Stable near-rational sunspot equilibria*

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Abstract

We introduce a new class of solutions to nonlinear forward-looking models called near-rational sunspot equilibria (NRSE). NRSE are natural nonlinear extensions of the usual sunspot equilibria associated with the linearized version of the economy, and are near-rational in that agents use the optimal linear forecasting model when forming expectations. Generic results for existence and stability under learning are established. NRSE in indeterminate nonlinear models are found to be stable under learning provided that the corresponding linearized model's minimal state variable solution is E-stable. NRSE are readily computable, and our results make it possible to use the standard linear tools to search for stable NRSE. We illustrate our results using a canonical nonlinear New Keynesian model.

1 Introduction

Dynamic macroeconomic models that include forward-looking agents may exhibit equilibrium multiplicity. In these cases there may exist rational expectations equilibria (REE) that depend upon extrinsic stochastic processes, that is, a sequence of shocks that influences the economy only because agents condition expectations on these shocks. Importantly, this dependency is self-fulfilling: it exists only because agents think it exists. Equilibria that depend upon such extrinsic shocks are called sunspot equilibria, with the shocks themselves referred to as the sunspots.

The possibility that competitive rational expectations models can have self-fulfilling solutions driven by extraneous stochastic processes was demonstrated by various authors. Following the

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early literature, which we discuss in Section 2, models with multiple REE have been found in many areas of applied macroeconomics and finance. For example, Farmer, Benhabib and coauthors developed an entire research program devoted to explaining business cycle co-movements through the incorporation of non-convexities into competitive DSGE models and through the analysis of the sunspot equilibria associated with the linearized versions of these models: see, for example, Farmer and Guo (1994) and Benhabib and Farmer (1994). Separately, a literature emerged warning of the dangers of sunspot equilibria resulting from poorly designed policy in DSGE models with price frictions, e.g. Clarida, Gali, and Gertler (2000) argued that passive monetary policy pre-1980 may have allowed for self-fulfilling sunspot fluctuations. Simultaneously, Sims (2001) provided a convenient took-kit to characterize the set of sunspot solutions in indeterminate linear models, and Lubik and Schorfheide (2004) developed techniques for testing for indeterminacy econometrically.

More recently, following the Great Recession, discussions of neo-Fisherian policies have raised the issues of indeterminacy and equilibrium selection in New Keynesian models when policy follows an interest-rate peg; see Cochrane (2017), Evans and McGough (2018b), Evans and McGough (2018a) and Garcia-Schmidt and Woodford (2019). The multiplicity generated by the zero lower bound to interest rates, emphasized by Benhabib, Schmitt-Grohe, and Uribe (2001), has also recently highlighted the potential relevance of sunspot equilibria, e.g. Mertens and Ravn (2014). In finance a recent strand of research relating to excess volatility due to asset price bubbles has stressed the possibility of stationary asset price bubbles driven by an extraneous exogenous process, which are in effect sunspots: see Martin and Ventura (2012), Gali (2014) and Miao, Shen, and Wang (2019). The possibility of multiplicity is also well-known in several other prominent areas, including labor search and monetary search models.[1]

While the issue of multiple equilibria in macroeconomics is salient and acute, reactions vary considerably. Multiplicity is viewed by some as awkward because it begs the issue of equilibrium selection. The implications of a model with indeterminacy, e.g. policy implications, may then depend critically on which equilibria are selected. Sunspot equilibria have an additional dimension of indeterminacy based on the selection of the exogenous sunspot variable itself.

One possible reaction to indeterminacy is to treat models with multiple equilibria as defective and thus to impose assumptions on models that shut down any multiplicity. However, since multiplicities frequently arise from market distortions that are plausibly present in the economy, we think that a better approach is to embrace multiplicity when it arises naturally and to study its implications for the behavior of the economy and for policy. At the same time, we believe significant discipline can and should be imposed following the adaptive learning literature. Stability of REE under least-squares learning provides a natural equilibrium-selection mechanism: see, for example, Bray and Savin (1986), Marcet and Sargent (1989), Evans (1985), Evans (1989) and Evans and Honkapohja (2001). Under the adaptive learning approach agents estimate and update statistical models for forecasting key variables relevant to their decision-making. When the model is determinate, for a wide range of models, though not in all cases, the unique nonexplosive REE

[1] Sunspots may appear to be closely related to "sentiments." In the literature sentiments has a range of interpretations that depend on the precise model under consideration. See, for example, Benhabib, Wang, and Wen (2015) and Xiong and Yan (2010).

will be stable under learning.[2] This naturally raises the question of whether sunspot equilibria can be stable under adaptive learning in the indeterminate case.

Woodford (1990) showed, for a nonlinear overlapping generations model with an indeterminate steady state, that if agents thought certain finite state Markov sunspot processes *might* be relevant for forecasting, these agents would learn that the sunspots *are* relevant: Woodford showed that under real-time adaptive learning the economy converged, in an appropriate sense, to the associated sunspot equilibrium. However, there are other models in which sunspot equilibria are not stable under learning; thus stability under learning of sunspot equilibria cannot be, in general, assumed, and techniques are needed for the assessment of their stability in particular models.

The stability under learning of an REE is generally governed by the E-stability principle emphasized in Evans and Honkapohja (2001). In many cases these stability conditions are straightforward to compute, and stability under learning then provides a selection criterion among the set of REE. Furthermore the set of stable sunspot equilibria, when they exist, will of course satisfy specific time-series properties.[3] This procedure to equilibrium selection thus provides considerable discipline in the cases of indeterminate models: (i) attention is restricted to sunspot equilibria that agents could plausibly coordinate on using an adaptive learning process, and (ii) these sunspot equilibria will satisfy autocorrelation restrictions – and in the multivariate case cross-equation restrictions. As with REE in the determinate case, under indeterminacy stable sunspot equilibria impose strong conditions on the data.

E-stability, together with its implication for adaptive learning, has played an important role recently in several strands of applied work. Milani (2007) and Slobodyan and Wouters (2012) have found that the implementation of adaptive learning in New Keynesians improves empirical fit. Adam, Marcet, and Nicolini (2016) show that adaptive learning models in finance can be used to explain a number of financial-market puzzles. Benhabib, Evans, and Honkapohja (2014) argue that with strong pessimistic expectations shocks, adaptive learning can lead to unstable dynamics, which require aggressive monetary and fiscal policy to avoid. Evans and McGough (2018b) criticize the neo-Fisherian view by exposing the instability under adaptive learning of interest-rate pegs. For new-Keynesian models more generally, the importance of adaptive learning for monetary policy has been emphasized in the survey paper by Eusepi and Preston (2018).

The current paper advances this literature in several key ways. Although Woodford (1990) and Evans and Honkapohja (1994) used a univariate nonlinear set-up satisfied by simple overlapping generations models, most of the results for stability under learning in stochastic multivariate settings are for linearized models. In addition, for technical reasons, Woodford (1990), Evans and Honkapohja (1994) and Evans and Honkapohja (2003) focused on nonlinear models without intrinsic exogenous shocks and with sunspots taking the form of a 2-state Markov process. However, most DSGE models include continuously measured AR(1) or VAR(1) exogenous stochastic shocks, and the sunspot equilibria usually considered in these models are driven by variables with

[2]See, for example, Bullard and Mitra (2002) and Bullard and Eusepi (2014).

[3]The E-stability Principle in effect identifies E-stability and stability under adaptive learning. Throughout the paper we will frequently use the generic term "stability" to refer to both notions.

continuous support. Meanwhile, the current trend in macroeconomic research for determinate models is to retain the nonlinear features of DSGE models and to study approximations more accurate than first order to the REE. In the approach of this paper, we retain the economic model's nonlinear structure and allow for continuously measured exogenous stochastic shocks. After presenting the theory we illustrate the power of our techniques by examining monetary policy in a simple, non-linear New-Keynesian model.

The principal goal of our paper is to point the way to how one can model and obtain results on adaptive learning in nonlinear models in which indeterminacy is present and agents may be coordinating on a continuously valued AR(1) or VAR(1) stationary sunspot variable. The key assumption required for our analysis is that agents form expectations using a linear forecasting equation.[4] We regard this as plausible: linear models comprise the benchmark forecasting technique used in applied econometrics, and so it seems natural to impose this restriction on the forecasting models used by our boundedly rational agents. Given this assumption we can then study near-rational sunspot equilibria, i.e. stochastic processes generated in our nonlinear model by agents using a linear forecast rule that depends on an observed sunspot variable. These solutions are equilibria in the sense that the forecast coefficients used by each agent are optimal provided that all other agents are using the same forecast coefficients.

The paper is organized as followed. Following discussion in Section 2 of existing issues and open questions, Section 3 provides a complete analysis of a generic forward-looking univariate nonlinear model, in which we obtain conditions for existence and stability under learning of near-rational sunspot equilibria (NRSE). The model is tractable: generic existence and E-stability conditions for NRSEs are easily checked, and we show that NRSEs resolve the outstanding issues listed in Section 2. In doing so we show how to link the properties of NRSEs to simple properties of the linearized model, and we illustrate the results for a standard overlapping generations model. In Section 4 we then show how it is possible to extend our procedure to handle implicit formulations, intrinsic stochastic shocks, multivariate settings, and models with lags. This Section thus strongly suggests that our approach has a range of application sufficient to include most current nonlinear DSGE models. Finally, in Section 5 we illustrate the generality of our approach by showing the existence of stable NRSE in a canonical bivariate nonlinear New Keynesian model. We conclude that our approach provides the tools for analyzing and assessing sunspot equilibria in modern macroeconomic settings.

2 Existing Results, Questions and Issues

The existence of rational sunspot equilibria in macroeconomic environments was first established by Shell (1977), Azariadis (1981), Cass and Shell (1983), Azariadis and Guesnerie (1986) and Guesnerie (1986).[5] These existence results were originally obtained in simple stylized models,

[4]The assumption that agents use a linear forecasting model is not as restrictive as it appears: the forecasting model is assumed to be linear in its parameters, but in principle it could condition on nonlinear functions of observables.

[5]See the extensive survey in Guesnerie and Woodford (1992).

such as the overlapping generations model of money, and the sunspot drivers were typically taken to be finite-state Markov processes; but generic results providing criteria for local equilibrium uniqueness have also been established. Blanchard and Kahn (1980) present a practical technique for determining whether a linear model is "determinate," i.e. has a unique non-explosive equilibrium, or "indeterminate," i.e. has multiple such solutions including sunspot equilibria. Woodford (1986) shows that local equilibrium uniqueness in a nonlinear model is implied by uniqueness in the linearized model. As emphasized in the introduction, subsequent research has established the possibility of sunspot equilibria in a wide range of DSGE models.

Before proceeding to our formal analysis, we situate our approach within the context of existing results and the apparent impediments to the general analysis of sunspot equilibria in nonlinear models. As foreshadowed in the Introduction, there is a tension in the literature between results obtained in linear versus nonlinear models, and this has led to several issues, including concerns about the plausibility of sunspot equilibria in linear models and the lack of general techniques in nonlinear models. We will show that our approach resolves these concerns, and in doing so we are able to link our findings to known results on existence and stability of sunspot equilibria in linearized models. A particularly important finding is that appropriate analysis of a linearized model provides useful information on existence and stability of the associated NRSEs in the corresponding nonlinear models. Because the analysis of linearized models is relatively straightforward, our approach provides a powerful tool for assessing the existence of stable NRSE in nonlinear models.

The methods of Blanchard and Kahn (1980) and Sims (2001) can be used to establish the existence of sunspot equilibria in linear models. Importantly, these existence results are constructive: the equilibria present in an easily analyzed VAR form; and, the extrinsic processes – the sunspots – characterizing the sunspot equilibria in these linear models have (or, at least, can have) continuous support, and are thus more general than the finite-state equilibria examined in Woodford (1990) and Evans and Honkapohja (1994). Some general results establishing the existence of sunspot equilibria in nonlinear models are also available: Woodford (1986) showed that equilibrium multiplicity in the linearized model implies local equilibrium multiplicity in the nonlinear model. Unlike their linearized counterparts, however, the sunspot equilibria associated with the nonlinear models are not easily analyzed: the existence result relies on an implicit function theorem and is not constructive in nature; indeed, given a nonlinear model, there is no general technique for establishing a closed-form representation, or even a numerical approximation of an equilibrium associated to a sunspot with continuous support.

Subsequent research on the stability under learning of constructible sunspot equilibria associated with specific *linearized* models has obtained varying results. While certain linear(ized) models are known to have stable sunspot equilibria, Evans and Honkapohja (2001) showed that the sunspot equilibria associated with the model examined by Farmer and Guo (1994), at least for the particular calibration used, were not stable under learning. Evans and McGough (2005a) and Duffy and Xiao (2007) extended this instability result to a host of non-convex RBC-type models.

Stability in linear models also depends upon the stochastic properties of the sunspot process associated with the equilibrium. For example, in a model previously thought to have no stable sunspot equilibria, Evans and McGough (2005c) found that the equilibria may be stable provided that the associated sunspot process exhibited the appropriate serial correlation, known as the "resonance frequency." Using this insight, Evans and McGough (2005b) established the existence of stable sunspot equilibria in a variety of New Keynesian specifications.

This research on sunspot equilibria and their stability under learning has raised a number of concerns, including three important issues we catalog here.

1. **No nonlinear equilibrium recursions.** The challenge of constructing and analyzing continuous support sunspot equilibria in nonlinear models is problematic not only for the modeler, but also (indeed, even more so) for the model's agents. If we, as theoretical economists, are unable to recursively represent a particular equilibrium and thereby capture the conditional distributions of the endogenous variables, how then do we imagine agents making optimal forecasts? And even if we wish to adopt a learning perspective, what forecasting model do we provide our agents?
2. **The knife-edge of resonance.** The discovery of resonance frequency sunspots has greatly expanded the literature's catalog of models exhibiting stable sunspot equilibria; however, some researchers have questioned reliance on the existence of extrinsic processes meeting the knife-edge resonance frequency condition.
3. **No general stability results.** Woodford's stability result has been extended to the general univariate, forward-looking case by Evans and Honkapohja (2003), provided that the sunspots are finite state. No stability results are available for equilibria in nonlinear models associated with sunspots that have continuous support. In particular, it is not known whether sunspot stability in a linearized model is, in general, even related to stability of sunspot equilibria in the nonlinear model.

In this paper, we develop a new equilibrium concept designed to simultaneously address the above questions and concerns. We take our cue from the literature on bounded rationality and embrace the possibility that our agents have insufficient information and/or cognitive capacity to uncover the economy's endogenous distributions. Instead, we assume agents use simple, linear forecasting models when forming expectations. If the linear forecasting model used by agents is optimal among all similarly specified linear models then the economy is in a *near-rational equilibrium*. If the linear model includes a conditional dependency upon a sunspot process then the economy is in a *near-rational sunspot equilibrium*.

While in a near-rational equilibrium the forecasting model used by agents is optimal within a restricted class, it is misspecified in the sense that superior forecasting models exist. How easy it would be for agents to detect this misspecification depends the specific circumstances. If, as is commonly assumed, the economic model's shocks have small support, then the linear forecasting models used by agents will be quite accurate; consequently, detecting the misspecification would require both considerable sophistication and a large sample size. Also, as noted above, in general the forecasting models need only be linear in parameters: our methods would allow agents to regress on nonlinear functions of observables, in particular on polynomial terms that might better approximate the nonlinear dynamics of the data-generating process implied by agents' beliefs.

While our equilibrium concept extends naturally to accommodate forecast models that are nonlinear in observables, for simplicity and tractability we do not pursue their analysis here. Finally, we note that the well-known trade-off between fit and precision may in any event incline agents to prefer parsimonious specifications for their forecasting models.

To show how the NRSE approach can address the enumerated concerns we first study in detail a univariate nonlinear model in which all three issues arise. Within this set-up, we establish in Section 3 a generic existence result: if the linearized model is indeterminate then NRSE exist. Importantly, while the existence result itself relies on a bifurcation argument and is thus not constructive in nature, NRSE are identified as fixed points of finite dimensional functions and thus easily computed; furthermore the associated equilibrium process has a computable, recursive structure and so is amenable to detailed analysis. This addresses point one.

The sunspot processes associated with NRSE are found to be natural generalizations of the linearized model's resonance frequency sunspots: the processes are serially correlated, with the required correlation converging to the associated resonance frequency as the model's curvature (nonlinearity) vanishes. However, for given curvature there is an open set of serial correlations corresponding to NRSEs. We conclude that the knife-edge resonance frequency condition is an artifact of the linearization, and point two is addressed.

The linear forecasting structure of an NRSE makes it amenable to stability analysis: simply provide agents with a linear perceived law of motion that precisely includes the conditioning variables in the NRSE. We find that if the linearized model is indeterminate and the steady state solution is stable under learning, then the NRSE are also stable under learning. In Evans and McGough (2011) we showed that, in this linearized model, indeterminacy together with stability of the steady state solution is equivalent to the existence of stable sunspot equilibria. This provides a link between the linear and nonlinear models: stable sunspot equilibria in the linear model imply stable NRSE in the nonlinear model. This addresses point three.

In summary, for a univariate forward-looking model we establish that all three concerns are addressed using the NRSE approach. Of course, realistic microfounded general equilibrium macroeconomic models go beyond the framework of Section 3 in several significant ways, raising the question of whether our results will generalize to applied DSGE models. To study this issue, we establish in Section 4 that our results appear to be quite general.

It is not our intention to develop our results within the broadest possible framework in part because of the tediousness of the exercise, and also in part because it not clear what the most useful framework is for applied work where tractability is critical. Instead, we extend our results along different dimensions separately, thus providing an architecture for future extensions should they become needed.

