cdot) \to D_{x_i}h(x,\cdot)$ almost everywhere. By the mean-value theorem, for almost all $\omega \in \Omega$, there is a δ_m with $|\delta_m| < |\Delta_m|$ such that

$$|h_n(x,\omega)| = |D_{x_i}h(x(\delta_m),\omega)| \leq G(\omega).$$

We may compute

$$\begin{aligned} D_{x_i}H(x) &= \lim_{m\to\infty} \Delta_m^{-1}(H(x(\Delta_m)) - H(x)) = \lim_{m\to\infty}\int_\Omega h_m(x,\omega)d\mu(\omega) \\ &= \int_\Omega \lim_{m\to\infty} h_m(x,\omega)d\mu(\omega) = \int_\Omega D_{x_i}h(x,\omega)d\mu(\omega), \end{aligned}$$

where the third equality follows from the dominated convergence theorem. The proof is completed by induction, recognizing that $D_{x_i}h(\cdot,\omega) \in C^{k-1}(U)$. ■

In the work below we will repeatedly be required to differentiate functions of the form H, constructed from functions of the form h, as defined in the lemma above. Our analysis will be local to a steady state, so that our sets U will have compact closure, thus giving the needed uniform bounds on D_xh, which themselves are assumed continuous.

Proof of Theorems 1 and 2. This analysis requires the computation of a host of derivatives, and we proceed with these computations now. Importantly, all derivatives of F are evaluated at zero and all partials (first and higher orders) of $\hat{F}$ and T are evaluated at $a = b = \xi = 0$. For notational ease, we will often omit the arguments. Note that when computing derivatives of $\hat{F}$, the variable η^ξ is taken as fixed.

Derivatives of $\hat{F}(a,b,\xi,\eta^{\xi}) = \int_{\Omega} F(a+b\lambda(\xi)\eta^{\xi}+b\varepsilon(\omega))d\mu(\omega)$

$$\hat{F}_a = \int_{\Omega} F' d\mu(\omega) = \beta \tag{1a}$$

$$\hat{F}_b = \int_{\Omega} F'(\beta^{-1}\eta^{\xi}+\varepsilon(\omega))d\mu(\omega) = \eta^{\xi} \tag{1b}$$

$$\hat{F}_{\xi} = \int_{\Omega} F' b\eta^{\xi} d\mu(\omega) = 0, \text{ since } b = 0. \tag{1c}$$

$$\hat{F}_{aa} = \int_{\Omega} F'' d\mu(\omega) = F''(0) \tag{1d}$$

$$\hat{F}_{ab} = \int_{\Omega} \lambda(\xi)\eta^{\xi} F'' d\mu(\omega) = \beta^{-1}\eta^{\xi} F''(0) \tag{1e}$$

$$\hat{F}_{bb} = \int_{\Omega} F''(\lambda(\xi)\eta^{\xi}+\varepsilon(\omega))^2 d\mu(\omega) = (\beta^{-2}\left(\eta^{\xi}\right)^2 + \sigma_{\varepsilon}^2)F''(0) \tag{1f}$$

$$\hat{F}_{\xi\xi} = \int_{\Omega} F''(b\eta^{\xi})^2 d\mu(\omega) = 0 \tag{1g}$$

$$\hat{F}_{a\xi} = \int_{\Omega} F'' b\eta^{\xi} d\mu(\omega) = 0 \tag{1h}$$

$$\hat{F}_{b\xi} = \int_{\Omega} (\eta^{\xi} F' + b\eta^{\xi} F'')d\mu(\omega) = \beta\eta^{\xi} \tag{1i}$$

$$\hat{F}_{bbb} = \int_{\Omega} F'''(\lambda(\xi)\eta^{\xi}+\varepsilon(\omega))^3 d\mu(\omega) = F'''(0)((\beta^{-1}\eta^{\xi})^3 + 3\beta^{-1}\eta^{\xi}\sigma_{\varepsilon}^2) \tag{1j}$$

Derivatives of $T^a(a,b,\xi) = \int_{\Omega} \hat{F}(a,b,\xi,\eta^{\xi}(\omega))d\mu(\omega)$

$$T^a_a = \int_{\Omega} \hat{F}_a d\mu(\omega) = \beta \tag{2a}$$

$$T^a_b = \int_{\Omega} \hat{F}_b d\mu(\omega) = \int_{\Omega} \eta^{\xi}(\omega)d\mu(\omega) = 0 \tag{2b}$$

$$T^a_{\xi} = \int_{\Omega} \hat{F}_{\xi} d\mu(\omega) = 0 \tag{2c}$$

$$T^a_{aa} = \int_{\Omega} \hat{F}_{aa} d\mu(\omega) = F''(0) \tag{2d}$$

$$T^a_{ab} = \int_{\Omega} \hat{F}_{ab} d\mu(\omega) = \int_{\Omega} \lambda(\xi)F''\eta^{\xi}(\omega)d\mu(\omega) = 0 \tag{2e}$$

$$T^a_{bb} = \int_{\Omega} \hat{F}_{bb} d\mu(\omega) = \int_{\Omega} (\beta^{-2}(\eta^{\xi}(\omega))^2 + \sigma_{\varepsilon}^2)F''(0)d\mu(\omega) = \sigma_{\eta}^2 F''(0) \tag{2f}$$

$$T^a_{\xi\xi} = \int_{\Omega} \hat{F}_{\xi\xi} d\mu(\omega) = 0 \tag{2g}$$

$$T^a_{\xi a} = \int_{\Omega} \hat{F}_{\xi a} d\mu(\omega) = 0 \tag{2h}$$

$$T^a_{\xi b} = \int_{\Omega} \hat{F}_{\xi b} d\mu(\omega) = \int_{\Omega} \beta\eta^{\xi}(\omega)d\mu(\omega) = 0 \tag{2i}$$

Derivatives of $T^b(a,b,\xi) = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \hat{F}(a,b,\xi,\eta^\xi(\omega))d\mu(\omega)$

$$T^b_a = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega) = 0 \tag{3a}$$

$$T^b_b = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega (\eta^\xi(\omega))^2 d\mu(\omega) = 1 \tag{3b}$$

$$T^b_\xi = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left[\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu(\omega) - \frac{\partial}{\partial\xi}\sigma^2_\eta(\xi)\int_\Omega \eta^\xi(\omega)\hat{F}d\mu(\omega)\right] = 0 \tag{3c}$$

$$T^b_{bb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{bb} d\mu(\omega) = \int_\Omega \eta^\xi(\omega)(\beta^{-2}(\eta^\xi(\omega))^2 + \sigma^2_\varepsilon)F''(0)d\mu(\omega) = \frac{F''(0)\mu_3^{\eta^\xi}}{\beta^2\sigma^2_\eta} \tag{3d}$$

$$T^b_{ab} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \beta^{-1}F''(\eta^\xi(\omega))^2 = \beta^{-1}F''(0) \tag{3e}$$

$$T^b_{aa} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{aa} d\mu(\omega) = 0 \tag{3f}$$

$$\begin{aligned} T^b_{\xi\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-4} \Bigg\{ \left(\sigma^2_\eta\right)^2 \Bigg[& \sigma^2_{\eta^\xi} \int_\Omega \left(2\hat{F}_\xi \frac{\partial}{\partial\xi}\eta^\xi(\omega) + \hat{F}\frac{\partial^2}{(\partial\xi)^2}\eta^\xi(\omega) + \hat{F}_{\xi\xi}\eta^\xi(\omega)\right) d\mu(\omega) \\ & + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu - \int_\Omega \frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} d\mu(\omega) \\ & - \int_\Omega \eta^\xi(\omega)\hat{F}_\xi \frac{\partial}{\partial\xi}\sigma^2_\eta(\xi)d\mu(\omega) - \int_\Omega \eta^\xi(\omega)\hat{F}\frac{\partial^2}{(\partial\xi)^2}\sigma^2_\eta(\xi)d\mu(\omega)\Bigg] \\ & -\sigma^2_{\eta^\xi}\int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right)\frac{\partial}{\partial\xi}(\sigma^2_\eta(\xi))^2 d\mu(\omega) + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega \eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega)\Bigg\} = 0 \end{aligned} \tag{3g}$$

$$T^b_{b\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left[\sigma^2_{\eta^\xi} \int_\Omega \left(\frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F}_b + \eta^\xi(\omega)\hat{F}_{b\xi}\right) d\mu(\omega) - \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega)\right] = \beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{3h}$$

$$T^b_{bbb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \hat{F}_{bbb}\eta^\xi(\omega)d\mu(\omega) = \frac{F'''(0)}{\sigma^2_\eta}\left(\beta^{-3}\sigma^4_\eta + 3\beta^{-1}\sigma^2_\eta\sigma^2_\varepsilon\right). \tag{3i}$$

Equation (3h) requires elaboration. Since

$$\begin{aligned} \frac{\partial}{\partial\xi}\eta^\xi(\omega) &= \lambda(\xi)^{-1}\sum_{m\geq 0} m\lambda(\xi)^m\varepsilon_m(\omega), \text{ and} \\ \hat{F}_b(\omega) &\equiv \hat{F}_b(a,b,\xi,\eta^\xi(\omega)) = \beta\lambda(\xi)\eta^\xi(\omega) = \beta\lambda(\xi)\sum_{k\geq 0}\lambda(\xi)^k\varepsilon_k(\omega), \end{aligned} \tag{4}$$

it follows that

$$\begin{aligned}
\int_\Omega \hat{F}_b(\omega)\frac{\partial}{\partial\xi}\eta^\xi(\omega)d\mu(\omega) &= \beta\int_\Omega\left(\sum_{k\geq 0}\lambda(\xi)^k\varepsilon_k(\omega)\right)\left(\sum_{m\geq 0}m\lambda(\xi)^m\varepsilon_m(\omega)\right)d\mu(\omega)\\
&= \beta\int_\Omega\sum_{k\geq 0}k\left(\lambda(\xi)^2\right)^k\varepsilon_k(\omega)^2d\mu(\omega)=\beta\lambda(\xi)^2\sum_{k\geq 0}k\left(\lambda(\xi)^2\right)^{k-1}\sigma_\varepsilon^2\\
&= \beta\lambda(\xi)^2\sigma_\varepsilon^2\frac{\partial}{\partial\lambda(\xi)^2}\sum_{k\geq 0}\left(\lambda(\xi)^2\right)^k=\\
&= \beta\lambda(\xi)^2\sigma_\varepsilon^2\frac{\partial}{\partial\lambda(\xi)^2}\left(1-\lambda(\xi)^2\right)^{-1}=\beta\left(\frac{\lambda(\xi)^2}{1-\lambda(\xi)^2}\right)\sigma_{\eta^\xi}^2.
\end{aligned}$$