In particular, in Section 4 we provide results establishing the existence of NRSE when the endogenous variable is implicitly defined, when the model has fundamental stochasticity, and when the model is multivariate – each of these results is demonstrated using the same proof strategy as the non-stochastic, univariate case, but each also holds its own special nuances. The remaining natural extension – the inclusion of a lagged endogenous variable – involves a significant technical barrier, so we consider this case numerically.

Our results suggest that there is a broader principle at work that governs whether stable NRSE will arise. When a linear model is determinate, the unique nonexplosive solution takes the form of a minimal state variable (MSV) solution, and MSV solutions also exist for linear models that are indeterminate.[6] Taken together, the theoretical results obtained in Sections 3 and 4 can be summarized as follows: if the linearized model is indeterminate and has an MSV solution that is stable under learning, then (generically) there exist NRSE that are stable under learning. We call this the *MSV Principle*, and we conjecture that it applies quite generally. This principle is important because it is straightforward both to check for indeterminacy of a linear model and to check for E-stability of an MSV solution.

In Section 5 we illustrate the MSV principle using a standard microfounded nonlinear bivariate New Keynesian model with a forward-looking interest-rate rule. Because the model is both multivariate and implicit, and because of the specific form of the interest-rate rule, the model does not fit into any of the specific extensions of Section 4; however, appealing to the MSV principle, we first examine the linearized model. As shown in Evans and McGough (2005b), we may pick policy parameters that generate an indeterminate steady state with an MSV solution of the linearized model that is E-stable and thus stable under learning. We may then construct a suitable sunspot variable and study numerically whether under learning there is convergence to an NRSE. The numerical findings indicate that there is indeed such convergence, consistent with the MSV principle. Our results confirm that the techniques provided in this paper make it possible to use standard tools for assessing indeterminacy and E-stability in linear models to assess the existence of learnable near-rational solutions for nonlinear models, in which economic fluctuations are driven in part by extraneous variables. These results appear scalable, in the sense that they could be extended and applied to larger computational macro models.

3 Near-rational Sunspot Equilibria (NRSE)

We start the formal analysis by developing our ideas in the simplest possible framework: a univariate nonlinear one-step-ahead forward-looking model with no exogenous shocks. This simple framework, consistent with the overlapping generations model of money, was used in the early theoretical literature to establish conditions for the existence of finite-state Markov sunspot equilibria and to study their stability under adaptive learning, e.g. Azariadis and Guesnerie (1986), Guesnerie (1986), Woodford (1990), Evans and Honkapohja (1994) and Evans and Honkapohja (2003). In revisiting this framework we are able to establish our main results: when the steady state is indeterminate, so that sunspot equilibria must exist, NRSE that depend on a stationary AR(1) sunspot process, must also exist; and that when the E-stability condition is also satisfied, which is easily determined, the NRSE are stable under adaptive learning; and, furthermore, all of the concerns

[6] We are using MSV in the sense of the primary criterion originally specified by McCallum (1983) for linear RE models: an MSV solution is one that depends linearly on a set of variables, and which is such that there does not exist a solution that depends linearly on a strict subset of those variables. See also Evans and Honkapohja (2001).

described in the preceding Section are readily addressed.

We examine the univariate case with considerable care in order to provide intuition for our results. Throughout we take as primitive a complete probability space (Ω, μ). Let $F : \mathbb{R} \to \mathbb{R}$ be C^k for $k \geq 4$, with an isolated fixed point normalized to be zero, i.e. $F(0) = 0$ and $F'(0) = \beta \neq 0$. Unless otherwise noted, $|\beta| > 1$. The abstract economy is taken as characterized by the following sequence of reduced-form equations:

$$y_t = E_t^* F(y_{t+1}). \tag{1}$$

Here E_t^* denotes the representative agent's subjective expectation based on their time t forecasting model of y_{t+1}. Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (1) satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

When agents satisfy the rational expectations hypothesis a rational expectations equilibrium (REE) of the model is any appropriately bounded stochastic process y_t satisfying (1) for $E_t^* = E_t$, where E_t denotes the true time t conditional expectation.[7] By the assumption $F(0) = 0$, it follows that $y_t = 0$ is a perfect foresight solution, which, clearly, is an MSV solution. Because we have assumed $|\beta| > 1$, we know from Woodford (1986) that the model is locally indeterminate: given any open neighborhood V of the origin, there is a non-MSV equilibrium (and in particular a sunspot equilibrium) with support in V; however, as noted in Section 2, these sunspot equilibria are, in general, difficult to characterize or even numerically approximate. Major attractions of the NRSE approach are that NRSE exist when sunspot equilibria exist and that it is straightforward to characterize NRSE and to assess their stability under adaptive learning. We begin with some preliminaries.

3.1 NRSE: Preliminaries

Our construction of near-rational sunspot processes for the nonlinear model (1) is motivated by the corresponding sunspots in the rational linear model. The linearized model associated to (1) is given by $y_t = \beta E_t^* y_{t+1}$. We define an REE of this model to be any stationary process y_t satisfying $y_t = \beta E_t y_{t+1}$, and we observe that $y_t = 0$ is the MSV solution. Now let ε_t be a zero-mean iid process, and with $\lambda = \beta^{-1}$, set $\eta_t = \lambda \eta_{t-1} + \varepsilon_t$. Then η_t is stationary provided that $|\beta| > 1$. Further, if $\hat{y}_t = \eta_t$ then $E_t \hat{y}_{t+1} = \lambda \eta_t$, so that $\hat{y}_t$ is a solution to the linearized model. The stochastic process η_t is usually referred to as a "sunspot" and the solution $\hat{y}_t = \eta_t$ as a sunspot equilibrium; $\hat{y}_t$ is an REE associated with the serially correlated sunspot process η_t. We note that $\hat{y}_t$ can be viewed as the sum of the MSV solution and a sunspot process with appropriate serial correlation. We conclude with the well-known result that if $|\beta| > 1$ then sunspot equilibria exist within the linearized model.

Returning to the nonlinear model (1) we now define our new equilibrium notion couched in the language and paradigms of bounded rationality. Similar to rational sunspot equilibria, the

[7] Our framework thus does not cover the "explosive" bubbles case.

equilibrium processes we identify will also depend upon extrinsic noise in a self-fulfilling manner: the dependence exists only if agents believe it exists. Unlike sunspot equilibria, however, the new equilibria are easily characterized, and amenable to both numerical and analytical examination.

We assume agents form expectations using linear forecasting models; and to impart discipline, we require in an NRSE that each agent's forecasting model is optimal among similarly specified linear models.

Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be an iid process in $L^\infty(\Omega)$ with zero mean and $\sigma_\varepsilon^2 > 0$. We assume further that, as a random variable, ε_t has compact support. Assume $\xi \in \mathbb{R}$ is such that $\lambda(\xi) = \beta^{-1} + \xi \in (-1, 1)$. It follows that

$$\eta_t^\xi = \sum_{k\geq 0} \lambda(\xi)^k \varepsilon_{t-k} \in L^\infty(\Omega).$$

The agents' Perceived Law of Motion (PLM), that is, the linear forecasting model used to form expectations, is given as

$$y_t = a + b\eta_t^\xi \tag{2}$$

$$\eta_t^\xi = \lambda(\xi)\eta_{t-1}^\xi + \varepsilon_t. \tag{3}$$

Observe that since $\eta_t^\xi \in L^\infty(\Omega)$, we have that for any continuous $f : \mathbb{R} \to \mathbb{R}$, $f \circ \eta_t^\xi(\cdot) \in L^\infty(\Omega)$; further, since η_t^ξ is stationary, it follows that for any s and t,

$$\int_\Omega f \circ \eta_t^\xi(\omega) d\mu(\omega) = \int_\Omega f \circ \eta_s^\xi(\omega) d\mu(\omega) = \int_\Omega f\left(\eta^\xi(\omega)\right) d\mu(\omega),$$

which exploits the time-invariant nature of the distribution against which the integral is taken. We will use this and similar observations repeatedly in the computations below, without further comment.

The PLM specifies E^*, yielding the following Actual Law of Motion (ALM):

$$y_t = \int_\Omega F(a + b\lambda(\xi)\eta_t^\xi + b\varepsilon_{t+1}(\omega)) d\mu(\omega) \equiv \hat{F}(a, b, \xi, \eta_t^\xi). \tag{4}$$

The ALM is the stochastic process (or "data generating process") that arises when agents form expectations based on a specified PLM.[8]

We may define the T-map $T(\cdot,\cdot,\xi) : \mathbb{R}^2 \to \mathbb{R}^2$ as the least squares projection of the ALM onto the span of $\{1, \eta_t^\xi\}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{T=T(a,b,\xi)} \begin{pmatrix} \int_\Omega \hat{F}(a,b,\xi,\eta^\xi(\omega)) d\mu(\omega) \\ \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) \hat{F}(a,b,\xi,\eta^\xi(\omega)) d\mu(\omega) \end{pmatrix} \equiv \begin{pmatrix} T^a(a,b,\xi) \\ T^b(a,b,\xi) \end{pmatrix}, \tag{5}$$

[8] The properties of $\hat{F}$, including its differentiability in a and b, as well as the properties of the many related functions, such as the various T-maps (e.g. equation (5)) are discussed in the Appendix and derived in the On-line Appendix.

where

$$\sigma^2_{\eta^\xi} = \int_\Omega \left(\eta^\xi(\omega)\right)^2 d\mu(\omega).$$

Observe that $T(0,0,\xi) = (0,0)'$.

Definition (NRSE). A non-trivial fixed point $\left(\bar{a},\bar{b}\right)$ of the T-map, i.e. with $\bar{b} \neq 0$, is a *near-rational sunspot equilibrium*.

This definition is natural because $T(\bar{a},\bar{b},\xi) = \left(\bar{a},\bar{b},\xi\right)$ implies that in the equilibrium in which all agents use a linear forecast rule with parameters $(\bar{a},\bar{b})$, any other choice of coefficients (a,b) would lead to forecasts with a larger mean square forecast error.

The NRSE concept can be viewed as a specific type of Restricted Perceptions Equilibrium (RPE). In an RPE agents use the optimal forecasting model within a pre-specified class. If this class does not include a forecasting model consistent with an REE then the RPE is not a rational expectations equilibrium. The RPE approach is quite general in the sense that it can be applied to PLMs characterized by omitted variables, omitted lags or misspecified functional forms: for its relation to adaptive learning see Evans and Honkapohja (2001) and for a general survey see Branch (2006).[9] In the current context the class of models is restricted to linear forecasting rules depending on a serially correlated sunspot.

We note that other authors have focused on the implications of linear forecasts in a nonlinear world. Hommes and Sorger (1998) consider a non-stochastic non-linear cobweb model in which agents use (linear) AR(1) models for forecasting, and require, in equilibrium, that the autocorrelogram of forecast models match the implied data. Hommes, Sorger, and Wagener (2013) extend these results to a stochastic environment by introducing white-noise shocks into the model. Branch and McGough (2005) demonstrate the existence of RPE in a non-linear forward model. The contribution of our paper is to extend the tractable analysis of self-fulfilling equilibrium dynamics to general nonlinear macroeconomic environments by leveraging adaptive learning and the RPE approach.

3.2 Illustrative special case: the simple cubic

The general examination of existence and stability of NRSE even in the univariate case requires a thorough and somewhat tedious two-dimensional bifurcation analysis. Before tackling the generic specification we provide intuition by restricting attention to the case in which F is cubic and symmetric about the origin:

$$F(y) = \beta y + \phi y^3. \tag{6}$$

We also assume here that $\mu_3^\varepsilon = \mu_3^{\eta^\xi} = 0$, where μ_n^x is the n^{th}-moment of x for $n > 2$. All of these assumptions will be relaxed in the general case examined below in Section 3.3.

[9]Hommes and Zhu (2014) show the possibility of multiple RPE arising when dynamics are underparamerized. Branch, McGough, and Zhu (2017) find that stable "sunspot RPE" exist in linear models for which no sunspot REE exist.

Recall $\lambda(\xi) = \beta^{-1} + \xi$. Assuming that agents use (2)-(3), a straightforward computation yields the following formulae for the T-map:

$$\begin{aligned} T^a(a,b,\xi) &= \beta a + \phi a^3 + 3\phi\sigma^2_{\eta^\xi} ab^2 \\ T^b(a,b,\xi) &= \beta\lambda(\xi)b + 3\phi\lambda(\xi)a^2 b + \phi\theta(\xi)b^3, \text{where} \\ \theta(\xi) &= \frac{\lambda(\xi)^3 \mu_4^{\eta^\xi}}{\sigma^2_{\eta^\xi}} + 3\lambda(\xi)\sigma^2_\varepsilon. \end{aligned}$$

Fixed points of the T-map correspond either to steady-state MSV solutions or to NRSE. The solution to $T^a = a$ is given by the line $a = 0$ and the (possibly empty) set $\mathscr{E}_a$, and the solution $T^b = b$ is given by the line $b = 0$ and the (possibly empty) set $\mathscr{E}_b$, where

$$\begin{aligned} \mathscr{E}_a &: \quad \frac{a^2}{\phi^{-1}(1-\beta)} + \frac{b^2}{\left(3\phi\sigma^2_{\eta^\xi}\right)^{-1}(1-\beta)} = 1 \\ \mathscr{E}_b &: \quad \frac{a^2}{-(3\phi\lambda(\xi))^{-1}\beta\xi} + \frac{b^2}{-(\phi\theta(\xi))^{-1}\beta\xi} = 1. \end{aligned}$$

Observing that for each of these equations the denominators have the same sign, it follows that when $\mathscr{E}_a$ and $\mathscr{E}_b$ are nonempty they are ellipses.

From the above it can be seen that the set of fixed points always includes the MSV solution $(0,0)$, and, when they exist, also includes the MSV solutions $\left(\pm\sqrt{\phi^{-1}(1-\beta)}, 0\right)$ and/or the NRSE given by $\left(0, \pm\sqrt{-(\phi\theta(\xi))^{-1}\beta\xi}\right)$. Additional solutions in which both components are nonzero may also exist. Recalling that the sunspot η is stationary when $|\beta| > 1$ and ξ is sufficiently small, and observing that the perturbation parameter ξ may be chosen to be either positive or negative, we conclude that $|\beta| > 1$ is in fact sufficient to guarantee existence of NRSE in this model. Importantly, there is an open set of "resonance frequencies" near β^{-1} for which NRSE exist: the "knife-edge of resonance" is indeed an artifact of the linearization. Of course our work allows us to conclude much more. We know exactly what the associated sunspots look like, and given the map F, we know how to compute the NRSE.

The T-maps defined above also allow us to address stability under learning. The precise connection will be made in Section 3.4 below. Here we simply note that the learning dynamics are characterized by the following differential equation system:

$$\dot{\gamma} = T(\gamma) - \gamma, \tag{7}$$

where $\gamma = (a,b)'$ and $T = \left(T^a, T^b\right)'$. It is worth observing that NRSE correspond to fixed points of this dynamic system, and this correspondence, together with bifurcation analysis, will be exploited to establish both existence and stability in the general case. To determine local stability of the NRSE for the case at hand we compute the eigenvalues of DT evaluated at the corresponding fixed points $\left(0, \pm\sqrt{-(\phi\theta(\xi))^{-1}\beta\xi}\right)$. An elementary calculation shows that the eigenvalues are given by

$$\beta - 1 - 3\beta\xi\sigma^2_{\eta^\xi}\theta(\xi)^{-1} \text{ and } -2\beta\xi.$$

Since these eigenvalues are necessarily real, a sufficient condition for stability is that they be negative. Noting that for ξ near zero the quantity $\sigma^2_{\eta^\xi}\theta(\xi)^{-1}$ is bounded and does not change sign, we see that for $\beta < 1$ the first term is negative for any small perturbation ξ. It follows that for small positive ξ, both eigenvalues are negative when $\beta < 1$. Finally, recalling that $|\beta| > 1$ is needed for existence of NRSE, we conclude that $\beta < -1$ is necessary and sufficient for existence of stable NRSE.

This stability result vindicates the MSV principle within the context of this simple cubic model. To see the connection, note that the linearized version of this model coincides with the linear model $y_t = \beta E^*_t y_{t+1}$ considered in Section 3.1. In the linearized model, the MSV solution is given by $y_t = 0$. Assessment of stability of this solution can be conducted by assuming agents use the PLM $y_t = a$ when forming forecasts. The associated T-map is easily seen to be given by $a \to \beta a$,

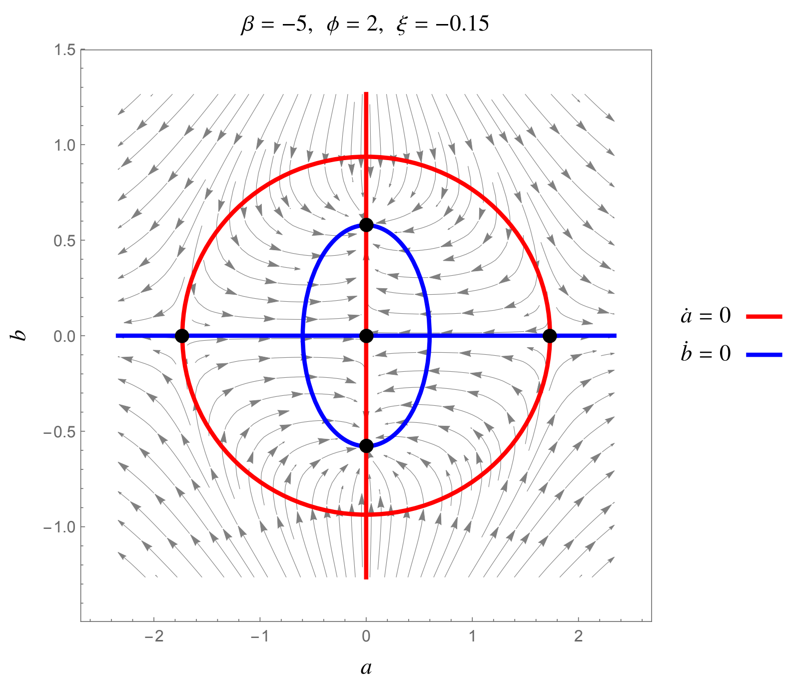

Figure 1. Cubic model with $\beta < -1$. Stable NRSE corresponding to fixed points with $a = 0, b \neq 0$.

and the learning dynamics are characterized by $\dot{a} = (\beta - 1)a$. This yields the well-known result that $\beta < 1$ guarantees stability of the MSV solution in the linearized model. Observing that the linearized model is indeterminate when $|\beta| > 1$, it follows that, according to the MSV principle, stable NRSE should exist in the cubic model when $\beta < -1$, which is precisely what we have established.

Global dynamics can be illustrated numerically. Figure 1 provides an example when $\beta < -1$. We observe five fixed points including, as expected, two stable NRSE. The remaining three fixed points, which all lie on the horizontal axis, correspond to fixed points of the cubic F, and thus to distinct perfect foresight steady-state equilibria of the cubic model. As is evident from the figure, these equilibria are not stable. Their instability in part reflects the inclusion of the sunspot variable in the PLM. In fact, if agents used the PLM $y_t = a$ then the steady state $y_t = 0$ is stable, as is evidenced in the figure by restricting attention to the dynamics on the horizontal axis.

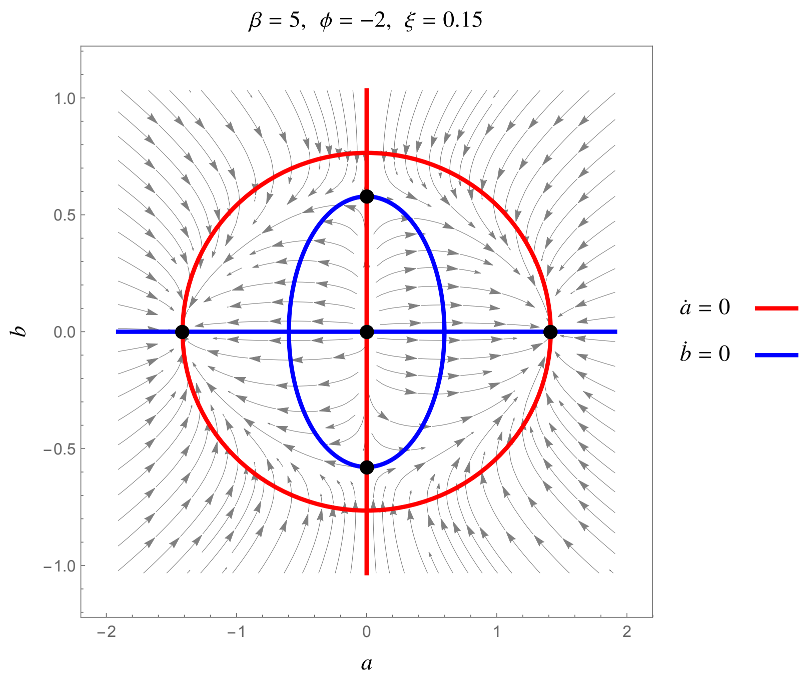

Figure 2. Cubic model with $\beta > 1$. Stable perfect foresight steady states corresponding to fixed points with $a \neq 0, b = 0$.

Figure 2 illustrates the case $\beta > 1$. As expected, neither the fixed points corresponding to NRSE, nor the steady state $y_t = 0$, are stable. However, the two non-zero perfect foresight steady-state equilibria are stable. This suggests the possibility of a more general phenomenon: in complex economic environments, agents may learn to coordinate on equilibria that are far from their initial priors (and perhaps not known to the economic modelers/policymakers).

3.3 Existence of NRSE

While the simple cubic of the preceding section conveys the key insights, it is of course important to show how these central results apply to the general nonlinear model (1). An economic example covered by this setup is given in Section 3.7.

The following is our main existence result, and is stated to emphasize the open set of resonance frequencies consistent with NRSE.

Theorem 1 (*Existence of NRSE*) *Assume that* $|\beta| > 1$ *and that either of the following two regularity conditions are met:*

Condition 1.1 $F''(0) \neq 0$ *and* $\mu_3^{\xi} \neq 0$

Condition 1.2 $\mathscr{BC} \equiv F'''(0)\left(\frac{3\sigma_\varepsilon^2}{\beta} + \frac{\mu_4^{\eta^\xi}}{\beta^3\sigma_{\eta^\xi}^2}\right) + \left(\frac{3(F''(0))^2}{(1-\beta)\beta}\right)\sigma_{\eta^\xi}^2 \neq 0$

Then NRSE exist: there exists a neighborhood V *of* β^{-1} *such that given any open set* $W \subset V$ *containing* β^{-1} *there exist* $\lambda(\xi) \in W$ *and* $(a,b) \in \mathbb{R}^2$ *with* $b \neq 0$ *satisfying* $T(a,b,\xi) = (a,b)'$.

The proof of Theorem 1 proceeds by conducting bifurcation analysis on the system (7). The argument is discussed in the Section A.2 of the Appendix, and the detailed proof of the result, as well as the proofs of all results in this paper, is contained in the On-line Appendix.

Theorem 1 generically addresses the first two concerns raised in the introduction and identified as motivating this effort. We now know when NRSE exist and what they look like. Further, we know that the resonance frequency restriction is an artifact of the linearization procedure: in fact, the sunspot's serial correlation acts a bifurcation parameter in the general case. Finally, and perhaps most interestingly, existence of NRSE obtains *if and only if* rational sunspot equilibria exist. This observation is particularly important from a practical perspective: assessing whether a given model may exhibit NRSE requires no new analytic tools.

Having established the generic existence of NRSE in the case $|\beta| > 1$, we now turn to the question of stability under learning.

3.4 Stability of NRSE

The existence of NRSE does not necessarily imply their importance. Can they arise in practice? This been a recurring issue in the sunspot literature and it is important to address it for NRSE. We follow the most widely used approach to assess the plausibility of sunspot equilibria by looking at whether they can emerge adaptively through statistical learning procedures. As is standard in the literature and natural given our assumptions regarding the forecasting behavior of agents, we have agents update their beliefs over time using recursive least squares: see Marcet and Sargent (1989) and Evans and Honkapohja (2001).

Let $\gamma_t = (a_t, b_t)'$ represent agents' beliefs conditional on information dated t and earlier. These beliefs evolve according to the following recursions:

$$\gamma_t = \gamma_{t-1} + \psi_t R_t^{-1} \begin{pmatrix} 1 \\ \eta_t^{\xi} \end{pmatrix} \left(\hat{F}\left(a_{t-1}, b_{t-1}, \xi, \eta_t^{\xi}\right) - \gamma_{t-1}' \begin{pmatrix} 1 \\ \eta_t^{\xi} \end{pmatrix} \right) \tag{8}$$

$$R_t = R_{t-1} + \psi_t \left(\begin{pmatrix} 1 \\ \eta_t^{\xi} \end{pmatrix} \begin{pmatrix} 1 & \eta_t^{\xi} \end{pmatrix} - R_{t-1} \right),$$

where R_t captures the sample second-moments matrix. Here ψ_t is referred to as the "gain" sequence and under least-squares learning we set $\psi_t = t^{-1}$. In many cases, including empirical applications, a constant gain $0 < \psi_t = \psi < 1$ is used.

The asymptotic behavior of this system may be analyzed by considering the differential equation system

$$\dot{\gamma} = R^{-1} \int_{\Omega} \left(\begin{pmatrix} 1 \\ \eta^{\xi}(\omega) \end{pmatrix} \hat{F}\left(a, b, \xi, \eta^{\xi}(\omega)\right) \right) d\mu(\omega) - R^{-1} M \gamma$$

$$\dot{R} = M - R,$$

where

$$M = \int_{\Omega} \left(\begin{pmatrix} 1 \\ \eta^{\xi}(\omega) \end{pmatrix} \begin{pmatrix} 1 & \eta^{\xi}(\omega) \end{pmatrix} \right) d\mu(\omega)$$

is the a.e. limit of R_t by the law of large numbers. It can be shown that the stability of this system at a given rest point (γ^*, M) is determined by the stability of the system (7), with the T-map given by equation (5). Since $\gamma^* \equiv (a^*, b^*)'$ corresponds to a fixed point of the T-map, it identifies an NRSE when $b^* \neq 0$. The theory of stochastic recursive algorithms tells us that if this fixed point is a Lyapunov stable rest point of (7), then an appropriately modified version of (8) will converge to it:[10] the associated NRSE is stable under learning. We note that the ordinary differential equation (ode) given by (7) corresponds to the usual E-stability differential equation, and thus, in the remainder of the paper, we will rely on E-stability when assessing the stability NRSE under learning.

If the model is linear then, as noted above, NRSE correspond to resonance frequency sunspot equilibria: $\lambda = \beta^{-1}$. Assuming agents know λ, it follows that $E_t^* y_{t+1} = a + b\lambda\eta_t$, so that the actual law of motion is given by

$$y_t = \beta a + b\eta_t.$$

We find that $T(a,b) = (\beta a, b)'$, so that the eigenvalues of DT are β and 1. We conclude that for the linear model sunspot stability obtains provided that $\beta < -1$.[11] An additional observation is warranted: if agents do not condition on the sunspot then the map from PLM to ALM is given

[10] To guarantee almost sure convergence, learning algorithms may, in some cases, require a projection facility: see Evans and Honkapohja (2001) for details.

[11] It is standard, in the stability analysis of sunspot equilibria associated to linear models, for the T-map to have at least one unit eigenvalue. This neutral stability reflects the (artificial) fact that, in a linear environment, any scalar multiple of a sunspot is again a sunspot. For a discussion, see Evans and McGough (2005a).

by $a \to \beta a$, so that the MSV solution is also E-stable exactly when $\beta < -1$. It is thus natural to view the sunspot equilibria as inheriting the stability of the MSV solution. This observation is an instance of the MSV principle in the linearized case.

Theorem 2 (*Stability of NRSE*) *Assume that $\beta < -1$ and that either condition 1.1 holds or that condition 1.2 holds with $\mathscr{B}\mathscr{C} < 0$. Then there exist NRSE that are stable under adaptive learning.*

When $\beta < -1$, the coefficients of $\sigma^2_{\eta^\xi}$ and $F'''(0)$, in the expression $\mathscr{B}\mathscr{C}$, are negative. This observation leads to the following corollary:

Corollary 1 (*Simple conditions for presence of stable NRSE*) *If $\beta < -1$ and either $F''(0) \neq 0$ or $F'''(0) > 0$ then stable NRSE exist.*

This result should be understood to mean that if the conditions of the corollary are met then stable NRSE exist for suitable choices of sunspot processes. More specifically, if $F'''(0) > 0$ then Condition 2 is met for any sunspot process with ξ near and on the appropriate side of β^{-1}, while if $F''(0) \neq 0$ then Condition 1 will be met for sunspots with ξ near and on the appropriate side of β^{-1} and $\mu_3^\varepsilon \neq 0$.

Returning now to the case of the simple cubic in Section 3.2, note that $F''(0) = 0$ and $F'''(0) = 6$. Thus if $|\beta| > 1$, condition 2 of Theorem 1 will be met for suitable sunspot processes η^ξ. Hence NRSE exist, and, by Corollary 1, they are stable if $\beta < -1$.

Theorems 1 and 2 provide vindication for resonance frequency sunspot equilibria: the knife-edge requirement needed in linear models is an artifact of the linearization and the tendency of resonance frequency sunspot equilibria to inherit the stability of the MSV solutions prevails in the nonlinear world. Put differently, by Theorem 2, E-stability of resonance frequency sunspot equilibria in the linear model guarantees the existence of stable NRSE in the nonlinear model (provided $F''(0) \neq 0$), which is a striking demonstration of the deep and broad reach of the E-stability principle. Theorem 2 also establishes the MSV principle in the context of the univariate forward-looking models studied in this Section: if $\beta < -1$ then the linearized model is indeterminate and its MSV solution is E-stable; theorem 2 shows that in this case stable NRSE exist.

3.5 Near-rational MSV solutions

The previous two subsections validate the MSV principle as it speaks to the existence of stable NRSE. In fact, in stochastic nonlinear models the MSV principle extends to near-rational solutions that only depend on the fundamentals. Specifically, if the MSV solution of the linear model is stable, then in the nonlinear model there are stable restricted perceptions equilibria associated with forecasting models that are linear in the fundamentals. We refer to these types of equilibria as near-rational MSV solutions, and we establish their existence and stability here within the context of a univariate model. This result is of interest in its own right, since boundedly-rational agents may plausibly use linear forecasting rules even in determinate nonlinear models, and it also provides a backdrop to our results on NRSE in stochastic models, which we study below in Section 4.2.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^k $(k \geq 4)$, with $F(0,0) = 0$ and $\beta = DF_y(0,0)$. Let $\zeta_t \in L^\infty(\Omega)$ be an iid process, let $0 < \rho < 1$, and let

$$v_t^\sigma(\omega) = \rho v_{t-1}^\sigma(\omega) + \sigma\zeta_t(\omega) = \sigma \sum_{k\geq 0} \rho^k \zeta_{t-k}(\omega) \in L^\infty(\Omega),$$

for $\sigma \in \mathbb{R}$. The model is given by $y_t = E_t^* F(y_{t+1}, v_{t+1}^\sigma)$. Given the specification of E^*, we are interested in solutions $\{y_t\}$ satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

For notational simplicity, we will suppress the dependence of v_t on σ. We assume agents use a PLM of the form $y_t = a + bv_t$, which yields the following ALM:

$$y_t = \int_\Omega F(a + b\rho v_t + b\sigma\zeta_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \equiv \tilde{F}(a,b,\sigma,v_t). \tag{9}$$

The corresponding T-map is given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{T=T(a,b,\sigma)} \begin{pmatrix} \int_\Omega \tilde{F}(a,b,\sigma,v(\omega))d\mu(\omega) \\ \left(\sigma_v^2(\sigma)\right)^{-1} \int_\Omega v(\omega)\tilde{F}(a,b,\sigma,v(\omega))d\mu(\omega) \end{pmatrix} \equiv \begin{pmatrix} T^a(a,b,\sigma) \\ T^b(a,b,\sigma) \end{pmatrix},$$

where

$$\sigma_v^2(\sigma) = \int_\Omega (v(\omega))^2 d\mu(\omega).$$

A fixed point of the T-map provides a near-rational MSV solution of this nonlinear model, and the stability of this solution is assessed by the associated differential equation system as above. We have the following existence and stability result.

Theorem 3 (*Near-rational MSV solutions*) *Assume* $DF_v(0,0) \neq 0$ *and* $\beta \neq 1$ *or* ρ^{-1}.

1. *For given* $|\sigma|$ *sufficiently small, there exists a unique near-rational MSV solution, i.e. there exists unique* $(a^*(\sigma), b^*(\sigma))' \in R^2$*, with* $b^*(\sigma) \neq 0$*, such that*

$$T(a^*(\sigma), b^*(\sigma), \sigma) = (a^*(\sigma), b^*(\sigma))'.$$

2. *If, in addition,* $\beta < 1$ *then this near-rational MSV solution is stable under adaptive learning.*

This result shows that the near-rational equilibrium approach is of interest beyond the study of sunspot equilibria. If agents use linear forecasting rules within a nonlinear set-up, and if the E-stability condition for the linearized model $\beta < 1$ is met, then the near-rational MSV solution is stable under least-squares learning whether or not the linearized model is determinate or indeterminate.

3.6 NRSE and REE

While we regard near-rational sunspot equilibria as a stand-alone equilibrium concept, it is natural to wonder about their connection to rational expectations equilibria. Establishing a formal connection requires taking a stand on the metric used for comparison, and is further complicated by the concepts' inherent multiplicities: even with a selected metric, which NRSE should be compared to which REE?

To make progress, we first characterize, to the extent possible, the REE local to the (indeterminate) steady state $y^* = 0$ of our model (1). Fix a martingale difference sequence (mds) $\hat{\varepsilon}_t$ with small support, and interpret it as the following rational forecast error: $\hat{\varepsilon}_t = F(y_t) - E_{t-1}F(y_t)$. It follows that the associated REE y_t must satisfy $F(y_t) = y_{t-1} + \hat{\varepsilon}_t$. Since $\hat{\varepsilon}_t$ has small support and $F'(0) \neq 0$, provided that $|y_{t-1}|$ is small, there is an open neighborhood U of the origin in $\mathbb{R}^2$, and a function $h : U \to \mathbb{R}$ so that $y_t = h(y_{t-1}, \hat{\varepsilon}_t)$. Furthermore, expanding h, we have that

$$y_t = \beta^{-1} y_{t-1} + \beta^{-1} \hat{\varepsilon}_t + \mathscr{O}\left(\| (y_{t-1}, \hat{\varepsilon}_t) \|^2\right),$$

which, by indeterminacy (i.e. $|\beta| > 1$) guarantees that $|y_{t-1}|$ will remain small if initialized near the origin. We conclude that the function h characterizes the REE associated to the mds $\hat{\varepsilon}_t$.[12] Conversely, all REE local to the steady state can be represented in this fashion: simply note that if y_t is an REE local to the steady state then, by setting $\hat{\varepsilon}_t = F(y_t) - E_{t-1}F(y_t)$. we may construct a function h so that $y_t = h(y_{t-1}, \hat{\varepsilon}_t)$.