Next,

$$\int_\Omega \eta^\xi(\omega)\hat{F}_{b\xi}d\mu(\omega) \;=\; \beta\int_\Omega\left(\eta^\xi(\omega)\right)^2d\mu(\omega)=\beta\sigma_{\eta^\xi}^2.$$

Finally,

$$\frac{\partial}{\partial\xi}\sigma_\eta^2(\xi) \;=\; \frac{\partial}{\partial\xi}\left(\frac{\sigma_\varepsilon^2}{1-\lambda(\xi)^2}\right)=\frac{2\lambda(\xi)\sigma_\varepsilon^2}{\left(1-\lambda(\xi)^2\right)^2}=2\left(\frac{\lambda(\xi)}{1-\lambda(\xi)^2}\right)\sigma_{\eta^\xi}^2,$$

so that

$$\frac{\partial}{\partial\xi}\sigma_\eta^2(\xi)\int_\Omega\eta^\xi(\omega)\hat{F}_bd\mu(\omega) \;=\; 2\left(\frac{\lambda(\xi)}{1-\lambda(\xi)^2}\right)\left(\sigma_{\eta^\xi}^2\right)^2.$$

Thus

$$\begin{aligned}
T^b_{b\xi} &= \left(\sigma_{\eta^\xi}^2\right)^{-2}\left[\sigma_{\eta^\xi}^2\left(\beta\left(\frac{\lambda(\xi)^2}{1-\lambda(\xi)^2}\right)\sigma_{\eta^\xi}^2+\beta\sigma_{\eta^\xi}^2\right)-2\left(\frac{\lambda(\xi)}{1-\lambda(\xi)^2}\right)\left(\sigma_{\eta^\xi}^2\right)^2\right]\\
&= \frac{\beta-2\lambda(\xi)}{1-\lambda(\xi)^2}=\beta\left(\frac{\beta^2-2}{\beta^2-1}\right).
\end{aligned}$$

This completes our computation of the needed derivatives.

We now turn to the body of the argument, which requires bifurcation analysis of the following dynamic system:

$$\begin{pmatrix}\dot{a}\\ \dot{b}\\ \dot{\xi}\end{pmatrix}=\begin{pmatrix}T^a(a,b,\xi)\\ T^b(a,b,\xi)\\ 0\end{pmatrix}-\begin{pmatrix}a\\ b\\ 0\end{pmatrix}\equiv H(a,b,\xi). \tag{5}$$

We may write decompose this system in to first, and higher-order terms:

$$H(a,b,\xi)=\begin{pmatrix}\beta-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ \xi\end{pmatrix}+\begin{pmatrix}f(a,b,\xi)\\ g(a,b,\xi)\\ 0\end{pmatrix},$$

where f and g are $\mathscr{O}\left(\|(a,b,\xi)\|^2\right)$, and given by $f = T^a - \beta a$ and $g = T^b - b$. By the center manifold theorem, the orbit structure of the dynamic system determined by (5) is topologically equivalent to the projection of the system on to the parameter-dependent center manifold, which may be expressed by a $C^4(V)$ function: $a = h(b,\xi)$, where $V \subset \mathbb{R}^2$ is an open region containing the rest point. The remainder of the proof involves two steps: computing the center manifold; and conducting bifurcation analysis of the projected system.

Computing the center

A closed form representation of h is not available, but we may use the invariance of the center manifold together with a Taylor expansion of h to establish a sufficient approximation. By (5), we have that

$$\dot{a} = (\beta - 1)h(b,\xi) + f(h(b,\xi),b,\xi).$$

Differentiating $a = h(b,\xi)$ with respect to time, we get $\dot{a} = h_b\dot{b} + h_\xi\dot{\xi}$. Using (5) and that $\dot{\xi} = 0$, we also have

$$\dot{a} = h_b(b,\xi)g(h(b,\xi),b,\xi).$$

Thus h is characterized by the functional equation

$$\mathrm{L}(b,\xi) \equiv (\beta - 1)h(b,\xi) + f(h(b,\xi),b,\xi) = h_b(b,\xi)g(h(b,\xi),b,\xi) \equiv R(b,\xi)$$

This functional equation, together with the implicit function theorem, may be used to approximate h: simply compute the Taylor expansions of L and R, equate like terms, and solve the coefficients in the Taylor expansion of h.

Since the center manifold is tangent to the eigenspaces of the linear component of H, it follows that $h_b(0,0) = h_\xi(0,0) = 0$. Also, the origin is a steady state: $h(0,0) = 0$. Thus, we may write

$$h(b,\xi) = \tfrac{1}{2}\cdot(h_{bb}\cdot b^2 + h_{\xi\xi}\cdot\xi^2) + h_{b\xi}\cdot\xi\cdot b + \mathscr{O}\left(\|(b,\xi)\|^3\right).$$

Here, all derivatives are evaluated at $(0,0)$. As notation, we also write

$$\begin{aligned} L(b,\xi) &= L_b\cdot b + L_\xi\cdot\xi + \tfrac{1}{2}\cdot(L_{bb}\cdot b^2 + L_{\xi\xi}\cdot\xi^2) + L_{b\xi}\cdot b\cdot\xi + \mathscr{O}\left(\|(b,\xi)\|^3\right), \\ R(b,\xi) &= R_b\cdot b + R_\xi\cdot\xi + \tfrac{1}{2}\cdot(R_{bb}\cdot b^2 + R_{\xi\xi}\cdot\xi^2) + R_{b\xi}\cdot b\cdot\xi + \mathscr{O}\left(\|(b,\xi)\|^3\right). \end{aligned}$$

Noting that, for example, $\frac{\partial}{\partial b}f = f_a\cdot h_b + f_b$, we compute

$$L_b = (\beta-1)h_b + f_a\cdot h_b + f_b \tag{6a}$$

$$L_{bb} = (\beta-1)h_{bb} + h_{bb}\cdot f_a + h_b\cdot f_{ab} + h_b\cdot f_{ab} + f_{bb} \tag{6b}$$

$$R_b = h_{bb}\cdot g + h_b\cdot(g_a\cdot h_b + g_b) \tag{6c}$$

$$R_{bb} = h_{bbb}\cdot g + 2h_{bb}\cdot(g_a\cdot h_b + g_b) + h_b\cdot\tfrac{\partial}{\partial b}(g_a\cdot h_b + g_b). \tag{6d}$$

Since $f, g,$ and h are zero at the origin and have no first order terms, we see $h_{bb} = \frac{f_{bb}}{1-\beta}$. Further, since $f_{bb} = T^a_{bb}$, it follows from (2f) that

$$h_{bb} = \left(\frac{F''(0)}{1-\beta}\right)\sigma^2_{\eta^\xi}.$$

As we will determine below, other second-order terms of h are not needed for the bifurcation analysis, and so our computation of the center manifold approximation is complete.

Bifurcation analysis

The local dynamics of (5) are topologically equivalent to the suspension of the projected system by the associated saddle. Intuitively this means that the dynamic system (5) may be decomposed into hyperbolic and center components; and, locally, the orbits of the decomposed systems, appropriately joined, are appropriately isomorphic to the orbits of the original system. In particular, if the projected system undergoes a particular bifurcation then so too does the system (5). The projected system is given by

$$\dot{b} = g(h(b,\xi),b,\xi) \equiv G(b,\xi). \tag{7}$$

To conduct bifurcation analysis, the higher-order derivatives of G are needed. That $G(0,0) = 0$ is immediate. Since $g = T^b - b$ we have that

$$g_{aa} = T^b_{aa} = 0 \tag{8a}$$

$$g_{ab} = T^b_{ab} = \beta^{-1}F''(0) \tag{8b}$$

$$g_{bb} = T^b_{bb} = \frac{F''(0)\mu_3^{\eta^\xi}}{\beta^2\sigma^2_{\eta^\xi}} \tag{8c}$$

$$g_{b\xi} = T^b_{b\xi} = \beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{8d}$$

$$g_{bbb} = T^b_{bbb} = \frac{F'''(0)}{\sigma^2_{\eta^\xi}}\left(\beta^{-3}\mu_4^\xi + 3\beta^{-1}\sigma^2_{\eta^\xi}\sigma^2_\varepsilon\right). \tag{8e}$$

Using our information about h, we compute

$$G_b = g_a \cdot h_b + g_b = 0 \tag{9a}$$
$$G_\xi = g_a \cdot h_\xi + g_\xi = 0 \tag{9b}$$
$$G_{bb} = g_a \cdot h_{bb} + h_b \cdot (g_{aa}\cdot h_b + g_{ab}) + g_{ab}\cdot h_b + g_{bb} = g_{bb} \tag{9c}$$
$$G_{b\xi} = h_b \cdot \tfrac{\partial}{\partial\xi} g_a + g_a \cdot h_{b\xi} + g_{ba}\cdot h_\xi + g_{b\xi} = g_{b\xi} \tag{9d}$$
$$\begin{aligned} G_{bbb} = {} & g_a h_{bbb} + 2h_{bb}\cdot(g_{aa}\cdot h_b + g_{ab}) + h_b \cdot \tfrac{\partial}{\partial b}(g_{aa}\cdot h_b + g_{ab}) \\ & + \ g_{ab}\cdot h_{bb} + h_b \cdot \tfrac{\partial}{\partial b} g_{ab} + h_b \cdot g_{bba} + g_{bbb} = 3h_{bb}\cdot g_{ab} + g_{bbb}, \end{aligned} \tag{9e}$$

where, in each computation, the second equality follows from the work just above and that h and g have no first order terms.