The characterization of REE by the function h provides the connection between REE and NRSE. In particular, note that, to first order, *any* mds $\hat{\varepsilon}_t$ induces an REE with serial correlation given by β^{-1}, and the serial correlation of *any* NRSE is a perturbation of this same value β^{-1}. Thus, to-first-order/up-to-perturbation, the correlograms of all REE and all NRSE are the same.

3.7 A simple economic example

Here, to provide an illustration of how to apply our results, we develop a simple economy that fits the hypotheses of Theorems 1 and 2.

Consider an overlapping-generations environment in which there is a continuum of agents born at each time t indexed by $\upsilon_t \in \Upsilon$. Each agent lives two periods, works when young and consumes when old. The population is constant at unit mass. Each agent owns a production technology that is linear in labor and produces a common, perishable consumption good. The agent can sell his produced good in a competitive market for a quantity of fiat currency, anticipating that he will be able to use this currency when old to purchase goods for consumption.

While we will focus on the homogeneous case, it remains important, especially in models with boundedly rational decision-making, to distinguish agent-level and aggregate variables. For this

[12] Note that, provided F is sufficiently smooth, h can be approximated to arbitrarily high order by expanding each side of $F(h(y_{t-1}, \hat{\varepsilon}_t)) = y_{t-1} + \hat{\varepsilon}_t$ around $(0,0)$ and equating coefficients.

reason, we will retain agent-specific indexes for our initial analysis. Thus let $\upsilon_t \in \Upsilon$ be the index of an agent born in time t. His problem is given by

$$\begin{aligned} \max_{\substack{c_{t+1}(\upsilon_t), n_t(\upsilon_t) \\ M_t(\upsilon_t)}} \quad & E^*(\upsilon_t)\left(u(c_{t+1}(\upsilon_t)) - \chi(n_t(\upsilon_t))\right) \\ \text{subject to} \quad & n_t(\upsilon_t) = q_t M_t(\upsilon_t) \text{ and } c_{t+1}(\upsilon_t) = q_{t+1} M_t(\upsilon_t) \end{aligned} \tag{10}$$

Here, $n_t(\upsilon_t)$ is the agent's labor supply when young as well as his output. Also, q_t is the time t goods price of money and $c_{t+1}(\upsilon_t)$ is the agent's planned consumption when old. The expectations operator $E^*(\upsilon_t)(\cdot)$ denotes the expectation of agent υ_t at time t, taken with respect to his subjective beliefs conditional on the information available to him. This information includes $n_t(\upsilon_t)$, $M_t(\upsilon_t)$ and current and lagged values of q_t.

The first order condition is given by

$$\chi'(n_t(\upsilon_t)) = E^*(\upsilon_t)\left(\frac{q_{t+1}}{q_t} u'(c_{t+1}(\upsilon_t))\right), \tag{11}$$

and to make our model particularly tractable, we assume that $\chi' = 1$ and $u(c) = \frac{1}{1-\sigma}\left(c^{1-\sigma} - 1\right)$. With simplification, we obtain agent υ_t's decision rules:

$$\begin{aligned} n_t(\upsilon_t) &= \left(q_t^{\sigma-1} E^*(\upsilon_t)\left(q_{t+1}^{1-\sigma}\right)\right)^{\frac{1}{\sigma}} \\ M_t(\upsilon_t) &= \left(\frac{1}{q_t} E^*(\upsilon_t)\left(q_{t+1}^{1-\sigma}\right)\right)^{\frac{1}{\sigma}}; \end{aligned}$$

and we note that, as is natural, the quantity of money demanded by agent υ_t at time t, depends on, among other things, the price at time t.

Assuming a constant (unit) supply of money, we obtain the market-clearing condition

$$\int_\Upsilon M_t(\upsilon_t) d\upsilon_t = 1,$$

which yields

$$q_t = \left(\int_\Upsilon \left(E^*(\upsilon_t)\left(q_{t+1}^{1-\sigma}\right)\right)^{\frac{1}{\sigma}} d\upsilon_t\right)^{\sigma}. \tag{12}$$

Equation (12) characterizes the equilibrium price path.

If agents are homogeneous, the model reduces to $q_t = E_t^* q_{t+1}^{1-\sigma}$, which is consistent with the framework considered in Section 3. If all agents have rational expectations then $q = 1$ is the unique, non-autarky, perfect-foresight steady state. The system may be log-linearized around this steady state to yield $\log q_t = (1-\sigma) E_t \log q_{t+1}$. The steady state is indeterminate if $\sigma > 2$: in this case the expectational feedback parameter is negative and sunspot equilibria exist in both the linearized and nonlinear models.

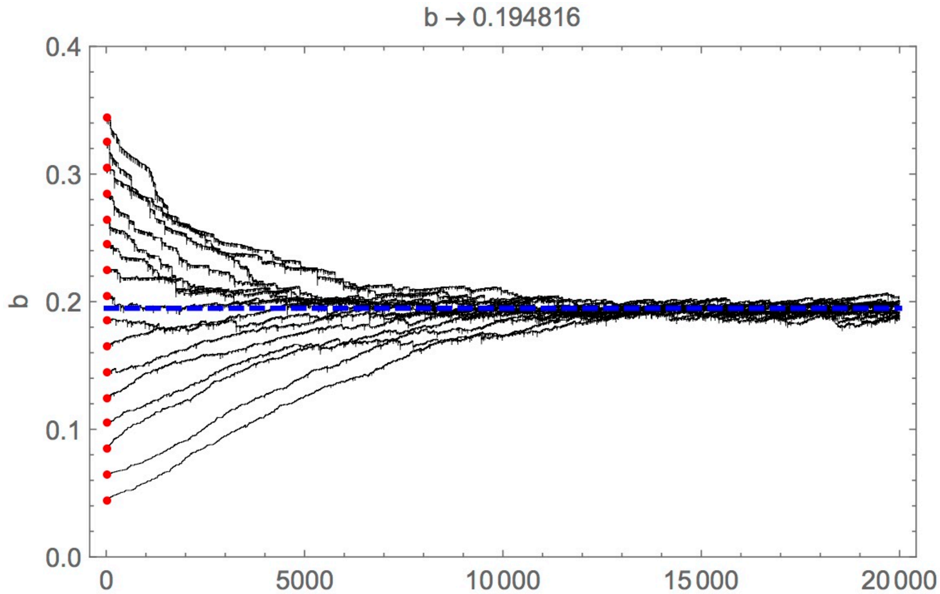

Figure 3. NRSE learning dynamics in the overlapping generations model
PLM: $y_t = a + b\eta_t$, convergence to an NRSE

To apply our theorems, let $F(y) = (y+1)^{1-\sigma} - 1$, so that the model becomes $y_t = E_t F(y_{t+1})$, with $y = q - 1$. We compute $F'(0) = 1 - \sigma$ and $F''(0) = \sigma(\sigma - 1)$, so that, by Corollary 1, stable NRSE exist provided $\sigma > 2$.

To assess this claim numerically, we calibrate the model by setting $\sigma = 2.5$, and, since $F''(0) > 0$, we select a negative perturbation ($\xi < 0$), so that the NRSE is stable. Then, choosing an asymmetric iid martingale difference sequence ε_t, we simulate the real-time learning dynamics corresponding to a variety of initial conditions: see Figure 3, which plots the dynamics of b_t, the time t-value of the sunspot coefficient in the agent's forecasting model.[13] We observe convergence to the estimated NRSE value of $b^* = .195$.[14]

4 Stable NRSE: extensions

The reduced form model (1) served as a platform to discuss and provide intuition for our main existence and stability results; however, most applied macro models do not present so simply. Ideally, the theory of NRSE should be developed against a sequence of reduced-form equations of

[13] For this Figure we use the following specification for the sunspot process: $\varepsilon_t \in \{-.475, .025\}$ is iid with $\Pr(\varepsilon_t = .025) = .95$, and $\xi = -.0175$. Since $\beta = -1.5$ this gives $\lambda(\xi) = -.684$. We use a constant gain of $\psi = 0.015$.

[14] It can be shown that if $2 < \sigma < \frac{1}{4}\left(5 + \sqrt{17}\right) \approx 2.28$ then the sunspot's stochastic driver ε_t can be taken as symmetric.

the form

$$E_t^* F(y_t, y_{t+1}, y_{t-1}, v_t) = 0, \tag{13}$$

where $y_t \in \mathbb{R}^n$ is endogenous, $v_t \in \mathbb{R}^m$ is a stationary exogenous process, and $F : \mathbb{R}^{3n} \oplus \mathbb{R}^m \to \mathbb{R}^n$ is C^k for $k \geq 4$; however, results for models at this level of generality are not yet available. To make some progress, and to show how modifications of our underlying framework and arguments apply in more general settings, in this Section we consider, separately, a variety of extensions suggested by the model (13). Because the development and argument structure are similar to the work done in Section 3, our discussions here will be considerably more brief.

4.1 Stable NRSE: the implicit case

In many modeling environments, the time t endogenous variable is defined only implicitly in terms of expectations of future variables. For example, in some set-ups Euler equations cannot easily or naturally be transformed into explicit equations, and it is important to know if our results extend to implicit equation frameworks.

To consider this case, let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^k $(k \geq 4)$, with $F(0,0) = 0$, $F_1(0,0) \neq 0$, and $\beta = -F_2(0,0)/F_1(0,0)$, where, in this section, F_i is the partial of F with respect to the i-th variable. The sequence of reduced-form equations is given by

$$E_t^* F(y_t, y_{t+1}) = 0. \tag{14}$$

Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (14) satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

As in the previous section, let $\{\varepsilon_t\} \subset L^\infty(\Omega)$ be a zero-mean iid process with compact support, and assume $\xi \in \mathbb{R}$ is such that $\lambda(\xi) = \beta^{-1} + \xi \in (-1,1)$. The agents' PLM is given as

$$\begin{aligned} y_t &= a + b\eta_t^\xi \\ \eta_t^\xi &= \lambda(\xi)\eta_{t-1}^\xi + \varepsilon_t, \end{aligned}$$

which, by specifying E^*, gives the following implicitly defined ALM:

$$\tilde{F}\left(y_t, a, b, \xi, \eta_t^\xi\right) \equiv \int_\Omega F\left(y_t, a + b\lambda(\xi)\eta_t^\xi + b\varepsilon(\omega)\right) d\mu(\omega) = 0.$$

Noting that $\tilde{F}(0,0,0,\xi,\eta_t^\xi) = 0$ and that, evaluated at $y_t = a = b = 0$, we have $\tilde{F}_y = F_1(0,0) \neq 0$, the implicit function theorem implies that locally the ALM may be written $y_t = \hat{F}\left(a, b, \xi, \eta_t^\xi\right)$, where we are now assuming that the support of ε_t is such that η^ξ remains in the domain of $\hat{F}$ for small ξ.

With $\hat{F}$ so defined, we may proceed just as in the previous case by defining the T-map $T(\cdot,\cdot,\xi) : \mathbb{R}^2 \to \mathbb{R}^2$ as the projection of the ALM onto the span of $\{1, \eta_t^\xi\}$. This again yields the formula in equation (5). Again observe that $T(0,0,\xi) = (0,0)'$. An NRSE of this model is a non-trivial fixed point of this T-map.

The following result provides conditions for existence and stability. The complicated expression corresponding to $\mathscr{IC}$ is given by equation (46f) in the Online Appendix. All derivatives are evaluated at zero.

Theorem 4 (*Extension: implicit case*) *Assume that $|\beta| > 1$.*

- ***Existence.*** *Assume that either of the following two regularity conditions is met:*

  Condition 4.1 $(2\beta F_{12} + F_{22})\,\mu_3^{\varepsilon} \neq \beta F_{112}\mu_4^{\eta^{\xi}}$
  Condition 4.2 $\mathscr{IC} \neq 0$

  Then NRSE exist. Specifically, there exists a neighborhood V of β^{-1} such that given any open set $W \subset V$ containing β^{-1} there is a $\lambda(\xi) \in W$ and $(a,b) \in \mathbb{R}^2$ with $b \neq 0$ satisfying $T(a,b,\xi) = (a,b)'$.

- ***Stability.*** *Assume that $\beta < -1$ and that either condition 4.1 holds or that condition 4.2 holds with $\mathscr{IC} < 0$. Then the NRSE are stable under adaptive learning.*

We observe that existence, and stability in case $\beta < -1$, are generic in the sense that they obtain for appropriate ε_t if $F_{112} \neq 0$. We note also that setting $F_1 = -1$ and $F_{1*} = 0$ corresponds to the previous case in which $y_t = E_t^* \check{F}(y_{t+1})$ (for appropriate $\check{F}$); and, the conditions we obtain here reduce to the conditions found in Theorems 1 and 2. In summary, apart from the specific regularity assumptions needed to rule out non-generic cases, our results on NRSEs extend to the implicit case.

4.2 Stable NRSE: the stochastic case

The benchmark set-up of Section 3 does not include intrinsic stochastic shocks. Most modern macroeconomic models include exogenous stochastic shocks to variables such as productivity, preferences and policy, and RE solutions will then depend on the realized shocks. Near-rational solutions depending on these shocks were already briefly considered in Section 3.5. We now turn to existence and stability of NRSE associated with the near-rational MSV solution of the stochastic model presented in that Section and reproduced here for convenience: $y_t = E_t^* F(y_{t+1}, v_{t+1})$.

Agents are assumed to use a PLM of the form $y_t = a + bv_t + c\eta_t^{\xi}$. In what follows, unless otherwise specified, derivatives are evaluated at

$$(a,b,c,\xi) = (a^*, b^*, 0, 0).$$

As argued in the proof of Theorem 5, by choosing $|\sigma|$ small we may assume that $DF_\star(y,v) \approx DF_\star(0,0)$ for $\star = y, v, yy$, etc. Thus, we may assume, for the remainder of this section, that $|\beta| > 1$, whence we may choose σ small enough that $|DF_y| > 1$.

Turning first to expectations, the PLM is given by

$$\begin{aligned} y_t &= a + bv_t + c\eta_t^{\xi} \\ v_t &= \rho v_{t-1} + \sigma\zeta_t \\ \eta_t^{\xi} &= \lambda(\xi)\eta_{t-1}^{\xi} + \varepsilon_t, \end{aligned}$$

where $0 < \rho < 1$ and $\lambda(\xi) = \beta^{-1} + \xi \in (-1, 1)$. We further assume that $\zeta_t \perp \varepsilon_s$ for all t, s. For fixed small σ, the ALM is given by $y_t = \hat{F}\left(a, b, c, \xi, v_t, \eta_t^{\xi}\right)$ where

$$\hat{F} = \int_{\Omega} F(a + b\rho v_t + b\sigma\zeta_{t+1}(\omega) + c\lambda(\xi)\eta_t^{\xi} + c\varepsilon_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega). \tag{15}$$

Exploiting independence, the T-map is given by

$$\begin{aligned} a &\xrightarrow{T^a(a,b,c,\xi)} \int_{\Omega} \hat{F}\left(a, b, c, \xi, v(\omega), \eta^{\xi}(\omega)\right) d\mu(\omega) \\ b &\xrightarrow{T^b(a,b,c,\xi)} \frac{1}{\sigma_v^2}\int_{\Omega} v(\omega)\hat{F}\left(a, b, c, \xi, v(\omega), \eta^{\xi}(\omega)\right) d\mu(\omega) \\ c &\xrightarrow{T^c(a,b,c,\xi)} \frac{1}{\sigma_{\eta^{\xi}}^2}\int_{\Omega} \eta^{\xi}(\omega)\hat{F}\left(a, b, c, \xi, v(\omega), \eta^{\xi}(\omega)\right) d\mu(\omega) \end{aligned}$$

Finally, let $(a^*(\sigma), b^*(\sigma)) = (a^*, b^*)$ be the near-rational MSV solution corresponding to σ, and note, using (15), that

$$T(a^*, b^*, 0, \xi) = (a^*, b^*, 0)'.$$

A non-trivial (i.e. $c \neq 0$) fixed point of the T-map is an NRSE.

Theorem 5 (*Extension: intrinsic stochasticity*) *Assume* $DF_v(0,0) \neq 0$, $|\beta| > 1$, $\beta\rho \neq 1$, *and that* $|\sigma|$ *is sufficiently small.*

- ***Existence.*** *Assume that either of the following two regularity conditions are met:*

  Condition 5.1 $DF_{yy} \neq 0$ *and* $\mu_3^{\varepsilon} \neq 0$

  Condition 5.2 $\mathscr{SC} \equiv DF_{yyy}\left(\frac{3\sigma_{\xi}^2}{\beta} + \frac{\mu_4^{\eta^{\xi}}}{\beta^3\sigma_{\eta^{\xi}}^2}\right) + \frac{3\left(DF_{yy}\right)^2\left(\sigma_{\eta^{\xi}}^2 + \beta^2\sigma_{\xi}^2\right)}{(1-\beta)\beta^3} \neq 0$

  Then NRSE exist. Specifically, there exists a neighborhood V of β^{-1} *so that given any open set* $W \subset V$ *containing* β^{-1} *there is a* $\lambda(\xi) \in W$ *and* $(a, b, c) \in \mathbb{R}^3$ *with* $c \neq 0$ *satisfying* $T(a, b, c, \xi) = (a, b, c)'$.

- ***Stability.*** *Assume further that* $\beta < -1$ *and that either condition 5.1 holds or that condition 5.2 holds with* $\mathscr{SC} < 0$. *Then the NRSE are stable under adaptive learning.*

Theorem 5 tells us that our existence and stability results for NRSEs also hold when fundamental exogenous shocks are present. In particular provided $\beta < -1$, so that the linearized model is indeterminate and the E-stability condition is satisfied, then generically stable NRSE exist in a stochastic set-up.