Since $G = G_b = G_\xi = 0$, and $G_{b\xi}$ is generically non-zero, we can assess the type of bifurcation by looking at the higher order terms in b. In particular, the type of bifurcation experienced by the system (7) depends on whether $G_{bb} = 0$. Noting g_{bb} is proportional to $\mu_3^{\eta^\xi}F''(0)$, assuming non-trivial second-order curvature in F, we see whether $g_{bb} = 0$ depends, generically, on whether $E(\varepsilon_t^3) = 0$.

Case 1: $E(\varepsilon_t^3) = 0$.

Since G_b, G_ξ and $G_{bb} = 0$, and $G_{b\xi} \neq 0$ the system undergoes a pitchfork bifurcation as ξ crosses zero provided that $G_{bbb} \neq 0$. Simplifying G_{bbb}, we get the following regularity condition:

$$G_{bbb} = F'''(0)\left(\frac{3\sigma_\varepsilon^2}{\beta} + \frac{\mu_4^{\eta^\xi}}{\beta^3\sigma_{\eta^\xi}^2}\right) + \left(\frac{3\,(F''(0))^2}{(1-\beta)\beta}\right)\sigma_{\eta^\xi}^2, \tag{10}$$

where we note that under the assumptions of the proposition, G_{bbb} is generically non-zero in that the set of all such parameters for which the condition (10) is not satisfied has Lebesgue measure zero in parameter space. We conclude that if $E(\varepsilon_t^3) = 0$ then the projected system undergoes a pitchfork bifurcation as ξ crosses zero, indicating the emergence of two additional fixed points: see chapter 3 of Wiggins (1990) for the relevant results in bifurcation theory used here and below.

Case 2: $E(\varepsilon_t^3) \neq 0$.

In this case we have $G_b = 0$, $G_\xi = 0$, and $G_{b\xi} \neq 0$. Since

$$G_{bb} = \frac{F''(0)\mu_3^{\eta^\xi}}{\beta^2\sigma_{\eta^\xi}^2}$$

is generically non-zero, we conclude that if $E(\varepsilon_t^3) \neq 0$ then the projected system undergoes a transcritical bifurcation as ξ crosses zero, indicating the emergence of two additional fixed points.

The proof of existence is completed by noting that in both cases, non-trivial fixed points of the projected system emerge as a result of a bifurcation, and further that the local dynamics of the projected system are topologically equivalent to the dynamics of the original system.

Turning now to stability, we recall from the body that stability under adaptive learning is governed by the E-stability ode (5); thus we are interesting in knowing when the bifurcation results in two new fixed points of (5), at least one of which is Lyapunov stable. Again, because, locally, the dynamics of (5) are topologically equivalent to suspension of the projected system by the associated saddle, stability of the post-bifurcation fixed points entails two requirements: first, the associated saddle must be stable, that is, $\beta - 1 < 0$; and second, the emergent fixed points of the projected system (7) must be Lyapunov stable. In case $E(\xi_t^3) \neq 0$, the bifurcation is transcritical in nature, so that we may simply choose an appropriate perturbation μ to obtain a stable fixed point. In case $E(\xi_t^3) = 0$, additional restrictions are required: the new fixed points inherit the stability of the origin. Thus stability of the new fixed points – the NRSE – requires in this case that $G_{bbb} < 0$, which yields the additional non-generic condition identified in the theorem. Note that we may still conclude that if $\beta < -1$ and $F''(0) \neq 0$ then stable NSRE exist. ∎

In the remaining sections of this Appendix, we general the model and conduct the associated bifurcation analysis. While the details of the arguments are model-specific, the proof strategy remains the same throughout. The arguments given below will be considerably more brief than provided in the proof of Theorem 1, and we will reference this proof when steps are skipped.

Proof Theorem 3. Again, we begin with derivatives.

Derivatives of $\tilde{F}\left(y_t, a, b, \xi, \eta_t^{\xi}\right) \equiv \int_{\Omega} F\left(y_t, a + b\lambda(\xi)\eta_t^{\xi} + b\varepsilon(\omega)\right) d\mu(\omega$

$$\tilde{F}_y = \int_{\Omega} F_1 d\mu(\omega) = F_1 \tag{11a}$$

$$\tilde{F}_a = \int_{\Omega} F_2 d\mu(\omega) = F_2 \tag{11b}$$

$$\tilde{F}_b = \int_{\Omega} F_2(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega)) d\mu(\omega) = \lambda(\xi)\eta^{\xi} F_2 \tag{11c}$$

$$\tilde{F}_{\xi} = \int_{\Omega} F_2 b\eta^{\xi} d\mu(\omega) = 0 \tag{11d}$$

$$\tilde{F}_{yy} = \int_{\Omega} F_{11} d\mu(\omega) = F_{11} \tag{11e}$$

$$\tilde{F}_{ya} = \int_{\Omega} F_{12} d\mu(\omega) = F_{12} \tag{11f}$$

$$\tilde{F}_{yb} = \int_{\Omega} \lambda(\xi)\eta^{\xi} F_{12} d\mu(\omega) = \beta^{-1}\eta^{\xi} F_{12} \tag{11g}$$

$$\tilde{F}_{y\xi} = \int_{\Omega} F_{11} b\eta^{\xi} d\mu(\omega) = 0 \tag{11h}$$

$$\tilde{F}_{aa} = \int_{\Omega} F_{22} d\mu(\omega) = F_{22} \tag{11i}$$

$$\tilde{F}_{ab} = \int_{\Omega} \lambda(\xi)\eta^{\xi} F_{22} d\mu(\omega) = \beta^{-1}\eta^{\xi} F_{22} \tag{11j}$$

$$\tilde{F}_{a\xi} = \int_{\Omega} F_{22} b\eta^{\xi} d\mu(\omega) = 0 \tag{11k}$$

$$\tilde{F}_{bb} = \int_{\Omega} F_{22}\left(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega)\right)^2 d\mu(\omega) = \left(\beta^{-2}\left(\eta^{\xi}\right)^2 + \sigma_{\varepsilon}^2\right) F_{22} \tag{11l}$$

$$\tilde{F}_{\xi\xi} = \int_{\Omega} F_{22}(b\eta^{\xi})^2 d\mu(\omega) = 0 \tag{11m}$$

$$\tilde{F}_{b\xi} = \int_{\Omega} (\eta^{\xi} F_2 + b\eta^{\xi} F_{22}) d\mu(\omega) = \beta\eta^{\xi} \tag{11n}$$

$$\tilde{F}_{bbb} = \int_{\Omega} F_{222}(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega))^3 d\mu(\omega) = F_{222}((\beta^{-1}\eta^{\xi})^3 + 3\beta^{-1}\eta^{\xi}\sigma_{\varepsilon}^2) \tag{11o}$$

$$\tilde{F}_{ybb} = \int_{\Omega} F_{122}\left(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega)\right)^2 d\mu(\omega) = \left(\beta^{-2}\left(\eta^{\xi}\right)^2 + \sigma_{\varepsilon}^2\right) F_{122} \tag{11p}$$

Derivatives of $\tilde{F}\left(\hat{F}\left(a,b,\xi,\eta^{\xi}\right),a,b,\xi,\eta^{\xi}\right)=0$

$$\hat{F}_a=-\frac{\tilde{F}_a}{\tilde{F}_y}=\beta \tag{12a}$$

$$\hat{F}_b=-\frac{\tilde{F}_b}{\tilde{F}_y}=\eta^{\xi} \tag{12b}$$

$$\hat{F}_{\xi}=-\frac{\tilde{F}_{\xi}}{\tilde{F}_y}=0 \tag{12c}$$

$$\hat{F}_{aa}=-\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{aa}+2\hat{F}_a\tilde{F}_{ya}+\hat{F}_a^2\tilde{F}_{yy}\right)=-\left(\frac{\beta F_{112}}{F_1}\right)\eta^{\xi}-\frac{2\beta F_{12}+F_{22}}{F_1}\equiv\Phi_{aa}^1\cdot\eta^{\xi}+\Phi_{aa}^0 \tag{12d}$$

$$\hat{F}_{ab}=-\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{ab}+\hat{F}_b\tilde{F}_{ya}+\hat{F}_a\tilde{F}_{yb}+\hat{F}_a\hat{F}_b\tilde{F}_{yy}\right)=-\left(\frac{F_{112}}{F_1}\right)\left(\eta^{\xi}\right)^2-\left(\frac{2\beta F_{12}+F_{22}}{\beta F_1}\right)\eta^{\xi}\equiv\Phi_{ab}^2\cdot\left(\eta^{\xi}\right)^2+\Phi_{ab}^1\cdot\eta^{\xi} \tag{12e}$$

$$\hat{F}_{bb}=-\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{bb}+2\hat{F}_b\tilde{F}_{yb}+\hat{F}_b^2\tilde{F}_{yy}\right)=-\left(\frac{F_{112}}{\beta F_1}\right)\left(\eta^{\xi}\right)^3-\left(\frac{2\beta F_{12}+F_{22}}{\beta^2 F_1}\right)\left(\eta^{\xi}\right)^2-\frac{F_{22}\sigma_{\varepsilon}^2}{F_1} \tag{12f}$$

$$\equiv\Phi_{bb}^3\cdot\left(\eta^{\xi}\right)^3+\Phi_{bb}^2\cdot\left(\eta^{\xi}\right)^2+\Phi_{bb}^0 \tag{12g}$$

$$\hat{F}_{b\xi}=-\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{b\xi}+\hat{F}_{\xi}\tilde{F}_{yb}+\hat{F}_b\tilde{F}_{y\xi}+\hat{F}_b\hat{F}_{\xi}\tilde{F}_{yy}\right)=\beta\eta^{\xi} \tag{12h}$$

$$\hat{F}_{\xi\xi}=0 \tag{12i}$$

$$\hat{F}_{bbb}=-\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{bbb}+3\hat{F}_{bb}\tilde{F}_{yb}+3\hat{F}_{bb}\hat{F}_b\tilde{F}_{yy}+3\hat{F}_b\tilde{F}_{ybb}+3\hat{F}_b^2\tilde{F}_{yyb}+\hat{F}_b^3\tilde{F}_{yyy}\right) \tag{12j}$$

$$\begin{aligned}&=\left(\frac{3F_{112}^2}{\beta^2F_1^2}\right)\left(\eta^{\xi}\right)^5+\left(\frac{9\beta F_{12}F_{112}+3F_{22}F_{112}}{\beta^3F_1^2}\right)\left(\eta^{\xi}\right)^4+\left(\frac{\beta^3 6\beta F_{12}^2-F_1F_{111}-3\beta F_1F_{122}+3F_{12}F_{22}-F_1F_{222}}{\beta^3F_1^2}\right)\left(\eta^{\xi}\right)^3\\&\quad+\left(\frac{3\beta^2F_{22}F_{112}\sigma_{\varepsilon}^2-3\beta^3F_1F_{11}}{\beta^3F_1^2}\right)\left(\eta^{\xi}\right)^2+\left(\frac{3\beta^2F_{12}F_{22}\sigma_{\varepsilon}^2-3\beta^3F_1F_{122}\sigma_{\varepsilon}^2-3\beta^2F_1F_{222}\sigma_{\varepsilon}^2}{\beta^3F_1^2}\right)\eta^{\xi}\\&\equiv\Phi_{bbb}^5\cdot\left(\eta^{\xi}\right)^5+\Phi_{bbb}^4\cdot\left(\eta^{\xi}\right)^4+\Phi_{bbb}^3\cdot\left(\eta^{\xi}\right)^3+\Phi_{bbb}^2\cdot\left(\eta^{\xi}\right)^2+\Phi_{bbb}^1\cdot\eta^{\xi}+\Phi_{bbb}^0\end{aligned} \tag{12k}$$