4.3 Stable NRSE: the multivariate case

Modern macroeconomic models are essentially multivariate, i.e. they have multiple endogenous variables. Even the simplest canonical textbook New Keynesian framework specifies a bivariate model of output and inflation, and medium-scale DSGE models include many endogenous variables.

In principle, there is no difficulty conducting the above analysis in higher dimensions, though in practice the work is somewhat more tedious; and, two distinct cases arise, depending on the nature of the model's roots. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be C^k where $k \geq 4$. Assume $F(0) = 0$ and $DF(0) \in \mathbb{R}^{n\times n}$ is diagonalizable. Write $DF(0) = S \cdot \oplus_{i=1}^n \beta_i \cdot S^{-1}$, with $\beta_i \in \mathbb{C}$ the eigenvalues of $DF(0)$. The economic model is given by

$$y_t = E_t^* F(y_{t+1}). \tag{16}$$

Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (16) satisfying $y_{it} \in L^\infty(\Omega)$, and $\sup_t \|y_{it}\|_\infty < \infty$.

So that the model is indeterminate, we assume at least one root, which we label as β_n, lies outside the unit circle. We make the further assumption that $\beta_n \in \mathbb{R}$. This is for simplicity, as the analysis is considerably more involved if all roots that lie outside S^1 are complex: the sunspot is necessarily a two-dimensional VAR(1) process, and co-dimension-2 bifurcation analysis is required.[15]

Working as before, assume $\xi \in \mathbb{R}$ is such that $\lambda(\xi) = \beta_n^{-1} + \xi \in (-1,1)$. The agents' PLM is given as

$$\begin{aligned} y_t &= a + b\eta_t^\xi \\ \eta_t^\xi &= \lambda(\xi)\eta_{t-1}^\xi + \varepsilon_t, \end{aligned}$$

with $a, b \in \mathbb{R}^n$. Writing $F = (F^1, \ldots, F^n)'$, the ALM is given by

$$y_{it} = \int_\Omega F^i\left(a + b\lambda(\xi)\eta_t^\xi + b\varepsilon_{t+1}(\omega)\right) d\mu(\omega) \equiv \hat{F}^i\left(a, b, \xi, \eta_t^\xi\right).$$

The T-map is given by

$$\begin{aligned} a_i &\to \int_\Omega \hat{F}^i\left(a, b, \xi, \eta^\xi(\omega)\right) d\mu(\omega) \\ b_i &\to \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i\left(a, b, \xi, \eta^\xi(\omega)\right) d\mu(\omega). \end{aligned}$$

It is immediate that $(a,b) = (0,0) \in \mathbb{R}^n \oplus \mathbb{R}^n$ is a fixed point of the T-map. A fixed point with non-zero b is an NRSE. The next theorem establishes existence and stability of NRSE in the simpler,

[15] Preliminary results indicate that, in the complex case, appropriate perturbation of the sunspot process's covariance matrix results in a Bogdanov-Takens bifurcation, from which a stable NRSE emerges. We also note that if $DF(0)$ has $m \leq n$ eigenvalues lying outside the unit circle, then sunspot processes up to dimension m may exist. We are developing these results in current work.

transcritical case, which occurs when $\mu_3^\xi \neq 0$. As notation, let S_n be the n^{th}-column of S (i.e. an eigenvector associated to β_n), $S^{-1} = (S^{ij})$ and D^2F^i the Hessian of F^i evaluated at zero.

Theorem 6 (*Extension: multivariate case*) *Let $DF(0) = S \cdot \oplus_{i=1}^n \beta_i \cdot S^{-1}$ with $\beta_n \in \mathbb{R}$ and $|\beta_n| > 1$.*

- ***Existence.*** *Assume the following regularity conditions hold:*

  Condition 6.1 $\mu_3^\xi \neq 0$

  Condition 6.2 $\sum_{i=1}^n S^{ni} \left(S_n' \cdot D^2F^i \cdot S_n \right) \neq 0$

  Then NRSE exist. Specifically, there exists a neighborhood V of β_n^{-1} so that given any open set $W \subset V$ containing β_n^{-1} there is a $\lambda(\xi) \in W$ and $(a,b) \in \mathbb{R}^n \oplus \mathbb{R}^n$ with $b \neq 0$ satisfying $T(a,b,\xi) = (a,b)'$.

- ***Stability.*** *Assume further that $Re(\beta_i) < 1$ for all $i = 1,\ldots,n$ and $\frac{Re(\beta_i)}{\beta_n} < 1$ for all $i = 1,\ldots,n-1$. Then the NRSE are E-stable.*

We remark that the second regularity condition for existence (above) can be viewed as generic in the following sense: S is invertible (and thus the S_{ij} and S^{ij} are not all zero) and S is a first-order term whereas the D^2F^i are second-order. Also, we note that the stability condition provided in this theorem is sufficient, but not necessary: indeed in this multivariate setting stable NRSE exist under many other constellations of conditions.

Theorem 6 demonstrates that it is possible to extend existence and stability results to multivariate models.

4.4 Stable NRSE: the case with lags

As a final extension, we consider a univariate reduced-form model with an endogenous lag. This, too, is an important extension because many macroeconomic models, and all serious DSGE models, are both forward-looking and backward-looking. Lagged effects arise, for example, from capital accumulation, adjustment costs, indexation and policy inertia. To investigate this in a simple set-up we revert to a univariate non-stochastic model.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^k $(k \geq 4)$, with $F(0,0) = 0$, $F_1(0,0) = \beta \neq 0$, and $F_2(0,0) = \delta \neq 0$, where, in this section, F_i is the partial of F with respect to the i-th variable. The sequence of reduced-form equations is given by

$$y_t = E_t^* F(y_{t+1}, y_{t-1}), \tag{17}$$

and we assume the model is subject to an initial condition $y_t = y_0$ at $t = 0$. Given the specification of E^*, we are interested in solutions $\{y_t\}$ to (17) satisfying $y_t \in L^\infty(\Omega)$, and $\sup_t \|y_t\|_\infty < \infty$.

The linearized, RE version of this model is given by

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1}. \tag{18}$$

It can be shown that the model (18) is determinate when $|\beta+\delta|<1$ and that the nonexplosive solution takes the form $y_t=\varphi_1 y_{t-1}$ where φ_1 is given below and $|\varphi_1|<1$. Clearly this is an MSV solution.

The model (18) is indeterminate and has real roots provided $|\beta+\delta|>1$, $|\delta|<|\beta|$ and $\beta\delta<\frac{1}{4}$, and we assume these conditions hold throughout the rest of this section. In this case both roots

$$\varphi_1=\frac{1-\sqrt{1-4\beta\delta}}{2\beta} \text{ and } \varphi_2=\frac{1+\sqrt{1-4\beta\delta}}{2\beta}$$

of the associated quadratic $\beta\varphi^2-\varphi+\delta=0$ have absolute values less than one and there are two MSV solutions $y_t=\varphi_i y_{t-1}$ for $i=1,2$. Under the information assumptions provided below, it can be shown that the solution $y_t=\varphi_1 y_{t-1}$ is E-stable when $\beta<0$, while the solution $y_t=\varphi_2 y_{t-1}$ is never E-stable.

As in previous sections, let $\{\varepsilon_t\}\subset L^\infty(\Omega)$ be a zero-mean iid process with compact support. Evans and McGough (2005c) showed that the process given by

$$\begin{aligned} y_t &= \varphi_1 y_{t-1}+\eta_t \\ \eta_t &= \varphi_2\eta_{t-1}+\varepsilon_t \end{aligned}$$

is a stationary sunspot equilibrium of (18); and further, if $\beta<0$ then this REE is stable under adaptive learning.

As we did in Section 3, we use perturbations of sunspots in the linear model to generate NRSE in the nonlinear model. Let $\xi\in\mathbb{R}$ be such that $\lambda_2(\xi)=\varphi_2+\xi\in(-1,1)$. The agents' PLM is given as

$$\begin{aligned} y_t &= a+by_{t-1}+c\eta_t^\xi \\ \eta_t^\xi &= \lambda_2(\xi)\eta_{t-1}^\xi+\varepsilon_t, \end{aligned}$$

which, by specifying E^*, gives the following ALM:

$$y_t=\int_\Omega F\left((1+b)a+b^2y_{t-1}+c\left(\lambda_2(\xi)+b\right)\eta_t^\xi+c\varepsilon(\omega),y_{t-1}\right)d\mu(\omega). \tag{19}$$

Here we are assuming, as in common in the literature, that when agents form expectations their information set includes y_{t-1} and η_t, but not y_t. Note that the PLMs used by the agents for forecasting are linear and take the form of the MSV solution plus the serially correlated sunspot variable.

The next step in the analysis would normally be to define the T-map, but this requires knowledge of the asymptotic distribution of the regressors for fixed beliefs (a,b,c). Unfortunately, given the presence of y_{t-1}, this distribution is endogenous to beliefs, which appears to be a formidable technical impediment. Based on our work thus far, and invoking the MSV principle, the following conjecture seems reasonable:

Conjecture 1 (*Extension: lagged case*) *Assume that* $|\beta+\delta|>1$, $|\delta|<|\beta|$ *and* $\beta\delta<\frac{1}{4}$.

- ***Existence.*** *NRSE generically exist.*
- ***Stability.*** *If, in addition, $\beta < 0$, then E-stable NRSE generically exist.*

To provide support for this conjecture, we present numerical results. First, observe that a "sample-version" of a T-map may be defined. Specifically, for fixed beliefs (a,b,c), we may draw a sequence of N shocks $\left\{\eta_t^\xi\right\}_{t=0}^N$, and using quadrature to evaluate (19), compute the associated endogenous realizations $\{y_t\}_{t=0}^N$, where y_0 and η_0^ξ are taken as given. The sample T-map is given by simply using these data to regress y_t on y_{t-1}, η_t^ξ and a constant. If the sample size N is large enough (and if the associated asymptotic distributions exist, etc.) then the sample T-map should well-approximate the true T-map, which means a fixed point of the sample T-map should well-approximate an NRSE. Finally, if the NRSE is E-stable, it is expected that iteration of the sample T-map, possibly modified to include a damping factor, should converge to a fixed point.

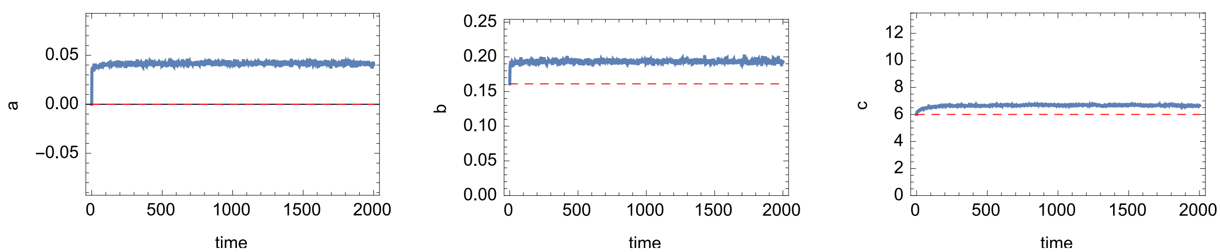

Figure 4. NRSE in a model with lags. PLM: $y_t = a + by_{t-1} + c\eta_t$.
Iterations of the T-map converging toward an NRSE.

Precisely this experiment is carried out in Figure 4. The map F used to construct this figure has linear terms $\beta = -1.5$ and $\delta = .2$ and an ad-hoc quadratic form to capture the nonlinearity. Thus the linear model is indeterminate, and, according to the conjecture, we expect stable NRSE to exist. Sample size is set at 3000, and the shock ε_t is uniformly distributed on $[-.1,.1]$. The initial conditions for beliefs, as indicated by the red, dashed lines, correspond to the linear REE values, with c set arbitrarily at 6. The sample T-map is then iterated, and the "time-plot" is provided in the Figure. We see convincing evidence of rapid convergence to non-REE values, suggesting the presence of a stable NRSE.[16]

We may also conduct the analogous real-time learning simulation – See Figure 5. In this case, as new data become available, beliefs are updated over time using recursive least squares. As above, the dynamics are initialized at the linear REE values, and for this simulation a decreasing gain algorithm is used.[17] The red, dashed lines in the first two panels identify the fixed point of

[16]That the sample T-map never settles down to a fixed point is a reflection of the finite sample properties of the map.

[17]We use a decreasing gain sequence with $\psi_t = t^{-0.8}$ rather than $\gamma = t^{-1}$ in order to increase the speed of convergence. The sunspot process is symmetric and sets $\xi = -0.02$ so that $\lambda = -0.85$.

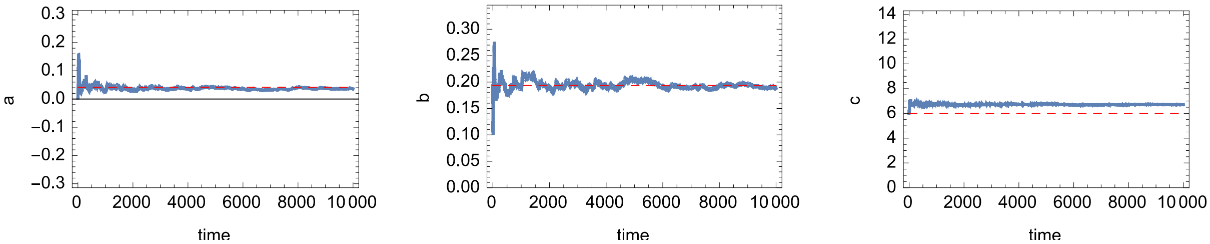

Figure 5. NRSE in a model with lags. PLM: $y_t = a + by_{t-1} + c\eta_t$. Real time adaptive learning of NRSE.

the sample T-map, and in the last panel, the red, dashed line corresponds to the initial condition for beliefs c. We note that convergence appears to obtain to the fixed point of the sample T-map identified in Figure 4, thus supporting our conjecture. Thus, while the presence of lagged endogenous variables seriously complicates the theoretical analysis, we conjecture that the MSV principle extends to this case.

5 Example: NRSE in a New Keynesian Model

We examine the existence and stability of near-rational sunspot equilibria in a standard model of monopolistic competition with Rotemberg-style price frictions. We assume a policy rule specifying that the nominal interest rate respond to expectations of inflation and output gap. The model's reduced-form equations may be written

$$y_t^{-\sigma} = \beta \cdot R_t E_t^* \pi_{t+1}^{-1} y_{t+1}^{-\sigma} \tag{20}$$

$$\gamma \cdot \pi_t(\pi_t - \pi^*) = \beta \cdot \gamma \cdot E_t^* \pi_{t+1}(\pi_{t+1} - \pi^*) + \left(\frac{\nu}{\alpha}\right) y_t^{\frac{1+\chi}{\alpha}} + (1-\nu) y_t^{1-\sigma} \tag{21}$$

$$R_t = R^* \left(E_t^* \left(\frac{\pi_{t+1}}{\pi^*}\right)\right)^{\alpha_\pi \cdot \pi^*} \left(E_t^* \left(\frac{y_{t+1}}{y^*}\right)\right)^{\alpha_y \cdot y^*} e^{v_t}, \tag{22}$$

$$v_t = \rho v_{t-1} + u_t, \tag{23}$$

where $0 < \rho < 1$ and u_t is white noise. Here equation (20) is the nonlinear IS relation, equation (21) is the nonlinear Phillips curve, and (22) is the Taylor rule, with policy shock v_t. Also, y^* is steady-state output, and R^* is the interest-rate target, chosen to satisfy $\beta R^* = \pi^*$. The details are provided in the On-line Appendix. The MSV solution posits a dependence of y_t, π_t and R_t only on v_t. If the steady state (y^*, π^*, R^*) is determinate then this MSV solution is locally the unique REE of the model; if the steady state is indeterminate there are also local stationary sunspot equilibria.

To assess existence and stability NRSE associated with this model, we simplify the reduced-form system. Letting $x = (y, \pi)'$, we can write (20)-(22) as

$$F(x_t, v_t) = G_1\left(E_t^* x_{t+1}\right) \cdot G_2\left(E_t^* x_{t+1}\right) \cdot E_t^* G_3(x_{t+1}), \tag{24}$$

for appropriate F and G_i. In doing so, however, we see that none of our results directly applies to (24); and further, there is no obvious way to manipulate the model so that our results would apply.

One way to proceed would be to work out the existence and stability results that do apply to systems of the form (24), and we anticipate that, using the techniques developed in this paper, establishing the desired results would be straightforward, if quite tedious. We suggest an alternative approach based on the MSV principle. Specifically, for a given calibration, we assess whether the linearized model is indeterminate and whether its minimal state variable solution is E-stable; if both of these conditions are met then the MSV principle suggests that then there are stable near-rational sunspot equilibria of the nonlinear model. We can then use simulations to examine the veracity of the principle for the case at hand.

We adopt a standard calibration, with $R^* = \beta^{-1}$ and $\pi^* = 1$, and with the policy parameters set aggressively to induce stable indeterminacy. In the On-line Appendix, we show that under this calibration the linearized model is indeterminate and the MSV solution is stable under adaptive learning; therefore, the MSV principle applies: stable near-rational sunspot equilibria should exist. To assess existence via simulation, we compute the model's resonance frequency λ, i.e. the serial correlation which, when perturbed, excites the existence of stable near-rational sunspots – see the On-line Appendix for this computation, as well as for the stochastic properties of the sunspot process used for our numerical work. We assume that the policy shock is observable and, when making forecasts, agents in the economy use a PLM of the form

$$x_t = a + b\eta_t + cv_t, \text{ where } \eta_t = (\lambda + \xi)\eta_{t-1} + \varepsilon_t, \tag{25}$$

where ε_t is mean-zero white noise. Here the 2×1 vectors $a = (a_y, a_\pi)'$, $b = (b_y, b_\pi)'$ and $c = (c_y, c_\pi)'$ capture agents beliefs, and they are assumed updated by recursive least-squares. A complete description of the model's dynamics is provided in the On-line Appendix.[18]

Figure 6 shows the results of a simulation in which agents do not believe in sunspots. As expected, the MSV solution is stable under learning: if agents do not condition on a sunspot then the estimated coefficients converge to their near-rational MSV values identified as dashed horizontal lines.

If, on the other hand, agents think that the sunspot may be relevant, and thus include it in their PLM, then the economy converges to an NRSE, as is evidenced in Figure 7.[19] Here we extend the simulation in order to illustrate convergence. Note that the sunspot equilibrium obtains even though agents are initially skeptical that sunspots have any forecasting value and thus place initially no weight on the sunspot, i.e. we set their initial estimates b equal to zero.

[18]For our calibration we set $\beta = .96, \nu = 1.5, \gamma = 5, \alpha_\pi = 5, \alpha_y = 5, \chi = .25, \alpha = .75, \rho = 0.5$, and $\sigma = 1$.For the learning algorithm we choose constant gain $\psi = 0.1$. The intrinsic innovation u_t is uniform on $[-0.1, 0.1]$. The serial correlation of the sunspot $\lambda(\xi) = -0.632$, and the mds ε_t is mean zero with asymmetric support $[-0.0000825, 0.0004125]$.

[19]To save space we only show the trajectory for the estimates of b; the trajectories for (a, c) are very similar to those of Figure 6.

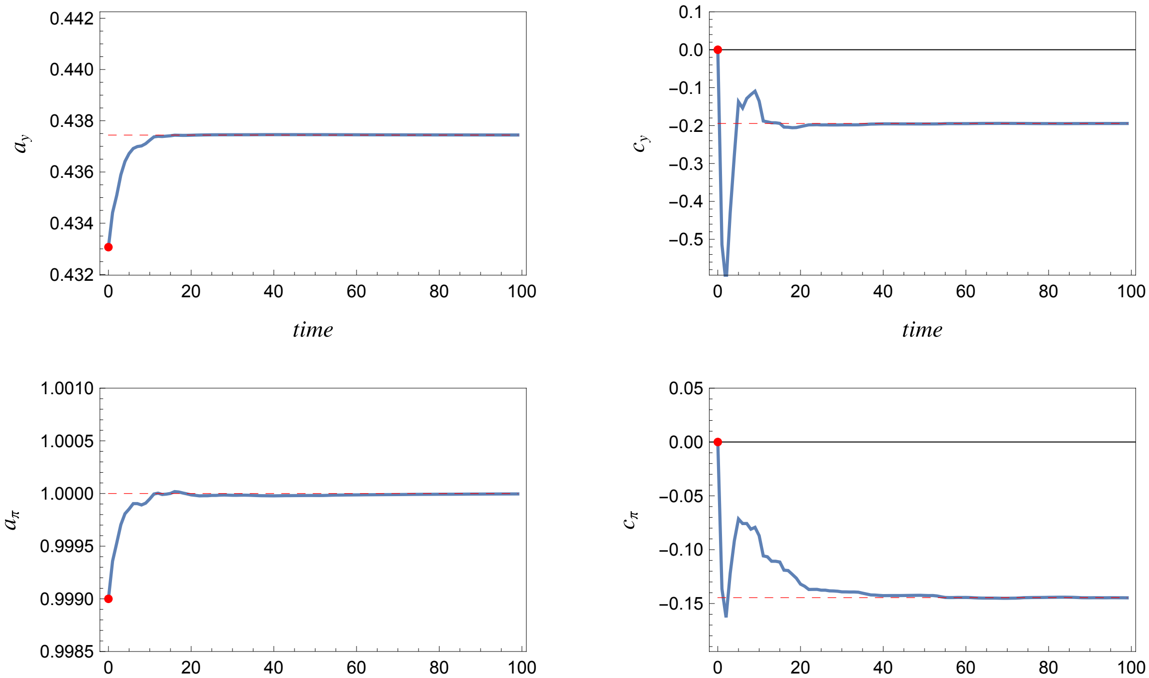

Figure 6. Stable near-rational MSV in New Keynesian model.
PLMs: $y_t = a_y + c_y v_t$ and $\pi_t = a_\pi + c_\pi v_t$.

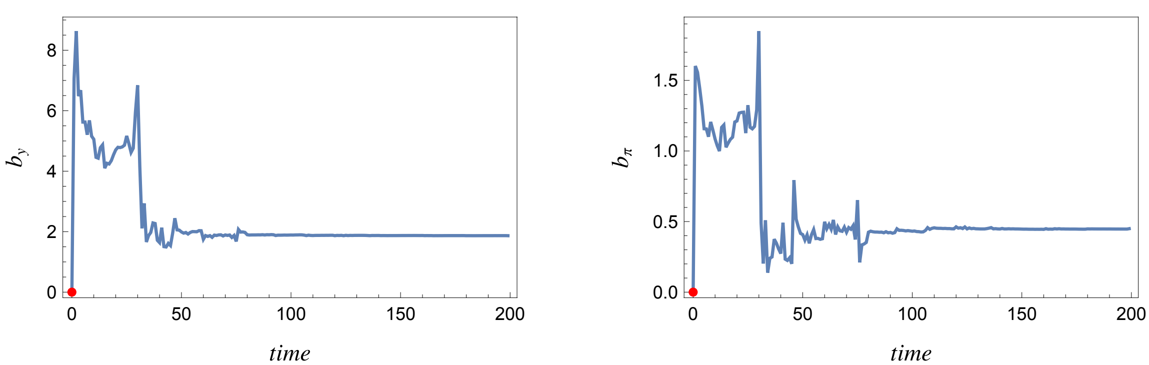

Figure 7. Stable NRSE in New Keynesian model.
PLMs: $y_t = a_y + b_y \eta_t + c_y v_t$ and $\pi_t = a_\pi + b_\pi \eta_t + c_\pi v_t$.

To illustrate the economic effects of coordination on an NRSE we compare the densities of output for the near-rational MSV and NRSE based on 20,000 observations drawn from the asymptotic distributions. See Figure 8, where we have normalized the densities to have maximum height

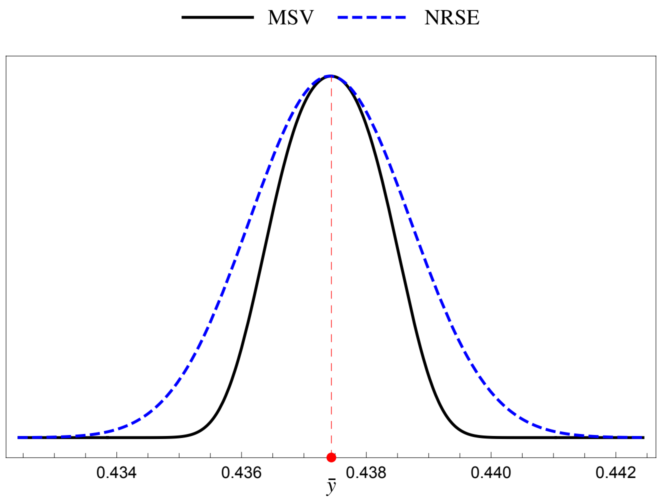

Figure 8. Comparison of output densities of near-rational MSV and NRSE in New Keynesian model.

equal to one. As might be anticipated, output in an NRSE exhibits greater volatility. Inflation and interest-rate volatility (not shown) also display greater volatility in the NRSE. Finally, it is worth observing that the standard advice for policymakers still applies: by guarding against indeterminacy they can eliminate the excess volatility associated with NRSE.

6 Conclusion

According to Blanchard, "... the world economy is pregnant with multiple equilibria – self-fulfilling outcomes of pessimism or optimism, with major macroeconomic implications."[20] This conclusion, and others like it, makes imperative understanding when and how sunspot equilibria, which represent and characterize the class of stationary multiple equilibria, are consistent with the dynamic stochastic general equilibrium modeling paradigm of the macroeconomic literature.

Investigations of sunspot equilibria in mainstream models have met with a variety of obstacles. Most notably, and as indicated in the Introduction, sunspot equilibria in nonlinear models have complicated stochastic structure, making them difficult for researchers and economic agents to model, and thus rendering stability analysis impossible.

Our embrace of a linear-forecasting framework allows us to circumvent this obstacle while preserving natural, agent-level behavior. We establish the existence of (near-rational) sunspot equilibria that have simple recursive stochastic structure. By providing agents an understanding of this

[20] IMF blog, http://blog-imfdirect.imf.org/2011/12/21/2011-in-review-four-hard-truths/

structure, we are then able to assess stability under adaptive learning, and indeed establish generic stability results.

Importantly, our results provide methods for assessing existence and stability of NRSE in nonlinear models using determinacy and E-stability tools for linear models. Furthermore, the MSV principle provides a convenient computational method for searching for stable NRSE in nonlinear models: simply look for stable MSV solutions to associated indeterminate linearized models.

Appendix

In this Appendix we discuss the generalization of Leibniz's rule needed for our analysis and the bifurcation argument underlying the proofs of Theorems 1 and 2 (as well as all other bifurcation-based results). All formal proofs are provided in the On-line Appendix.

A.1 Leibniz's Rule

We need to be able to differentiate a variety of functions defined in terms of Lesbesgue integrals. For this, we require the following simple generalization of Leibniz's rule.

Lemma 1 (Leibniz's Rule) *Let $k \geq 4$, $U \subset \mathbb{R}^n$ be open and $h : U \times \Omega \to \mathbb{R}$ have the following properties:*

1. *For all $x \in U$, $h(x, \cdot) \in L^\infty(\Omega)$*
2. *For almost all $\omega \in \Omega$, $h(\cdot, \omega) \in C^k(U)$*
3. *There exists $G \in L^1(\Omega)$ so that for all $x \in U$, $|D_{x_i} h(x, \omega)| \leq G(\omega)$ for almost all $\omega \in \Omega$.*

If $H : U \to \mathbb{R}$ is given by $H(x) = \int_\Omega h(x, \omega) d\mu(\omega)$ then $H \in C^k(U)$ and

$$D_{x_i} H(x) = \int_\Omega D_{x_i} h(x, \omega) d\mu(\omega).$$

While surely well known, for completeness, we present the proof of this Lemma in the On-line Appendix. Here, we outline the simple argument for the application of Lemma 1 in our analysis. Consistent with the notation from the main text, consider the following ALM, as given by $\hat{F}$:

$$y_t = \int_\Omega F(a + b\lambda(\xi)\eta_t^\xi + b\varepsilon_{t+1}(\omega)) d\mu(\omega) \equiv \hat{F}(a, b, \xi, \eta_t^\xi).$$

Since F is continuous, it follows that $\hat{F}(a, b, \xi, \eta_t^\xi(\cdot)) \in L^\infty(\Omega)$ for all t. Further, the analysis below will be local to the steady state $(0,0,0)'$, thus we may assume the existence of an open neighborhood $U \subset \mathbb{R}^3$ of the steady state, with compact closure, so that $\hat{F} : U \times \Omega \to \mathbb{R}$; and since $\hat{F}(\cdot, \eta_t^\xi(\omega))$ is $C^4(U)$, the compact closure of U provides the uniform bounds on the various partials needed to apply Lemma 1.

A.2 Outline of the bifurcation argument

The bifurcation argument used to establish many of the existence and stability results in this paper is standard, and outlined here. Recall that we let $\gamma = (a, b)'$. Write $H(\gamma, \xi) = T(\gamma, \xi) - \gamma$. The first step involves the decomposition of H into first and higher-order terms. We require the following derivatives, which are all evaluated at the origin:

$$
\begin{aligned}
\hat{F}_a &= \int_\Omega F' d\mu(\omega) = \beta \\
\hat{F}_b &= \int_\Omega F' \cdot (\lambda(\xi)\eta^\xi + \varepsilon(\omega)) d\mu(\omega) = \beta\lambda(\xi)\eta^\xi \\
T^a_a &= \int_\Omega \hat{F}_a d\mu(\omega) = \beta \\
T^a_b &= \int_\Omega \hat{F}_b d\mu(\omega) = \int_\Omega \eta^\xi(\omega) d\mu(\omega) = 0 \\
T^b_a &= \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega) = 0 \\
T^b_b &= \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega) = \tfrac{\beta\lambda(\xi)}{\sigma^2_{\eta^\xi}} \int_\Omega (\eta^\xi(\omega))^2 d\mu(\omega) = \beta\lambda(\xi).
\end{aligned}
$$

Noting that $DH = DT - I_2$ and that $\beta\lambda(\xi) - 1 = \beta\xi$ we have that

$$
H(\gamma,\xi) = \begin{pmatrix} \beta - 1 & 0 \\ 0 & \beta\xi \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} f(a,b,\xi) \\ g(a,b,\xi) \end{pmatrix}, \tag{26}
$$

where f and g are $\mathscr{O}\left(\|(a,b,\xi)\|^2\right)$.

As noted in Sections 3.3 and 3.4, the system $\dot{\gamma} = H(\gamma,\xi)$ is central to our analysis. It is evident that this system bifurcates at $\xi = 0$. To assess the nature of this bifurcation, we appeal to the center manifold theorem. This theorem guarantees the existence of a sufficiently smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ characterizing an invariant, parameter-dependent manifold, that is, a differentiable subset $W_c(\xi)$ of $\mathbb{R}^2$, tangent to the b-axis, so that

- For ξ and b near zero, $W_c(\xi)$ is the graph of $a = h(b,\xi)$.
- $W_c(\xi)$ is invariant under the action of H, that is, the trajectory of γ implied by $\dot{\gamma} = H(\gamma,\xi)$ remains in $W_c(\xi)$ if it is initialized in $W_c(\xi)$.

The invariance of the center manifold may be used to specify a functional equation characterizing h. Specifically, by definition, $\dot{a} = (\beta - 1)a + f(a,b,\xi)$; and, on $W_c(\xi)$, $a = h(b,\xi)$, so that

$$
\dot{a} = h_b(b,\xi) \cdot \dot{b} = h_b(b,\xi)\left(\beta\xi b + g(a,b,\xi)\right).
$$

We conclude that h must satisfy the functional equation

$$
(\beta - 1)h(b,\xi) + f\left(h(b,\xi),b,\xi\right) = h_b(b,\xi)\left(\beta\xi b + g\left(h(b,\xi),b,\xi\right)\right).
$$

Using this equation together with the implicit function theorem allows for the computation of the Taylor expansion of h to arbitrary order.

The importance of the manifold $W_c(\xi)$ follows from a corollary to the center manifold theorem which states that the dynamic behavior of the two-dimensional system $\dot{\gamma} = H(\gamma)$ is locally equivalent in a natural sense to its behavior on $W_c(\xi)$; and, using h, this behavior is captured by the univariate system

$$\dot{b} = \beta\xi b + g(h(b,\xi), b, \xi).$$

The proofs in the On-line Appendix involve fleshing out the details of this analysis.

References

ADAM, K., A. MARCET, AND J. P. NICOLINI (2016): "Stock Market Volatility and Learning," *Journal of Finance*, 71, 33–82.

AZARIADIS, C. (1981): "Self-Fulfilling Prophecies," *Journal of Economic Theory*, 25, 380–396.

AZARIADIS, C., AND R. GUESNERIE (1986): "Sunspots and Cycles," *Review of Economic Studies*, 53, 725–737.

BENHABIB, J., G. W. EVANS, AND S. HONKAPOHJA (2014): "Liquidity Traps and Expectation Dynamics: Fiscal Stimulus or Fiscal Austerity?," *Journal of Economic Dynamics and Control*, 45, 220–238.

BENHABIB, J., AND R. A. FARMER (1994): "Indeterminacy and Increasing Returns," *Journal of Economic Theory*, 63, 19–41.

BENHABIB, J., S. SCHMITT-GROHE, AND M. URIBE (2001): "The Perils of Taylor Rules," *Journal of Economic Theory*, 96, 40–69.

BENHABIB, J., P. WANG, AND Y. WEN (2015): "Sentiments and Aggregate Demand Fluctuations," *Econometrica*, 83(2), 549–585.

BISCHI, G. I., C. CHIARELLA, AND I. SUSHKO (eds.) (2013): *Global analysis of dynamic models in economic and finance: essays in honour of Laura Gardini*. Springer, Berlin.

BLANCHARD, O., AND C. KAHN (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48, 1305–1311.

BRANCH, W. (2006): "Restricted Perceptions Equilibria and Learning in Macroeconomics," in Colander (2006), pp. 135–160.

BRANCH, W. A., AND B. MCGOUGH (2005): "Consistent expectations and misspecification in stochastic non-linear economies," *Journal of Economic Dynamics and Control*, 29(4), 659 – 676.

BRANCH, W. A., B. MCGOUGH, AND M. ZHU (2017): "Statistical Sunspots," Mimeo.

Bray, M., and N. Savin (1986): "Rational Expectations Equilibria, Learning, and Model Specification," *Econometrica*, 54, 1129–1160.

Bullard, J., and S. Eusepi (2014): "When Does Determinacy Imply Expectational Stability?," *International Economic Review*, 55, 1–22.

Bullard, J., and K. Mitra (2002): "Learning About Monetary Policy Rules," *Journal of Monetary Economics*, 49, 1105–1129.

Cass, D., and K. Shell (1983): "Do Sunspots Matter?," *Journal of Political Economy*, 91, 193–227.

Clarida, R., J. Gali, and M. Gertler (2000): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115, 147–180.

Cochrane, J. H. (2017): "The New-Keynesian Liquidity Trap," *Journal of Monetary Economics*, 92, 47–63.