Derivatives of $T^a(a,b,\xi)=\int_{\Omega}\hat{F}(a,b,\xi,\eta^{\xi}(\omega))d\mu(\omega)$

$$T_a^a=\int_{\Omega}\hat{F}_a d\mu(\omega)=\beta \tag{13a}$$

$$T_b^a=\int_{\Omega}\hat{F}_b d\mu(\omega)=\int_{\Omega}\eta^{\xi}(\omega)d\mu(\omega)=0 \tag{13b}$$

$$T_{\xi}^a=\int_{\Omega}\hat{F}_{\xi} d\mu(\omega)=0 \tag{13c}$$

$$T_{aa}^a=\int_{\Omega}\hat{F}_{aa} d\mu(\omega)=\int_{\Omega}\left(\Phi_{aa}^1\cdot\eta^{\xi}(\omega)+\Phi_{aa}^0\right)d\mu(\omega)=\Phi_{aa}^0 \tag{13d}$$

$$T_{ab}^a=\int_{\Omega}\hat{F}_{ab} d\mu(\omega)=\int_{\Omega}\left(\Phi_{ab}^2\cdot\left(\eta^{\xi}(\omega)\right)^2+\Phi_{ab}^1\cdot\eta^{\xi}\omega)\right)d\mu(\omega)=\Phi_{ab}^2\cdot\sigma_{\eta^{\xi}}^2 \tag{13e}$$

$$T_{bb}^a=\int_{\Omega}\hat{F}_{bb} d\mu(\omega)=\int_{\Omega}\left(\Phi_{bb}^3\cdot\left(\eta^{\xi}(\omega)\right)^3+\Phi_{bb}^2\cdot\left(\eta^{\xi}(\omega)\right)^2+\Phi_{bb}^0\right)d\mu(\omega)=\Phi_{bb}^3\cdot\mu_3^{\eta^{\xi}}+\Phi_{bb}^2\cdot\sigma_{\eta^{\xi}}^2+\Phi_{bb}^0 \tag{13f}$$

$$T_{\xi b}^a=\int_{\Omega}\hat{F}_{\xi b} d\mu(\omega)=0 \tag{13g}$$

Derivatives of $T^b(a,b,\xi) = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \hat{F}(a,b,\xi,\eta^\xi(\omega)) d\mu(\omega)$

$$T^b_a = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \hat{F}_a d\mu(\omega) = 0 \tag{14a}$$

$$T^b_b = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega (\eta^\xi(\omega))^2 d\mu(\omega) = 1 \tag{14b}$$

$$T^b_\xi = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left[\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F} \frac{\partial}{\partial \xi} \eta^\xi(\omega) + \eta^\xi(\omega) \hat{F}_\xi\right) d\mu(\omega) - \frac{\partial}{\partial \xi} \sigma^2_\eta(\xi) \int_\Omega \eta^\xi(\omega) \hat{F} d\mu(\omega)\right] = 0 \tag{14c}$$

$$T^b_{bb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \hat{F}_{bb} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \left(\Phi^3_{bb} \cdot \left(\eta^\xi(\omega)\right)^3 + \Phi^2_{bb} \cdot \left(\eta^\xi(\omega)\right)^2 + \Phi^0_{bb}\right) d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \left(\Phi^3_{bb} \sigma^4_{\eta^\xi} + \Phi^2_{bb} \mu_3^{\eta^\xi}\right) \tag{14d}$$

$$T^b_{ab} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \left(\Phi^2_{ab} \cdot \left(\eta^\xi(\omega)\right)^2 + \Phi^1_{ab} \cdot \eta^\xi(\omega)\right) d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \left(\Phi^2_{ab} \mu_3^{\eta^\xi} + \Phi^1_{ab} \sigma^2_{\eta^\xi}\right) \tag{14e}$$

$$T^b_{aa} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \hat{F}_{aa} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \left(\Phi^1_{aa} \cdot \eta^\xi(\omega) + \Phi^0_{aa}\right) d\mu(\omega) = \Phi^1_{aa} \tag{14f}$$

$$\begin{aligned} T^b_{\xi\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-4} \Bigg\{ & \left(\sigma^2_\eta\right)^2 \Bigg[\sigma^2_{\eta^\xi} \int_\Omega \left(2\hat{F}_\xi \frac{\partial}{\partial \xi} \eta^\xi(\omega) + \hat{F} \frac{\partial^2}{(\partial \xi)^2} \eta^\xi(\omega) + \hat{F}_{\xi\xi} \eta^\xi(\omega)\right) d\mu(\omega) \\ & + \frac{\partial}{\partial \xi} \sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F} \frac{\partial}{\partial \xi} \eta^\xi(\omega) + \eta^\xi(\omega) \hat{F}_\xi\right) d\mu - \int_\Omega \frac{\partial}{\partial \xi} \eta^\xi(\omega) \hat{F} \frac{\partial}{\partial \xi} \sigma^2_{\eta^\xi} d\mu(\omega) \\ & - \int_\Omega \eta^\xi(\omega) \hat{F}_\xi \frac{\partial}{\partial \xi} \sigma^2_\eta(\xi) d\mu(\omega) - \int_\Omega \eta^\xi(\omega) \hat{F} \frac{\partial^2}{(\partial \xi)^2} \sigma^2_\eta(\xi) d\mu(\omega)\Bigg] \\ & - \sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F} \frac{\partial}{\partial \xi} \eta^\xi(\omega) + \eta^\xi(\omega) \hat{F}_\xi\right) \frac{\partial}{\partial \xi} (\sigma^2_\eta(\xi))^2 d\mu(\omega) + \frac{\partial}{\partial \xi} \sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega) \hat{F} \frac{\partial}{\partial \xi} (\sigma^2_{\eta^\xi})^2 d\mu(\omega) \Bigg\} = 0 \end{aligned} \tag{14g}$$

$$T^b_{b\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left[\sigma^2_{\eta^\xi} \int_\Omega \left(\frac{\partial}{\partial \xi} \eta^\xi(\omega) \hat{F}_b + \eta^\xi(\omega) \hat{F}_{b\xi}\right) d\mu(\omega) - \frac{\partial}{\partial \xi} \sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega) \hat{F}_b d\mu(\omega)\right] = \beta \left(\frac{\beta^2 - 2}{\beta^2 - 1}\right) \tag{14h}$$

$$T^b_{bbb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \hat{F}_{bbb} \eta^\xi(\omega) d\mu(\omega) \tag{14i}$$

$$= \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \left(\Phi^5_{bbb} \cdot \left(\eta^\xi(\omega)\right)^5 + \Phi^4_{bbb} \cdot \left(\eta^\xi(\omega)\right)^4 + \Phi^3_{bbb} \cdot \left(\eta^\xi(\omega)\right)^3 + \Phi^2_{bbb} \cdot \left(\eta^\xi(\omega)\right)^2 + \Phi^1_{bbb} \cdot \eta^\xi(\omega) + \Phi^0_{bbb}\right) \eta^\xi(\omega) d\mu(\omega) \tag{14j}$$

$$= \tfrac{1}{\sigma^2_{\eta^\xi}} \left(\Phi^5_{bbb} \cdot \sigma^6_{\eta^\xi} + \Phi^4_{bbb} \cdot \sigma^5_{\eta^\xi} + \Phi^3_{bbb} \cdot \mu^\xi_4 + \Phi^2_{bbb} \cdot \mu_3^{\eta^\xi} + \Phi^1_{bbb} \cdot \sigma^2_{\eta^\xi}\right) \tag{14k}$$

We now turn to the body of the argument, which, as before, requires bifurcation analysis of the system (5). The center manifold may be characterized locally as a C^4 function: $a = h(b,\xi)$, which satisfies the following functional equation:

$$(\beta - 1)h(b,\xi) + f(h(b,\xi), b, \xi) = h_b(b,\xi) g(h(b,\xi), b, \xi).$$

Working as before, we find that

$$h_{bb} = \tfrac{1}{1-\beta} f_{bb} = \tfrac{1}{1-\beta} T^a_{bb} = \tfrac{1}{1-\beta} \left(\Phi^3_{bb} \cdot \mu_3^{\eta^\xi} + \Phi^2_{bb} \cdot \sigma^2_{\eta^\xi} + \Phi^0_{bb}\right).$$

The projected system is given by

$$\dot{b} = g(h(b,\xi), b, \xi) \equiv G(b,\xi). \tag{15}$$

To conduct bifurcation analysis, the higher-order derivatives of G are needed. That $G(0,0)=0$ is immediate. Since $g=T^b-b$ we have that

$$g_{aa}=T^b_{aa}=\Phi^1_{aa} \tag{16a}$$

$$g_{ab}=T^b_{ab}=\frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^2_{ab}\mu_3^{\eta^\xi}+\Phi^1_{ab}\sigma^2_{\eta^\xi}\right) \tag{16b}$$

$$g_{bb}=T^b_{bb}=\frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\sigma^4_{\eta^\xi}+\Phi^2_{bb}\mu_3^{\eta^\xi}\right) \tag{16c}$$

$$g_{b\xi}=T^b_{b\xi}=\beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{16d}$$

$$g_{bbb}=T^b_{bbb}=\frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^5_{bbb}\cdot\sigma^6_{\eta^\xi}+\Phi^4_{bbb}\cdot\sigma^5_{\eta^\xi}+\Phi^3_{bbb}\cdot\mu_4^{\xi}+\Phi^2_{bbb}\cdot\mu_3^{\eta^\xi}+\Phi^1_{bbb}\cdot\sigma^2_{\eta^\xi}\right). \tag{16e}$$

Using our information about h, we compute

$$G_b=g_a\cdot h_b+g_b=0 \tag{17a}$$

$$G_\xi=g_a\cdot h_\xi+g_\xi=0 \tag{17b}$$

$$G_{bb}=g_a\cdot h_{bb}+h_b\cdot(g_{aa}\cdot h_b+g_{ab})+g_{ab}\cdot h_b+g_{bb}=g_{bb}=\frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\sigma^4_{\eta^\xi}+\Phi^2_{bb}\mu_3^{\eta^\xi}\right) \tag{17c}$$

$$G_{b\xi}=h_b\cdot\frac{\partial}{\partial\xi}g_a+g_a\cdot h_{b\xi}+g_{ba}\cdot h_\xi+g_{b\xi}=\beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{17d}$$

$$\begin{aligned}G_{bbb}&=g_ah_{bbb}+2h_{bb}\cdot(g_{aa}\cdot h_b+g_{ab})+h_b\cdot\frac{\partial}{\partial b}(g_{aa}\cdot h_b+g_{ab})\\&\quad+\ g_{ab}\cdot h_{bb}+h_b\cdot\frac{\partial}{\partial b}\,g_{ab}+h_b\cdot g_{bba}+g_{bbb}=3h_{bb}\cdot g_{ab}+g_{bbb} &(17e)\\&=\frac{1}{(1-\beta)\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\cdot\mu_3^{\eta^\xi}+\Phi^2_{bb}\cdot\sigma^2_{\eta^\xi}+\Phi^0_{bb}\right)\left(\Phi^2_{ab}\mu_3^{\eta^\xi}+\Phi^1_{ab}\sigma^2_{\eta^\xi}\right)\\&\quad+\frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^5_{bbb}\cdot\mu_6^{\eta^\xi}+\Phi^4_{bbb}\cdot\mu_5^{\eta^\xi}+\Phi^3_{bbb}\cdot\mu_4^{\eta^\xi}+\Phi^2_{bbb}\cdot\mu_3^{\eta^\xi}+\Phi^1_{bbb}\cdot\sigma^2_{\eta^\xi}\right)\equiv\mathscr{I}\mathscr{C} &(17f)\end{aligned}$$

where, in each computation, the second equality follows from the work just above and that h and g have no first order terms.

Since $G=G_b=G_\xi=0$, and $G_{b\xi}$ is generically non-zero, the type of bifurcation experienced by the projected system depends on whether $G_{bb}=0$. Noting that $G_{bb}=\frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\mu_4^{\eta^\xi}+\Phi^2_{bb}\mu_3^{\eta^\xi}\right)$ and that $\Phi^3_{bb}=-\frac{F_{112}}{\beta F_1}$ and $\Phi^2_{bb}=\frac{2\beta F_{12}+F_{22}}{\beta^2 F_1}$, we have two cases:

<u>Case 1:</u> $2\beta F_{12}+F_{22}\neq 0$ and $\mu_3^\varepsilon\neq 0$, or $F_{112}\neq 0$;

In this case we have $G_b=0$, $G_\xi=0$, $G_{b\xi}\neq 0$, and $G_{bb}\neq 0$; thus the projected system undergoes a transcritical bifurcation as ξ crosses zero, indicating the emergence of two additional fixed points.

<u>Case 2:</u> $2\beta F_{12}+F_{22}=0$ or $\mu_3^\varepsilon=0$, and $F_{112}=0$ and $\mathscr{I}\mathscr{C}\neq 0$.

Since G_b, G_ξ and $G_{bb}=0$, and $G_{b\xi}\neq 0$ the system undergoes a pitchfork bifurcation as ξ crosses zero provided that $G_{bbb}\neq 0$, thus $\mathscr{I}\mathscr{C}$ must be non-zero.

The remainder of the proof is completed as before, with stability in Case 1 requiring that $\mathscr{I}\mathscr{C}$ be negative. ■

Proof of Lemma 2. The natural approach is to consider a perturbation of σ near zero; the technical challenge is that the T-map is not defined for $\sigma = 0$. To side-step the complication that $\sigma_v^2 \to 0$ as $\sigma \to 0$, define $\hat{v}_t = \sigma^{-1} v_t$, and notice that $\hat{v}_t = \rho \hat{v}_{t-1} + \zeta_t$. Now consider the new function $\hat{F}$, defined as

$$y_t = \int_\Omega F(a + b\rho\hat{v}_t + b\zeta_{t+1}(\omega), \rho\sigma\hat{v}_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \equiv \hat{F}(a,b,\sigma,\hat{v}_t).$$

Projecting this process onto the span of $(1,\hat{v}_t)$ yields the following map, which we label $\hat{T}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{\hat{T}=\hat{T}(a,b,\sigma)} \begin{pmatrix} \int_\Omega \hat{F}(a,b,\sigma,\hat{v}(\omega))d\mu(\omega) \\ \left(\sigma_{\hat{v}}^2\right)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}(a,b,\sigma,\hat{v}(\omega))d\mu(\omega) \end{pmatrix} \equiv \begin{pmatrix} \hat{T}^a(a,b,\sigma) \\ \hat{T}^b(a,b,\sigma) \end{pmatrix}.$$

By construction, $\hat{T}$ is defined, k-times differentiable, and has a fixed point at $(0,0)'$ when $\sigma = 0$.

Let $H = \hat{T} - (a,b)'$. We need some more derivatives.

<u>Derivatives of $\hat{F}$ and H</u>

$$\hat{F}_a = DF_y \equiv \beta \tag{18a}$$
$$\hat{F}_b = \rho\beta\hat{v} \tag{18b}$$
$$\hat{F}_\sigma = DF_v \cdot \rho\hat{v} \tag{18c}$$
$$H_a^a = \int_\Omega \hat{F}_a d\mu(\omega) - 1 = \beta - 1 \tag{18d}$$
$$H_b^a = \int_\Omega \hat{F}_b d\mu(\omega) = \beta\rho \int_\Omega \hat{v}(\omega)\mu(\omega) = 0 \tag{18e}$$
$$H_a^b = \left(\sigma_{\hat{v}}^2\right)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}_a d\mu(\omega) = \left(\sigma_{\hat{v}}^2\right)^{-1} \beta \int_\Omega \hat{v}(\omega)d\mu(\omega) = 0 \tag{18f}$$
$$H_b^b = \left(\sigma_{\hat{v}}^2\right)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}_b d\mu(\omega) - 1 = \left(\sigma_{\hat{v}}^2\right)^{-1} \rho DF_y \int_\Omega \hat{v}(\omega)^2 d\mu(\omega) = \beta\rho - 1 \tag{18g}$$
$$H_\sigma^a = \int_\Omega \hat{F}_\sigma d\mu(\omega) = \rho DF_v \int_\Omega \hat{v}(\omega)d\mu(\omega) = 0 \tag{18h}$$
$$H_\sigma^b = \left(\sigma_{\hat{v}}^2\right)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}_\sigma d\mu(\omega) = \left(\sigma_{\hat{v}}^2\right)^{-1} \rho DF_v \int_\Omega \hat{v}(\omega)^2 d\mu(\omega) = DF_v \cdot \rho\hat{v} \tag{18i}$$

From these computations, we find that

$$DH_{(a,b)'}(0,0,0) = \begin{pmatrix} \beta - 1 & 0 \\ 0 & \beta\rho - 1 \end{pmatrix}, \text{ and } DH_\sigma(0,0,0) = \begin{pmatrix} 0 \\ \rho DF_v \end{pmatrix}.$$

We conclude that the implicit function theorem applies to the system of equations $H = 0$, and that $\frac{\partial b^*}{\partial \sigma} = (1-\beta\rho)^{-1}\rho DF_v \neq 0$.[1]

[1] We observe that given a linear model $y_t = \beta E_t y_{t+1} + \rho DF_v v_t$, the REE is given by $y_t = b^* v_t$ with $b^* = (1-\beta\rho)^{-1}\rho DF_v$.