Colander, D. (2006): *Post Walrasian Macroeconomics*. Cambridge, Cambridge, U.K.

Duffy, J., and W. Xiao (2007): "Instability of Sunspot Equilibria in Real Business Cycle Models under Adaptive Learning," *Journal of Monetary Economics*, 54, 879–903.

Eusepi, S., and B. Preston (2018): "The Science of Monetary Policy: An Imperfect Knowledge Perspective," *Journal of Economic Literature*, 56, 3–59.

Evans, G. W. (1985): "Expectational Stability and the Multiple Equilibria Problem in Linear Rational Expectations Models," *The Quarterly Journal of Economics*, 100, 1217–1233.

——— (1989): "The Fragility of Sunspots and Bubbles," *Journal of Monetary Economics*, 23, 297–317.

Evans, G. W., and S. Honkapohja (1994): "On the Local Stability of Sunspot Equilibria under Adaptive Learning Rules," *Journal of Economic Theory*, 64, 142–161.

——— (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.

——— (2003): "Existence of Adaptively Stable Sunspot Equilibria near an Indeterminate Steady State," *Journal of Economic Theory*, 111, 125–134.

Evans, G. W., and B. McGough (2005a): "Indeterminacy and the Stability Puzzle in Non-Convex Economies," *The B.E. Journal of Macroeconomics* (Contributions), 5, Iss. 1, Article 8.

——— (2005b): "Monetary Policy, Indeterminacy and Learning," *Journal of Economic Dynamics and Control*, 29, 1809–1840.

——— (2005c): "Stable Sunspot Solutions in Models with Predetermined Variables," *Journal of Economic Dynamics and Control*, 29, 601–625.

——— (2011): "Representations and Sunspot Stability," *Macroeconomic Dynamics*, 15, 80–92.

——— (2018a): "Equilibrium Selection, Observability and Backward-Stable Solutions," *Journal of Monetary Economics*, 98, 1–10.

——— (2018b): "Interest-Rate Pegs in New Keynesian Models," *Journal of Money, Credit and Banking*, 50, 939–965.

FARMER, R. E., AND J.-T. GUO (1994): "Real Business Cycles and the Animal Spirits Hypothesis," *The Journal of Economic Theory*, 63, 42–72.

GALI, J. (2014): "Monetary Policy and Rational Asset Price Bubbles," *American Economic Review*, 104, 721–752.

GARCIA-SCHMIDT, M., AND M. WOODFORD (2019): "Are Low Interest Rates Deflationary? A Paradox of Perfect-Foresight Analysis," *American Economic Review*, 109, 86–120.

GUESNERIE, R. (1986): "Stationary Sunspot Equilibria in an N-commodity World," *Journal of Economic Theory*, 40, 103–128.

GUESNERIE, R., AND M. WOODFORD (1992): "Endogenous Fluctuations," in Laffont (1992), chap. 6, pp. 289–412.

HOMMES, C., G. SORGER, AND F. WAGENER (2013): "Consistency of linear forecasts in a nonlinear stochastic economy," in Bischi, Chiarella, and Sushko (2013), pp. 229–287.

HOMMES, C. H., AND G. SORGER (1998): "Consistent Expectations Equilibria," *Macroeconomic Dynamics*, 2, 287–321.

HOMMES, C. H., AND M. ZHU (2014): "Behavioral Learning equilibria," *Journal of Economic Theory*, 150, 778–814.

LAFFONT, J.-J. (ed.) (1992): *Advances in Economic Theory: Sixth World Congress. Volume 2*. Cambridge University Press, Cambridge, UK.

LUBIK, T. A., AND F. SCHORFHEIDE (2004): "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *American Economic Review*, 94, 190–217.

MARCET, A., AND T. J. SARGENT (1989): "Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

MARTIN, A., AND J. VENTURA (2012): "Economic Growth with Bubbles," *American Economic Review*, 102, 3033–3058.

McCallum, B. T. (1983): “On Nonuniqueness in Linear Rational Expectations Models: An Attempt at Perspective,” *Journal of Monetary Economics*, 11, 139–168.

Mertens, K., and M. O. Ravn (2014): “Fiscal Policy in an Expectation Driven Liquidity Trap,” *Review of Economic Studies*, 81, 1637–1667.

Miao, J., Z. Shen, and P. Wang (2019): “Monetary Policy and Rational Asset Price Bubbles: Comment,” *American Economic Review*, 109, 1969–1990.

Milani, F. (2007): “Expectations, Learning and Macroeconomic Persistence,” *Journal of Monetary Economics*, 54, 2065–2082.

Shell, K. (1977): “Monnaie et Allocation Intertemporelle,” Working paper, CNRS Seminaire de E.Malinvaud, Paris.

Sims, C. A. (2001): “Solving Linear Rational Expectations Models,” *Computational Economics*, 20, 1–20.

Slobodyan, S., and R. Wouters (2012): “Estimating a Medium-Scale DSGE Model with Expectations Based on Small Forecasting Models,” *American Economic Journal: Macroeconomics*, 4, 65–101.

Woodford, M. (1986): “Stationary Sunspot Equilibria: The Case of Small Fluctuations Around a Deterministic Steady State,” manuscript, University of Chicago and New York University.

——— (1990): “Learning to Believe in Sunspots,” *Econometrica*, 58, 277–307.

Xiong, W., and H. Yan (2010): “Heterogeneous Expectations and Bond Markets,” *Review of Financial Studies*, 23(4), 1433–1466.

On-line Appendix

Proof of Lemma 1. Fix $x \in U$, $i \in \{1, \ldots, n\}$, and let Δ_m be a real sequence converging to zero such that

$$x(\Delta_m) = (x_1, \ldots, x_{i-1}, x_i + \Delta_m, x_{i+1}, \ldots, x_n) \in U.$$

Define $h_n(x, \omega) = \Delta_m^{-1}(h(x(\Delta_m) - h(x))$. Since $L^\infty(\Omega) \subset L^1(\Omega)$ it follows that $h_n(x, \cdot) \in L^1(\Omega)$ and $h_n(x, \cdot) \to D_{x_i}h(x, \cdot)$ almost everywhere. By the mean-value theorem, for almost all $\omega \in \Omega$, there is a δ_m with $|\delta_m| < |\Delta_m|$ such that

$$|h_n(x, \omega)| = |D_{x_i}h(x(\delta_m), \omega)| \leq G(\omega).$$

We may compute

$$\begin{aligned} D_{x_i}H(x) &= \lim_{m\to\infty} \Delta_m^{-1}(H(x(\Delta_m)) - H(x)) = \lim_{m\to\infty} \int_\Omega h_m(x, \omega) d\mu(\omega) \\ &= \int_\Omega \lim_{m\to\infty} h_m(x, \omega) d\mu(\omega) = \int_\Omega D_{x_i}h(x, \omega) d\mu(\omega), \end{aligned}$$

where the third equality follows from the dominated convergence theorem. The proof is completed by induction, recognizing that $D_{x_i}h(\cdot, \omega) \in C^{k-1}(U)$. ■

In the work below we will repeatedly be required to differentiate functions of the form H, constructed from functions of the form h, as defined in the lemma above. Our analysis will be local to a steady state, so that our sets U will have compact closure, thus giving the needed uniform bounds on $D_x h$, which themselves are assumed continuous.

Proof of Theorems 1 and 2. This analysis requires the computation of a host of derivatives, and we proceed with these computations now. Importantly, all derivatives of F are evaluated at zero and all partials (first and higher orders) of $\hat{F}$ and T are evaluated at $a = b = \xi = 0$. For notational ease, we will often omit the arguments. Note that when computing derivatives of $\hat{F}$, the variable η^ξ is taken as fixed.

Derivatives of $\hat{F}(a,b,\xi,\eta^{\xi}) = \int_{\Omega} F(a + b\lambda(\xi)\eta^{\xi} + b\varepsilon(\omega))d\mu(\omega)$

$$\hat{F}_a = \int_{\Omega} F' d\mu(\omega) = \beta \tag{27a}$$

$$\hat{F}_b = \int_{\Omega} F' \cdot (\beta^{-1}\eta^{\xi} + \varepsilon(\omega))d\mu(\omega) = \eta^{\xi} \tag{27b}$$

$$\hat{F}_{\xi} = \int_{\Omega} F' b\eta^{\xi} d\mu(\omega) = 0, \text{ since } b = 0. \tag{27c}$$

$$\hat{F}_{aa} = \int_{\Omega} F'' d\mu(\omega) = F''(0) \tag{27d}$$

$$\hat{F}_{ab} = \int_{\Omega} \lambda(\xi)\eta^{\xi} F'' d\mu(\omega) = \beta^{-1}\eta^{\xi} F''(0) \tag{27e}$$

$$\hat{F}_{bb} = \int_{\Omega} F''(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega))^2 d\mu(\omega) = (\beta^{-2}\left(\eta^{\xi}\right)^2 + \sigma_{\varepsilon}^2)F''(0) \tag{27f}$$

$$\hat{F}_{\xi\xi} = \int_{\Omega} F''(b\eta^{\xi})^2 d\mu(\omega) = 0 \tag{27g}$$

$$\hat{F}_{a\xi} = \int_{\Omega} F'' b\eta^{\xi} d\mu(\omega) = 0 \tag{27h}$$

$$\hat{F}_{b\xi} = \int_{\Omega} (\eta^{\xi} F' + b\eta^{\xi} F'')d\mu(\omega) = \beta\eta^{\xi} \tag{27i}$$

$$\hat{F}_{bbb} = \int_{\Omega} F'''(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega))^3 d\mu(\omega) = F'''(0)((\beta^{-1}\eta^{\xi})^3 + 3\beta^{-1}\eta^{\xi}\sigma_{\varepsilon}^2) \tag{27j}$$

Derivatives of $T^a(a,b,\xi) = \int_{\Omega} \hat{F}(a,b,\xi,\eta^{\xi}(\omega))d\mu(\omega)$

$$T_a^a = \int_{\Omega} \hat{F}_a d\mu(\omega) = \beta \tag{28a}$$

$$T_b^a = \int_{\Omega} \hat{F}_b d\mu(\omega) = \int_{\Omega} \eta^{\xi}(\omega)d\mu(\omega) = 0 \tag{28b}$$

$$T_{\xi}^a = \int_{\Omega} \hat{F}_{\xi} d\mu(\omega) = 0 \tag{28c}$$

$$T_{aa}^a = \int_{\Omega} \hat{F}_{aa} d\mu(\omega) = F''(0) \tag{28d}$$

$$T_{ab}^a = \int_{\Omega} \hat{F}_{ab} d\mu(\omega) = \int_{\Omega} \lambda(\xi)F''\eta^{\xi}(\omega)d\mu(\omega) = 0 \tag{28e}$$

$$T_{bb}^a = \int_{\Omega} \hat{F}_{bb} d\mu(\omega) = \int_{\Omega} (\beta^{-2}(\eta^{\xi}(\omega))^2 + \sigma_{\varepsilon}^2)F''(0)d\mu(\omega) = \sigma_{\eta}^2 F''(0) \tag{28f}$$

$$T_{\xi\xi}^a = \int_{\Omega} \hat{F}_{\xi\xi} d\mu(\omega) = 0 \tag{28g}$$

$$T_{\xi a}^a = \int_{\Omega} \hat{F}_{\xi a} d\mu(\omega) = 0 \tag{28h}$$

$$T_{\xi b}^a = \int_{\Omega} \hat{F}_{\xi b} d\mu(\omega) = \int_{\Omega} \beta\eta^{\xi}(\omega)d\mu(\omega) = 0 \tag{28i}$$

Derivatives of $T^b(a,b,\xi) = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}(a,b,\xi,\eta^\xi(\omega))d\mu(\omega)$

$$T^b_a = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega) = 0 \tag{29a}$$

$$T^b_b = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega (\eta^\xi(\omega))^2 d\mu(\omega) = 1 \tag{29b}$$

$$T^b_\xi = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left[\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu(\omega) - \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F} d\mu(\omega)\right] = 0 \tag{29c}$$

$$T^b_{bb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{bb} d\mu(\omega) = \int_\Omega \eta^\xi(\omega)(\beta^{-2}(\eta^\xi(\omega))^2 + \sigma^2_\varepsilon)F''(0)d\mu(\omega) = \frac{F''(0)\mu_3^{\eta^\xi}}{\beta^2\sigma^2_\eta} \tag{29d}$$

$$T^b_{ab} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \beta^{-1}F''(\eta^\xi(\omega))^2 = \beta^{-1}F''(0) \tag{29e}$$

$$T^b_{aa} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{aa} d\mu(\omega) = 0 \tag{29f}$$

$$\begin{aligned}
T^b_{\xi\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-4} \Bigg\{ (\sigma^2_\eta)^2 \Bigg[& \sigma^2_{\eta^\xi} \int_\Omega \left(2\hat{F}_\xi \frac{\partial}{\partial\xi}\eta^\xi(\omega) + \hat{F}\frac{\partial^2}{(\partial\xi)^2}\eta^\xi(\omega) + \hat{F}_{\xi\xi}\eta^\xi(\omega)\right) d\mu(\omega) \\
& + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu - \int_\Omega \frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} d\mu(\omega) \\
& - \int_\Omega \eta^\xi(\omega)\hat{F}_\xi \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} d\mu(\omega) - \int_\Omega \eta^\xi(\omega)\hat{F}\frac{\partial^2}{(\partial\xi)^2}\sigma^2_{\eta^\xi} d\mu(\omega) \Bigg] \\
& - \sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) \frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega) + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega) \Bigg\} = 0
\end{aligned} \tag{29g}$$

$$T^b_{b\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left[\sigma^2_{\eta^\xi} \int_\Omega \left(\frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F}_b + \eta^\xi(\omega)\hat{F}_{b\xi}\right) d\mu(\omega) - \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega)\right] = \beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{29h}$$

$$T^b_{bbb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \hat{F}_{bbb}\eta^\xi(\omega)d\mu(\omega) = \frac{F'''(0)}{\sigma^2_\eta}\left(\beta^{-3}\sigma^4_\eta + 3\beta^{-1}\sigma^2_\eta\sigma^2_\varepsilon\right). \tag{29i}$$

Equation (29h) requires elaboration. Since

$$\begin{aligned}
\frac{\partial}{\partial\xi}\eta^\xi(\omega) &= \lambda(\xi)^{-1}\sum_{m\geq 0} m\lambda(\xi)^m\varepsilon_m(\omega), \text{ and} \\
\hat{F}_b(\omega) &\equiv \hat{F}_b(a,b,\xi,\eta^\xi(\omega)) = \beta\lambda(\xi)\eta^\xi(\omega) = \beta\lambda(\xi)\sum_{k\geq 0}\lambda(\xi)^k\varepsilon_k(\omega),
\end{aligned} \tag{30}$$

it follows that

$$\begin{aligned}
\int_\Omega \hat{F}_b(\omega)\frac{\partial}{\partial\xi}\eta^\xi(\omega)d\mu(\omega) &= \beta\int_\Omega\left(\sum_{k\geq 0}\lambda(\xi)^k\varepsilon_k(\omega)\right)\left(\sum_{m\geq 0}m\lambda(\xi)^m\varepsilon_m(\omega)\right)d\mu(\omega)\\
&= \beta\int_\Omega\sum_{k\geq 0}k\left(\lambda(\xi)^2\right)^k\varepsilon_k(\omega)^2d\mu(\omega)=\beta\lambda(\xi)^2\sum_{k\geq 0}k\left(\lambda(\xi)^2\right)^{k-1}\sigma_\varepsilon^2\\
&= \beta\lambda(\xi)^2\sigma_\varepsilon^2\frac{\partial}{\partial\lambda(\xi)^2}\sum_{k\geq 0}\left(\lambda(\xi)^2\right)^k=\\
&= \beta\lambda(\xi)^2\sigma_\varepsilon^2\frac{\partial}{\partial\lambda(\xi)^2}\left(1-\lambda(\xi)^2\right)^{-1}=\beta\left(\frac{\lambda(\xi)^2}{1-\lambda(\xi)^2}\right)\sigma^2_{\eta^\xi}.
\end{aligned}$$

Next,

$$\int_\Omega \eta^\xi(\omega)\hat{F}_{b\xi}d\mu(\omega) = \beta\int_\Omega\left(\eta^\xi(\omega)\right)^2d\mu(\omega)=\beta\sigma^2_{\eta^\xi}.$$

Finally,

$$\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} = \frac{\partial}{\partial\xi}\left(\frac{\sigma_\varepsilon^2}{1-\lambda(\xi)^2}\right)=\frac{2\lambda(\xi)\sigma_\varepsilon^2}{\left(1-\lambda(\xi)^2\right)^2}=2\left(\frac{\lambda(\xi)}{1-\lambda(\xi)^2}\right)\sigma^2_{\eta^\xi},$$

so that

$$\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega\eta^\xi(\omega)\hat{F}_bd\mu(\omega) = 2\left(\frac{\lambda(\xi)}{1-\lambda(\xi)^2}\right)\left(\sigma^2_{\eta^\xi}\right)^2.$$

Thus

$$\begin{aligned}
T^b_{b\xi} &= \left(\sigma^2_{\eta^\xi}\right)^{-2}\left[\sigma^2_{\eta^\xi}\left(\beta\left(\frac{\lambda(\xi)^2}{1-\lambda(\xi)^2}\right)\sigma^2_{\eta^\xi}+\beta\sigma^2_{\eta^\xi}\right)-2\left(\frac{\lambda(\xi)}{1-\lambda(\xi)^2}\right)\left(\sigma^2_{\eta^\xi}\right)^2\right]\\
&= \frac{\beta-2\lambda(\xi)}{1-\lambda(\xi)^2}=\beta\left(\frac{\beta^2-2}{\beta^2-1}\right).
\end{aligned}$$

This completes our computation of the needed derivatives.