We have demonstrated that for small σ, there exist $(\hat{a}(\sigma),\hat{b}(\sigma))'$, with $\hat{b}(\sigma) \neq 0$, such that $\hat{T}(\hat{a}(\sigma),\hat{b}(\sigma),\sigma) = (\hat{a}(\sigma),\hat{b}(\sigma))'$. The proof is completed by demonstrating that $T(\hat{a}(\sigma),\sigma^{-1}\hat{b}(\sigma),\sigma) = (\hat{a}(\sigma),\sigma^{-1}\hat{b}(\sigma))'$. To this end, first notice

$$\begin{aligned}
\tilde{F}(a,\sigma^{-1}b,\sigma,v_t) &= \int_\Omega F(a+\sigma^{-1}b\rho v_t + b\zeta_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \\
&= \int_\Omega F(a+b\rho\hat{v}_t + b\zeta_{t+1}(\omega), \rho\sigma\hat{v}_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \\
&= \hat{F}(a,b,\sigma,\hat{v}_t).
\end{aligned}$$

Using this, we compute

$$\begin{aligned}
T^a(a,\sigma^{-1}b,\sigma) &= \int_\Omega \tilde{F}(a,\sigma^{-1}b,\sigma,v(\omega))d\mu(\omega) \\
&= \int_\Omega \hat{F}(a,b,\sigma,\hat{v}(\omega))d\mu(\omega) = \hat{T}^a(a,b,\sigma).
\end{aligned}$$

Finally,

$$\begin{aligned}
T^b(a,\sigma^{-1}b,\sigma) &= \left(\sigma_v^2(\sigma)\right)^{-1}\int_\Omega v^\sigma(\omega)\tilde{F}(a,\sigma^{-1}b,\sigma,v^\sigma(\omega))d\mu(\omega) \\
&= \left(\sigma^2\sigma_{\hat{v}}^2(\sigma)\right)^{-1}\int_\Omega \sigma\hat{v}(\omega)\hat{F}(a,b,\sigma,v^\sigma(\omega))d\mu(\omega) \\
&= \sigma^{-1}\hat{T}^b(a,b,\sigma),
\end{aligned}$$

and the result follows. ■

Proof Theorem 4 In what follows, unless otherwise specified, derivatives are evaluated at

$$(a,b,c,\xi) = (a^*,b^*,0,0).$$

Recall that in the body we stated that by choosing $|\sigma|$ small we may assume that $DF_\star \approx DF_\star(0,0)$ for $\star = y,v,yy$, etc. To see this, first observe that if $(\hat{a}(\sigma),\hat{b}(\sigma))'$ is the fixed point of the map $\hat{T}$ (see proof of Lemma 2) then $\lim_{\sigma\to 0}(\hat{a}(\sigma),\hat{b}(\sigma)) = (0,0)$, and since $a^*(\sigma) = \hat{a}(\sigma)$, we may assume $|a^*(\sigma)|$ is small. Also, since $b^*(\sigma) = \frac{1}{\sigma}\hat{b}(\sigma)$, we may assume $|\sigma b^*(\sigma)|$ is small. Since $v_t = \sigma\hat{v}_t$ follows that for small $|\sigma|$,

$$DF_\star \equiv DF_\star(a^*(\sigma)+b^*(\sigma)\rho v_t + b^*(\sigma)\sigma\zeta_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega)) \approx DF_\star(0,0) \tag{19}$$

for $\star = y,v,yy$, etc.

Turning now the the main argument, the proof follows the same structure as the proof of Theorem 1, and because of this, we will be considerably more brief. Again, we require a host of derivatives.

Derivatives of $\hat{F} = \int_\Omega F(a + b\rho v_t + b\sigma\zeta_{t+1}(\omega) + c\lambda(\xi)\eta_t^\xi + c\varepsilon_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega)$

$$\hat{F}_a = \int_\Omega DF_y \cdot d\mu(\omega) = DF_y \tag{20a}$$

$$\hat{F}_b = \int_\Omega DF_y \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) = DF_y \cdot \rho v_t \tag{20b}$$

$$\hat{F}_c = \int_\Omega DF_y \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))d\mu(\omega) = DF_y \cdot \lambda(\xi)\eta_t^\xi \tag{20c}$$

$$\hat{F}_\xi = \int_\Omega DF_y \cdot c\eta_t^\xi d\mu(\omega) = 0 \tag{20d}$$

$$\hat{F}_{aa} = \int_\Omega DF_{yy} \cdot d\mu(\omega) = DF_{yy} \tag{20e}$$

$$\hat{F}_{ab} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) = DF_{yy} \cdot \rho v_t \tag{20f}$$

$$\hat{F}_{ac} = \int_\Omega DF_{yy} \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))d\mu(\omega) = DF_{yy} \cdot \lambda(\xi)\eta_t^\xi \tag{20g}$$

$$\hat{F}_{a\xi} = \int_\Omega DF_{yy} \cdot c\eta_t^\xi d\mu(\omega) = 0 \tag{20h}$$

$$\hat{F}_{bb} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))^2 d\mu(\omega) = DF_{yy} \cdot (\rho^2 v_t^2 + \sigma^2\sigma_\zeta^2) \tag{20i}$$

$$\hat{F}_{bc} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))(\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))\mu(\omega) = DF_{yy} \cdot \lambda(\xi)\rho\eta_t^\xi v_t \tag{20j}$$

$$\hat{F}_{b\xi} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))c\eta_t^\xi d\mu(\omega) = 0 \tag{20k}$$

$$\hat{F}_{cc} = \int_\Omega DF_{yy} \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))^2 d\mu(\omega) = DF_{yy} \cdot \left(\lambda(\xi)^2 \left(\eta_t^\xi\right)^2 + \sigma_\varepsilon^2\right) \tag{20l}$$

$$\hat{F}_{c\xi} = \int_\Omega \left(DF_{yy} \cdot c\eta_t^\xi \left(\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega)\right) + DF_y \cdot \eta_t^\xi\right) d\mu(\omega) = DF_y \cdot \eta_t^\xi \tag{20m}$$

$$\hat{F}_{\xi\xi} = \int_\Omega \left(DF_{yy} \cdot \left(c\eta_t^\xi\right)^2 + DF_y \cdot c \cdot \tfrac{\partial}{\partial\xi}\eta_t^\xi\right) d\mu(\omega) = 0 \tag{20n}$$

$$\hat{F}_{ccc} = \int_\Omega DF_{yyy} \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))^3 d\mu(\omega) = DF_{yyy} \cdot \left(\lambda(\xi)^3 \left(\eta_t^\xi\right)^3 + 3\lambda(\xi)\sigma_\varepsilon^2\eta_t^\xi + \mu_4^\varepsilon\right) \tag{20o}$$

Derivatives of $T^a = \int_\Omega \hat{F}(a,b,c,\xi,v(\omega),\eta^\xi(\omega))d\mu(\omega)$

$$T^a_a = \int_\Omega \hat{F}_a \cdot d\mu(\omega) = DF_y \tag{21a}$$

$$T^a_b = \int_\Omega \hat{F}_b \cdot d\mu(\omega) = DF_y \cdot \rho \int_\Omega v(\omega)d\mu(\omega) = 0 \tag{21b}$$

$$T^a_c = \int_\Omega \hat{F}_c \cdot d\mu(\omega) = DF_y \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{21c}$$

$$T^a_\xi = \int_\Omega \hat{F}_\xi \cdot d\mu(\omega) = 0 \tag{21d}$$

$$T^a_{aa} = \int_\Omega \hat{F}_{aa} \cdot d\mu(\omega) = DF_{yy} \tag{21e}$$

$$T^a_{ab} = \int_\Omega \hat{F}_{ab} \cdot d\mu(\omega) = DF_{yy} \cdot \rho \int_\Omega v(\omega)d\mu(\omega) = 0 \tag{21f}$$

$$T^a_{ac} = \int_\Omega \hat{F}_{ac} \cdot d\mu(\omega) = DF_{yy} \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{21g}$$

$$T^a_{a\xi} = \int_\Omega \hat{F}_{a\xi} \cdot d\mu(\omega) = 0 \tag{21h}$$

$$T^a_{bb} = \int_\Omega \hat{F}_{bb} \cdot d\mu(\omega) = DF_{yy} \int_\Omega \left(\rho^2 v(\omega)^2 + \sigma^2_\zeta\right) d\mu(\omega) = DF_{yy} \cdot \sigma^2_v \tag{21i}$$

$$T^a_{bc} = \int_\Omega \hat{F}_{bc} \cdot d\mu(\omega) = DF_{yy} \cdot \rho \cdot \lambda(\xi) \int_\Omega v(\omega)\eta^\xi(\omega)d\mu(\omega) = 0 \tag{21j}$$

$$T^a_{b\xi} = \int_\Omega \hat{F}_{b\xi} d\mu(\omega) = 0 \tag{21k}$$

$$T^a_{cc} = \int_\Omega \hat{F}_{cc} \cdot d\mu(\omega) = DF_{yy} \int_\Omega \left(\lambda(\xi)^2 \eta^\xi(\omega)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF_{yy} \cdot \sigma^2_{\eta^\xi} \tag{21l}$$

$$T^a_{c\xi} = \int_\Omega \hat{F}_{c\xi} \cdot d\mu(\omega) = DF_y \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{21m}$$

Derivatives of $T^b = \frac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}(a,b,c,\xi,v(\omega),\eta^\xi(\omega))d\mu(\omega)$

$$T_a^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_a d\mu(\omega) = 0 \tag{22a}$$

$$T_b^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_y\cdot\rho\int_\Omega v(\omega)^2 d\mu(\omega) = DF_y\cdot\rho \tag{22b}$$

$$T_c^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_c d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_y\cdot\lambda(\xi)\int_\Omega v(\omega)\eta^\xi(\omega)d\mu(\omega) = 0 \tag{22c}$$

$$T_\xi^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_\xi d\mu(\omega) = 0 \tag{22d}$$

$$T_{aa}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{aa} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\int_\Omega v(\omega)d\mu(\omega) = 0 \tag{22e}$$

$$T_{ab}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\cdot\rho\int_\Omega v(\omega)^2 d\mu(\omega) = \rho\cdot DF_{yy} \tag{22f}$$

$$T_{ac}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{ac} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\cdot\lambda(\xi)\int_\Omega v(\omega)\eta^\xi(\omega)d\mu(\omega) = 0 \tag{22g}$$

$$T_{a\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{a\xi} d\mu(\omega) = 0 \tag{22h}$$

$$T_{bb}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{bb} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\int_\Omega v(\omega)\left(\rho^2 v(\omega)+\sigma^2\sigma_\zeta^2\right)d\mu(\omega) = DF_{yy}\cdot\rho^2\cdot\left(\frac{\mu_3^v}{\sigma_v^2}\right) \tag{22i}$$

$$T_{bc}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{bc} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\cdot\lambda(\xi)\cdot\rho\int_\Omega v(\omega)^2\eta^\xi(\omega)d\mu(\omega) = 0 \tag{22j}$$

$$T_{b\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{b\xi} d\mu(\omega) = 0 \tag{22k}$$

$$T_{cc}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{cc} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\int_\Omega v(\omega)\left(\lambda(\xi)^2\eta^\xi(\omega)^2+\sigma_\varepsilon^2\right)d\mu(\omega) = 0 \tag{22l}$$

$$T_{c\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{c\xi} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_y\int_\Omega v(\omega)\cdot\eta^\xi(\omega)d\mu(\omega) = 0 \tag{22m}$$

$$T_{\xi\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{\xi\xi} d\mu(\omega) = 0 \tag{22n}$$

$$\underline{\text{Derivatives of } T^c = \left(\sigma^2_{\eta^\xi}\right)^{-1} \int_\Omega \eta^\xi(\omega)\hat{F}(a,b,c,\xi,v(\omega),\eta^\xi(\omega))d\mu(\omega)}$$

$$T^c_a = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega) = 0 \tag{23a}$$

$$T^c_b = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_y \int_\Omega \eta^\xi(\omega)v(\omega)d\mu(\omega) = 0 \tag{23b}$$

$$T^c_c = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_c d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_y \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = 1 \tag{23c}$$

$$T^c_\xi = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F} \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F} d\mu(\omega)\right) = 0 \tag{23d}$$

$$T^c_{aa} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{aa} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} DF_{yy} \cdot \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{23e}$$

$$T^c_{ab} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \cdot \rho \int_\Omega \eta^\xi(\omega)v(\omega)d\mu(\omega) = 0 \tag{23f}$$

$$T^c_{ac} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ac} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = DF_{yy} \cdot \lambda(\xi) \tag{23g}$$

$$T^c_{a\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}_a \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_{a\xi}\right) d\mu(\omega) + \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega)\right) = 0 \tag{23h}$$

$$T^c_{bb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{bb} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \int_\Omega \eta^\xi(\omega) \left(\rho^2 v(\omega) + \sigma^2\sigma^2_\zeta\right) d\mu(\omega) = 0 \tag{23i}$$

$$T^c_{bc} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{bc} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \cdot \lambda(\xi) \cdot \rho \int_\Omega \eta^\xi(\omega)^2 v(\omega)d\mu(\omega) = 0 \tag{23j}$$

$$T^c_{b\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}_b \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_{b\xi}\right) d\mu(\omega) + \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega)\right) = 0 \tag{23k}$$

$$T^c_{cc} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{cc} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} DF_{yy} \int_\Omega \eta^\xi(\omega) \left(\lambda(\xi)^2\eta^\xi(\omega)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF_{yy} \cdot \lambda(\xi)^2 \left(\frac{\mu_3^{\eta^\xi}}{\sigma^2_{\eta^\xi}}\right) \tag{23l}$$

$$T^c_{c\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}_c \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_{c\xi}\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}_c d\mu(\omega)\right) = DF_y \left(\frac{DF_y^2 - 2}{DF_y^2 - 1}\right) \tag{23m}$$

$$\begin{aligned} T^c_{\xi\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-4} \Bigg\{ \left(\sigma^2_{\eta^\xi}\right)^2 \Bigg[&\sigma^2_{\eta^\xi} \int_\Omega \left(2\hat{F}_\xi \frac{\partial}{\partial\xi}\eta^\xi(\omega) + \hat{F}\frac{\partial^2}{(\partial\xi)^2}\eta^\xi(\omega) + \hat{F}_{\xi\xi}\eta^\xi(\omega)\right) d\mu(\omega) \\ &+ \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu - \int_\Omega \frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} d\mu(\omega) \\ &- \int_\Omega \eta^\xi(\omega)\hat{F}_\xi \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} d\mu(\omega) - \int_\Omega \eta^\xi(\omega)\hat{F}\frac{\partial^2}{(\partial\xi)^2}\sigma^2_\eta(\xi)d\mu(\omega)\Bigg] \\ &- \sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) \frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega) + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega)\Bigg\} = 0 \end{aligned} \tag{23n}$$

$$T^c_{ccc} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ccc} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yyy} \int_\Omega \left(\lambda(\xi)^3\eta^\xi(\omega)^4 + 3\lambda(\xi)\sigma^2_\varepsilon\eta^\xi(\omega)^2\right) d\mu(\omega) \tag{23o}$$

$$= \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yyy} \left(\lambda(\xi)^3\mu_4^\xi + \lambda(\xi)\sigma^2_{\eta^\xi}\sigma^2_\varepsilon\right) \tag{23p}$$

The computations (23d), (23h), (23k) require that at $c = 0$, $\hat{F}$ and its first partials are independent of η, and that $\int_\Omega \frac{\partial}{\partial\xi}\eta(\omega)d\mu(\omega) = 0$, which follows from equation (4). Also, (23m) follows from the same argument as (3h).

We turn now to the bifurcation analysis. Change coordinates: $\alpha = a - a^*, \gamma = b - b^*$, and consider the dynamic system

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\gamma} \\ \dot{c} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} T^a(\alpha + a^*, \gamma + b^*, c, \xi) \\ T^b(\alpha + a^*, \gamma + b^*, c, \xi) \\ T^c(\alpha + a^*, \gamma + b^*, c, \xi) \\ 0 \end{pmatrix} - \begin{pmatrix} \alpha + a^* \\ \gamma + b^* \\ c \\ 0 \end{pmatrix} \equiv H(\alpha, \gamma, c, \xi),$$

noting that the origin is a rest point. Following the usual proof strategy, write

$$H(\alpha, \gamma, c, \xi) = \begin{pmatrix} T^a_a & T^a_b, & T^a_c & T^a_\xi \\ T^b_a & T^b_b, & T^b_c & T^b_\xi \\ T^c_a & T^c_b, & T^c_c & T^c_\xi \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} f^1(\alpha, \gamma, c, \xi) \\ f^2(\alpha, \gamma, c, \xi) \\ g(\alpha, \gamma, c, \xi) \\ 0 \end{pmatrix},$$

where f^i and g are $\mathscr{O}\left(\|(a, b, c, \xi)\|^2\right)$. By appealing to our previous computations, we find that

$$\begin{pmatrix} T^a_a & T^a_b, & T^a_c & T^a_\xi \\ T^b_a & T^b_b, & T^b_c & T^b_\xi \\ T^c_a & T^c_b, & T^c_c & T^c_\xi \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} DF_y - 1 & 0 & 0 & 0 \\ 0 & \rho DF_y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} f^1(\alpha, \gamma, c, \xi) &= T^a(\alpha + a^*, \gamma + b^*, c, \xi) - DF_y \cdot \alpha - a^* \\ f^2(\alpha, \gamma, c, \xi) &= T^b(\alpha + a^*, \gamma + b^*, c, \xi) - \rho DF_y \cdot \gamma - b^* \\ g(\alpha, \gamma, c, \xi) &= T^c(\alpha + a^*, \gamma + b^*, c, \xi) - c. \end{aligned}$$

The center manifold is parameterized by $\alpha = h^\alpha(c, \xi)$ and $\gamma = h^\gamma(c, \xi)$; and, using invariance, these parameterizations satisfy the following functional equations:

$$L^\alpha(c, \xi) \equiv (DF_y - 1)h^\alpha + f^1(h^\alpha, h^\gamma, c, \xi) = h^\alpha_c \cdot g(h^\alpha, h^\gamma, c, \xi) \equiv R^\alpha(c, \xi) \tag{24}$$

$$L^\gamma(c, \xi) \equiv (\rho DF_y - 1)h^\gamma + f^2(h^\alpha, h^\gamma, c, \xi) = h^\gamma_c \cdot g(h^\alpha, h^\gamma, c, \xi) \equiv R^\gamma(c, \xi). \tag{25}$$

Computing as in (6), we find that

$$\begin{aligned} h^\alpha_{cc} &= \frac{f^1_{cc}}{1 - DF_y} = \frac{T^a_{cc}}{1 - DF_y} = \left(\frac{DF_{yy}}{1 - DF_y}\right) \sigma^2_{\eta^\xi} \\ h^\gamma_{cc} &= \frac{f^2_{cc}}{1 - \rho DF_y} = \frac{T^b_{cc}}{1 - \rho DF_y} = 0, \end{aligned}$$

and, as before, these are the only partials we require.

Projected onto the center, the dynamics take the form

$$\dot{c} = g(h^\alpha(c, \xi), h^\gamma(c, \xi), c, \xi) \equiv G(c, \xi).$$

Computing as in (17), we find $G_\star = 0$ and

$$G_{cc} = g_{cc} = T^c_{cc} = DF_{yy} \cdot DF_y^{-2} \left(\frac{\mu_3^{\eta^\xi}}{\sigma^2_{\eta^\xi}} \right) \tag{26a}$$

$$G_{c\xi} = g_{c\xi} = T^c_{c\xi} = DF_y \left(\frac{DF_y^2 - 2}{DF_y^2 - 1} \right) \tag{26b}$$

$$G_{ccc} = 3(h^\alpha_{cc} \cdot g_{ac} + h^\gamma_{cc} \cdot g_{bc}) + g_{ccc} = 3h^\alpha_{cc} \cdot T^c_{ac} + T^c_{ccc} = \frac{3DF_{yy}}{DF_y} \left(\frac{DF_{yy}}{1-DF_y} \right) \sigma^2_{\eta^\xi} + T^c_{ccc} \tag{26c}$$

$$= \frac{3DF_{yy}}{DF_y} \left(\frac{DF_{yy}}{1-DF_y} \right) \sigma^2_{\eta^\xi} + \frac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yyy} \left(DF_y^{-3} \mu_3^\varepsilon + DF_y^{-1} \sigma^2_{\eta^\xi} \sigma^2_\varepsilon \right). \tag{26d}$$

The proofs of existence and stability are complete arguing as in the proof of Theorem 1. ■

Proof of Theorem 5. First, we require some derivatives. As notation, write

$$T = \begin{pmatrix} T^a \\ T^b \end{pmatrix} \text{ and } DF = \left(DF^i_{y_j} \right).$$

We compute as follows:

Derivatives of $\hat{F}(a,b,\xi,\eta_t^\xi) = \int_\Omega (F^i(a_1 + b_1\lambda(\xi)\eta_t^\xi + b_1\varepsilon_{t+1}(\omega), a_2 + b_2\lambda(\xi)\eta_t^\xi + b_2\varepsilon_{t+1}(\omega))d\mu(\omega)$

$$\hat{F}^i_{a_j} = \int_\Omega DF^i_{y_j} \cdot d\mu(\omega) = DF^i_{y_j} \tag{27a}$$

$$\hat{F}^i_{b_j} = \int_\Omega DF^i_{y_j} \cdot \left(\lambda(\xi) \cdot \eta_t^\xi + \varepsilon_{t+1}(\omega) \right) d\mu(\omega) = DF^i_{y_j} \cdot \lambda(\xi) \cdot \eta_t^\xi \tag{27b}$$

$$\hat{F}^i_\xi = \int_\Omega \left(\sum_{j=1}^n DF^i_{y_j} \cdot b_j \right) \eta_t^\xi d\mu(\omega) = 0 \tag{27c}$$

$$\hat{F}^i_{a_j a_k} = \int_\Omega DF^i_{y_j y_k} \cdot d\mu(\omega) = DF^i_{y_j y_k} \tag{27d}$$

$$\hat{F}^i_{a_j b_k} = \int_\Omega DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \eta_t^\xi d\mu(\omega) = DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \eta_t^\xi \tag{27e}$$

$$\hat{F}^i_{a_j \xi} = \int_\Omega \left(\sum_{k=1}^n DF^i_{y_j y_k} \cdot b_k \right) \eta_t^\xi d\mu(\omega) = 0 \tag{27f}$$

$$\hat{F}^i_{b_j b_k} = \int_\Omega DF^i_{y_j y_k} \cdot \left(\lambda(\xi) \cdot \eta_t^\xi + \varepsilon_{t+1}(\omega) \right)^2 d\mu(\omega) = DF^i_{y_j y_k} \left(\left(\lambda(\xi) \cdot \eta_t^\xi \right)^2 + \sigma^2_\varepsilon \right) \tag{27g}$$

$$\hat{F}^i_{b_j \xi} = \int_\Omega \left(\left(\sum_{k=1}^n DF^i_{y_j y_k} \cdot b_k \right) \eta_t^\xi + DF^i_{y_j} \cdot \eta_t^\xi \right) d\mu(\omega) = DF^i_{y_j} \cdot \eta_t^\xi \tag{27h}$$

Derivatives of $T^{a_i} = \int_\Omega \hat{F}^i(a,b,\xi,\eta^\xi(\omega))d\mu(\omega)$

$$T^{a_i}_{a_j} = \int_\Omega \hat{F}^i_{a_j} d\mu(\omega) = DF^i_{y_j} \tag{28a}$$
$$T^{a_i}_{b_j} = \int_\Omega \hat{F}^i_{b_j} d\mu(\omega) = \int_\Omega DF^i_{y_j} \cdot \lambda(\xi) \cdot \eta^\xi(\omega) d\mu(\omega) = 0 \tag{28b}$$
$$T^{a_i}_{\xi} = \int_\Omega \hat{F}^i_{\xi} d\mu(\omega) = 0 \tag{28c}$$
$$T^{a_i}_{a_j a_k} = \int_\Omega \hat{F}^i_{a_j a_k} d\mu(\omega) = DF^i_{y_j y_k} \tag{28d}$$
$$T^{a_i}_{a_j b_k} = \int_\Omega \hat{F}^i_{a_j b_k} d\mu(\omega) = \int_\Omega DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \eta^\xi(\omega) d\mu(\omega) = 0 \tag{28e}$$
$$T^{a_i}_{a_j \xi} = \int_\Omega \hat{F}^i_{a_j \xi} d\mu(\omega) = 0 \tag{28f}$$
$$T^{a_i}_{b_j b_k} = \int_\Omega \hat{F}^i_{b_j b_k} d\mu(\omega) = \int_\Omega DF^i_{y_j y_k} \left(\lambda(\xi)^2 \cdot \eta^\xi(\omega)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF^i_{y_j y_k} \cdot \sigma^2_{\eta^\xi} \tag{28g}$$
$$T^{a_i}_{b_j \xi} = \int_\Omega \hat{F}^i_{b_j \xi} d\mu(\omega) = \int_\Omega DF^i_{y_j} \cdot \eta^\xi(\omega) d\mu(\omega) = 0 \tag{28h}$$

Derivatives of $T^{b_i} = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i(a,b,\xi,\eta^\xi(\omega))d\mu(\omega)$

$$T^{b_i}_{a_j} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{a_j} d\mu(\omega) = 0 \tag{29a}$$
$$T^{b_i}_{b_j} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{b_j} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF^i_{y_j} \cdot \lambda(\xi) \cdot \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = DF^i_{y_j} \cdot \lambda(\xi) \tag{29b}$$
$$T^{b_i}_{\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\eta^\xi(\omega)\hat{F}^i_\xi + \hat{F}^i \tfrac{\partial}{\partial\xi}\eta^\xi(\omega)\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}^i d\mu(\omega)\right) = 0 \tag{29c}$$
$$T^{b_i}_{a_j a_k} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{a_j a_k} d\mu(\omega) = 0 \tag{29d}$$
$$T^{b_i}_{a_j b_k} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{a_j b_k} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = DF^i_{y_j y_k} \cdot \lambda(\xi) \tag{29e}$$
$$T^{b_i}_{b_j b_k} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{b_j b_k} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) DF^i_{y_j y_k} \left(\left(\lambda(\xi)\cdot\eta^\xi(\omega)\right)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF^i_{y_j y_k} \cdot \lambda(\xi)^2 \left(\tfrac{\mu_3^{\eta^\xi}}{\sigma^2_{\eta^\xi}}\right) \tag{29f}$$
$$T^{b_i}_{b_j \xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\eta^\xi(\omega)\hat{F}^i_{b_j\xi} + \hat{F}^i_{b_j} \tfrac{\partial}{\partial\xi}\eta^\xi(\omega)\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{b_j} d\mu(\omega)\right) = DF^i_{y_j} \left(\frac{1-2\lambda(\xi)^2}{1-\lambda(\xi)^2}\right) \tag{29g}$$

The above computations show that $DT = DF \oplus \lambda(\xi)DF$. Next, let $\hat{S} = S \oplus S$, $\theta = (a', b')'$ and $\phi = \hat{S}^{-1}\theta$, and consider the dynamic system

$$\dot{\phi} = \hat{S}^{-1}T(\hat{S}\phi, \xi) - \phi = \hat{H}(\phi, \xi), \tag{30}$$

which is topologically equivalent to the E-stability differential equation of our economic model, except now, to first order, the dynamics are decoupled. In particular, after adjoining ξ as usual, we may write the dynamic system (30) as

$$
\begin{pmatrix} \dot{\phi}_1 \\ \vdots \\ \dot{\phi}_n \\ \dot{\phi}_{n+1} \\ \vdots \\ \dot{\phi}_{2n-1} \\ \dot{\phi}_{2n} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} \beta_1 - 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta_n - 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & \frac{\beta_1}{\beta_n} - 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{\beta_{n-1}}{\beta_n} - 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \\ \phi_{n+1} \\ \vdots \\ \phi_{2n-1} \\ \phi_{2n} \\ \xi \end{pmatrix} + \begin{pmatrix} f^1(\phi,\xi) \\ \vdots \\ f^n(\phi,\xi) \\ f^{n+1}(\phi,\xi) \\ \vdots \\ f^{2n-1}(\phi,\xi) \\ g(\phi,\xi) \\ 0 \end{pmatrix}, \tag{31}
$$

where f^i and g comprise higher-order terms.

The center manifold is parameterized by $\phi_i = h^i(\phi_{2n}, \xi)$ for $i = 1, \ldots 2n-1$. Invariance provides the following functional equations in ϕ_{2n} and ξ:

$$
h^i_{\phi_{2n}} \cdot g = f^i - D\hat{H}_{ii} \cdot h^i. \tag{32}
$$

These may be used to compute a second-order approximation to the h^i. Finally, the projected dynamics are given by

$$
\dot{\phi}_{2n} = g\left(h^1(\phi_{2n}, \xi), \ldots, h^{2n-1}(\phi_{2n}, \xi), \phi_{2n}, \xi\right) \equiv G(\phi_{2n}, \xi).
$$

We now turn to bifurcation analysis of $\dot{\phi}_{2n} = G(\phi_{2n}, \xi)$.

Note that G is second order: $G = G_\star = 0$. Thus, to show that a transcritical bifurcation occurs it suffices to show that $G_{\phi_{2n}\phi_{2n}}$ and $G_{\phi_{2n}\xi}$ are non-zero. Using $h^i = h^i_\star = 0$ we find that

$$
G_{\phi_{2n}\phi_{2n}} = g_{\phi_{2n}\phi_{2n}} \text{ and } G_{\phi_{2n}\xi} = g_{\phi_{2n}\xi},
$$

just as in previous arguments.

Recalling that $S^{-1} = (S^{ij})$ we find

$$
g(*, \phi_{2n}, \xi) = \sum_{i=1}^{n} S^{ni} \cdot T^{b_i}(*, b_1(\phi_{2n}), \ldots, b_n(\phi_{2n}), \xi),
$$

where $b_i(\phi_{2n}) = * + S_{in} \cdot \phi_{2n}$, and here and below an "$*$" captures terms that are not relevant to the local argument. We compute

$$
\begin{aligned}
g_{\phi_{2n}} &= \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot T_{b_j}^{b_i} \\
g_{\phi_{2n}\phi_{2n}} &= \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot \sum_{k=1}^{n} S_{kn} \cdot T_{b_j b_k}^{b_i} = \lambda(\xi)^2 \left(\frac{\mu_3^{\varepsilon}}{\sigma_{\eta^{\xi}}^2} \right) \sum_{i=1}^{n} S^{ni} \left(S_n' \cdot D^2 F^i \cdot S_n \right) \\
g_{\phi_{2n}\xi} &= \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot T_{b_j \xi}^{b_i} = \left(\frac{1 - 2\lambda(\xi)^2}{1 - \lambda(\xi)^2} \right) \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot DF_j^i \\
&= \left(\frac{1 - 2\lambda(\xi)^2}{1 - \lambda(\xi)^2} \right) \sum_{i=1}^{n} S^{ni} \cdot DF^i \cdot S_n = \left(\frac{1 - 2\lambda(\xi)^2}{1 - \lambda(\xi)^2} \right) \beta_n .
\end{aligned}
$$

Existence is now established as in case 1 of the proof of Theorem 1, and stability follows from the topological equivalence of (30) with the E-stability ode, together with the fact that, under the assumptions, the non-zero eigenvalues of $D\hat{H}$ are negative. ■