We now turn to the body of the argument, which requires bifurcation analysis of the following dynamic system:

$$\begin{pmatrix}\dot{a}\\ \dot{b}\\ \dot{\xi}\end{pmatrix}=\begin{pmatrix}T^a(a,b,\xi)\\ T^b(a,b,\xi)\\ 0\end{pmatrix}-\begin{pmatrix}a\\ b\\ 0\end{pmatrix}\equiv H(a,b,\xi). \tag{31}$$

We may write decompose this system in to first, and higher-order terms:

$$H(a,b,\xi)=\begin{pmatrix}\beta-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ \xi\end{pmatrix}+\begin{pmatrix}f(a,b,\xi)\\ g(a,b,\xi)\\ 0\end{pmatrix},$$

where f and g are $\mathscr{O}\left(\|(a,b,\xi)\|^2\right)$, and given by $f = T^a - \beta a$ and $g = T^b - b$. By the center manifold theorem, the orbit structure of the dynamic system determined by (31) is topologically equivalent to the projection of the system on to the parameter-dependent center manifold, which may be expressed by a $C^4(V)$ function: $a = h(b,\xi)$, where $V \subset \mathbb{R}^2$ is an open region containing the rest point. The remainder of the proof involves two steps: computing the center manifold; and conducting bifurcation analysis of the projected system.

Computing the center

A closed form representation of h is not available, but we may use the invariance of the center manifold together with a Taylor expansion of h to establish a sufficient approximation. By (31), we have that

$$\dot{a} = (\beta - 1)h(b,\xi) + f(h(b,\xi),b,\xi).$$

Differentiating $a = h(b,\xi)$ with respect to time, we get $\dot{a} = h_b\dot{b} + h_\xi\dot{\xi}$. Using (31) and that $\dot{\xi} = 0$, we also have

$$\dot{a} = h_b(b,\xi)g(h(b,\xi),b,\xi).$$

Thus h is characterized by the functional equation

$$\mathrm{L}(b,\xi) \equiv (\beta - 1)h(b,\xi) + f(h(b,\xi),b,\xi) = h_b(b,\xi)g(h(b,\xi),b,\xi) \equiv R(b,\xi)$$

This functional equation, together with the implicit function theorem, may be used to approximate h: simply compute the Taylor expansions of L and R, equate like terms, and solve the coefficients in the Taylor expansion of h.

Since the center manifold is tangent to the eigenspaces of the linear component of H, it follows that $h_b(0,0) = h_\xi(0,0) = 0$. Also, the origin is a steady state: $h(0,0) = 0$. Thus, we may write

$$h(b,\xi) = \tfrac{1}{2}\cdot(h_{bb}\cdot b^2 + h_{\xi\xi}\cdot\xi^2) + h_{b\xi}\cdot\xi\cdot b + \mathscr{O}\left(\|(b,\xi)\|^3\right).$$

Here, all derivatives are evaluated at $(0,0)$. As notation, we also write

$$\begin{aligned} L(b,\xi) &= L_b\cdot b + L_\xi\cdot\xi + \tfrac{1}{2}\cdot(L_{bb}\cdot b^2 + L_{\xi\xi}\cdot\xi^2) + L_{b\xi}\cdot b\cdot\xi + \mathscr{O}\left(\|(b,\xi)\|^3\right), \\ R(b,\xi) &= R_b\cdot b + R_\xi\cdot\xi + \tfrac{1}{2}\cdot(R_{bb}\cdot b^2 + R_{\xi\xi}\cdot\xi^2) + R_{b\xi}\cdot b\cdot\xi + \mathscr{O}\left(\|(b,\xi)\|^3\right). \end{aligned}$$

Noting that, for example, $\frac{\partial}{\partial b}f = f_a\cdot h_b + f_b$, we compute

$$L_b = (\beta - 1)h_b + f_a\cdot h_b + f_b \tag{32a}$$

$$L_{bb} = (\beta - 1)h_{bb} + h_{bb}\cdot f_a + h_b\cdot f_{ab} + h_b\cdot f_{ab} + f_{bb} \tag{32b}$$

$$R_b = h_{bb}\cdot g + h_b\cdot(g_a\cdot h_b + g_b) \tag{32c}$$

$$R_{bb} = h_{bbb}\cdot g + 2h_{bb}\cdot(g_a\cdot h_b + g_b) + h_b\cdot\tfrac{\partial}{\partial b}(g_a\cdot h_b + g_b). \tag{32d}$$

Since f, g, and h are zero at the origin and have no first order terms, we see $h_{bb} = \frac{f_{bb}}{1-\beta}$. Further, since $f_{bb} = T^a_{bb}$, it follows from (28f) that

$$h_{bb} = \left(\frac{F''(0)}{1-\beta}\right)\sigma^2_{\eta^\xi}.$$

As we will determine below, other second-order terms of h are not needed for the bifurcation analysis, and so our computation of the center manifold approximation is complete.

Bifurcation analysis

The local dynamics of (31) are topologically equivalent to the suspension of the projected system by the associated saddle. Intuitively this means that the dynamic system (31) may be decomposed into hyperbolic and center components; and, locally, the orbits of the decomposed systems, appropriately joined, are appropriately isomorphic to the orbits of the original system. In particular, if the projected system undergoes a particular bifurcation then so too does the system (31). The projected system is given by

$$\dot{b} = g(h(b,\xi),b,\xi) \equiv G(b,\xi). \tag{33}$$

To conduct bifurcation analysis, the higher-order derivatives of G are needed. That $G(0,0) = 0$ is immediate. Since $g = T^b - b$ we have that

$$g_{aa} = T^b_{aa} = 0 \tag{34a}$$

$$g_{ab} = T^b_{ab} = \beta^{-1}F''(0) \tag{34b}$$

$$g_{bb} = T^b_{bb} = \frac{F''(0)\mu_3^{\eta^\xi}}{\beta^2\sigma^2_{\eta^\xi}} \tag{34c}$$

$$g_{b\xi} = T^b_{b\xi} = \beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{34d}$$

$$g_{bbb} = T^b_{bbb} = \frac{F'''(0)}{\sigma^2_{\eta^\xi}}\left(\beta^{-3}\mu_4^\xi + 3\beta^{-1}\sigma^2_{\eta^\xi}\sigma^2_\varepsilon\right). \tag{34e}$$

Using our information about h, we compute

$$G_b = g_a \cdot h_b + g_b = 0 \tag{35a}$$

$$G_\xi = g_a \cdot h_\xi + g_\xi = 0 \tag{35b}$$

$$G_{bb} = g_a \cdot h_{bb} + h_b \cdot (g_{aa} \cdot h_b + g_{ab}) + g_{ab} \cdot h_b + g_{bb} = g_{bb} \tag{35c}$$

$$G_{b\xi} = h_b \cdot \tfrac{\partial}{\partial\xi} g_a + g_a \cdot h_{b\xi} + g_{ba} \cdot h_\xi + g_{b\xi} = g_{b\xi} \tag{35d}$$

$$\begin{aligned} G_{bbb} = {} & g_a h_{bbb} + 2h_{bb} \cdot (g_{aa} \cdot h_b + g_{ab}) + h_b \cdot \tfrac{\partial}{\partial b}(g_{aa} \cdot h_b + g_{ab}) \\ & + \ g_{ab} \cdot h_{bb} + h_b \cdot \tfrac{\partial}{\partial b}\, g_{ab} + h_b \cdot g_{bba} + g_{bbb} = 3h_{bb} \cdot g_{ab} + g_{bbb}, \end{aligned} \tag{35e}$$

where, in each computation, the second equality follows from the work just above and that h and g have no first order terms.

Since $G = G_b = G_\xi = 0$, and $G_{b\xi}$ is generically non-zero, we can assess the type of bifurcation by looking at the higher order terms in b. In particular, the type of bifurcation experienced by the system (33) depends on whether $G_{bb} = 0$. Noting g_{bb} is proportional to $\mu_3^{\eta^\xi} F''(0)$, assuming non-trivial second-order curvature in F, we see whether $g_{bb} = 0$ depends, generically, on whether $E(\varepsilon_t^3) = 0$.

Case 1: $E(\varepsilon_t^3) = 0.$

Since G_b, G_ξ and $G_{bb} = 0$, and $G_{b\xi} \neq 0$ the system undergoes a pitchfork bifurcation as ξ crosses zero provided that $G_{bbb} \neq 0$. Simplifying G_{bbb}, we get the following regularity condition:

$$G_{bbb} = F^{'''}(0) \left(\frac{3\sigma_\varepsilon^2}{\beta} + \frac{\mu_4^{\eta^\xi}}{\beta^3 \sigma_{\eta^\xi}^2} \right) + \left(\frac{3\,(F''(0))^2}{(1-\beta)\beta} \right) \sigma_{\eta^\xi}^2, \tag{36}$$

where we note that under the assumptions of the proposition, G_{bbb} is generically non-zero in that the set of all such parameters for which the condition (36) is not satisfied has Lebesgue measure zero in parameter space. We conclude that if $E(\varepsilon_t^3) = 0$ then the projected system undergoes a pitchfork bifurcation as ξ crosses zero, indicating the emergence of two additional fixed points: see chapter 3 of Wiggins (1990) for the relevant results in bifurcation theory used here and below.

Case 2: $E(\varepsilon_t^3) \neq 0.$

In this case we have $G_b = 0$, $G_\xi = 0$, and $G_{b\xi} \neq 0$. Since

$$G_{bb} = \frac{F''(0)\mu_3^{\eta^\xi}}{\beta^2 \sigma_{\eta^\xi}^2}$$

is generically non-zero, we conclude that if $E(\varepsilon_t^3) \neq 0$ then the projected system undergoes a transcritical bifurcation as ξ crosses zero, indicating the emergence of two additional fixed points.

The proof of existence is completed by noting that in both cases, non-trivial fixed points of the projected system emerge as a result of a bifurcation, and further that the local dynamics of the projected system are topologically equivalent to the dynamics of the original system.

Turning now to stability, we recall from the body that stability under adaptive learning is governed by the E-stability ode (31); thus we are interesting in knowing when the bifurcation results in two new fixed points of (31), at least one of which is Lyapunov stable. Again, because, locally, the dynamics of (31) are topologically equivalent to suspension of the projected system by the associated saddle, stability of the post-bifurcation fixed points entails two requirements: first, the associated saddle must be stable, that is, $\beta - 1 < 0$; and second, the emergent fixed points of the projected system (33) must be Lyapunov stable. In case $E(\xi_t^3) \neq 0$, the bifurcation is transcritical in nature, so that we may simply choose an appropriate perturbation μ to obtain a stable fixed point. In case $E(\xi_t^3) = 0$, additional restrictions are required: the new fixed points inherit the stability of the origin. Thus stability of the new fixed points – the NRSE – requires in this case that $G_{bbb} < 0$, which yields the additional non-generic condition identified in the theorem. Note that we may still conclude that if $\beta < -1$ and $F''(0) \neq 0$ then stable NSRE exist. ■

Proof Theorem 3. The natural approach is to consider a perturbation of σ near zero; the technical challenge is that the T-map is not defined for $\sigma = 0$. To side-step the complication that $\sigma_v^2 \to 0$ as $\sigma \to 0$, define $\hat{v}_t = \sigma^{-1} v_t$, and notice that $\hat{v}_t = \rho \hat{v}_{t-1} + \zeta_t$. Now consider the new function $\hat{F}$, defined as

$$y_t = \int_\Omega F(a + b\rho\hat{v}_t + b\zeta_{t+1}(\omega), \rho\sigma\hat{v}_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \equiv \hat{F}(a,b,\sigma,\hat{v}_t).$$

Projecting this process onto the span of $(1, \hat{v}_t)$ yields the following map, which we label $\hat{T}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{\hat{T}=\hat{T}(a,b,\sigma)} \begin{pmatrix} \int_\Omega \hat{F}(a,b,\sigma,\hat{v}(\omega))d\mu(\omega) \\ (\sigma_{\hat{v}}^2)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}(a,b,\sigma,\hat{v}(\omega))d\mu(\omega) \end{pmatrix} \equiv \begin{pmatrix} \hat{T}^a(a,b,\sigma) \\ \hat{T}^b(a,b,\sigma) \end{pmatrix}.$$

By construction, $\hat{T}$ is defined, k-times differentiable, and has a fixed point at $(0,0)'$ when $\sigma = 0$.

Let $H = \hat{T} - (a,b)'$. We need some more derivatives.

Derivatives of $\hat{F}$ and H

$$\hat{F}_a = DF_y \equiv \beta \tag{37a}$$

$$\hat{F}_b = \rho\beta\hat{v} \tag{37b}$$

$$\hat{F}_\sigma = DF_v \cdot \rho\hat{v} \tag{37c}$$

$$H_a^a = \int_\Omega \hat{F}_a d\mu(\omega) - 1 = \beta - 1 \tag{37d}$$

$$H_b^a = \int_\Omega \hat{F}_b d\mu(\omega) = \beta\rho \int_\Omega \hat{v}(\omega)\mu(\omega) = 0 \tag{37e}$$

$$H_a^b = (\sigma_{\hat{v}}^2)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}_a d\mu(\omega) = (\sigma_{\hat{v}}^2)^{-1} \beta \int_\Omega \hat{v}(\omega)d\mu(\omega) = 0 \tag{37f}$$

$$H_b^b = (\sigma_{\hat{v}}^2)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}_b d\mu(\omega) - 1 = (\sigma_{\hat{v}}^2)^{-1} \rho DF_y \int_\Omega \hat{v}(\omega)^2 d\mu(\omega) = \beta\rho - 1 \tag{37g}$$

$$H_\sigma^a = \int_\Omega \hat{F}_\sigma d\mu(\omega) = \rho DF_v \int_\Omega \hat{v}(\omega)d\mu(\omega) = 0 \tag{37h}$$

$$H_\sigma^b = (\sigma_{\hat{v}}^2)^{-1} \int_\Omega \hat{v}(\omega)\hat{F}_\sigma d\mu(\omega) = (\sigma_{\hat{v}}^2)^{-1} \rho DF_v \int_\Omega \hat{v}(\omega)^2 d\mu(\omega) = DF_v \cdot \rho\hat{v} \tag{37i}$$

From these computations, we find that

$$DH_{(a,b)'}(0,0,0) = \begin{pmatrix} \beta - 1 & 0 \\ 0 & \beta\rho - 1 \end{pmatrix}, \text{ and } DH_\sigma(0,0,0) = \begin{pmatrix} 0 \\ \rho DF_v \end{pmatrix}. \tag{38}$$

We conclude that the implicit function theorem applies to the system of equations $H = 0$, and that $\frac{\partial b^*}{\partial \sigma} = (1 - \beta\rho)^{-1}\rho DF_v \neq 0$.[21]

[21] We observe that given a linear model $y_t = \beta E_t y_{t+1} + \rho DF_v v_t$, the REE is given by $y_t = b^* v_t$ with $b^* = (1 - \beta\rho)^{-1}\rho DF_v$.

We have demonstrated that for small σ, there exist $(\hat{a}(\sigma), \hat{b}(\sigma))'$, with $\hat{b}(\sigma) \neq 0$, such that $\hat{T}(\hat{a}(\sigma), \hat{b}(\sigma), \sigma) = (\hat{a}(\sigma), \hat{b}(\sigma))'$. The proof of part 1 of the theorem is completed by demonstrating that $T(\hat{a}(\sigma), \sigma^{-1}\hat{b}(\sigma), \sigma) = (\hat{a}(\sigma), \sigma^{-1}\hat{b}(\sigma))'$. To this end, first notice

$$\begin{aligned}
\tilde{F}(a, \sigma^{-1}b, \sigma, v_t) &= \int_\Omega F(a + \sigma^{-1}b\rho v_t + b\zeta_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \\
&= \int_\Omega F(a + b\rho\hat{v}_t + b\zeta_{t+1}(\omega), \rho\sigma\hat{v}_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) \\
&= \hat{F}(a, b, \sigma, \hat{v}_t).
\end{aligned}$$

Using this, we compute

$$\begin{aligned}
T^a(a, \sigma^{-1}b, \sigma) &= \int_\Omega \tilde{F}(a, \sigma^{-1}b, \sigma, v(\omega))d\mu(\omega) \\
&= \int_\Omega \hat{F}(a, b, \sigma, \hat{v}(\omega))d\mu(\omega) = \hat{T}^a(a, b, \sigma),
\end{aligned}$$

and

$$\begin{aligned}
T^b(a, \sigma^{-1}b, \sigma) &= \left(\sigma_v^2(\sigma)\right)^{-1} \int_\Omega v^\sigma(\omega)\tilde{F}(a, \sigma^{-1}b, \sigma, v^\sigma(\omega))d\mu(\omega) \\
&= \left(\sigma^2\sigma_{\hat{v}}^2(\sigma)\right)^{-1} \int_\Omega \sigma\hat{v}(\omega)\hat{F}(a, b, \sigma, v^\sigma(\omega))d\mu(\omega) \\
&= \sigma^{-1}\hat{T}^b(a, b, \sigma).
\end{aligned}$$

Thus

$$T^a(a, \sigma^{-1}b, \sigma) = \hat{T}^a(a, b, \sigma) \text{ and } T^b(a, \sigma^{-1}b, \sigma) = \sigma^{-1}\hat{T}^b(a, b, \sigma). \tag{39}$$

and the result follows.

To establish part 2 of the theorem, we may combine equations (39) with (38) to obtain

$$\begin{aligned}
DT(a, b, \sigma) &= DT(a, \sigma^{-1}(\sigma b), \sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^{-1} \end{pmatrix} D\hat{T}(a, \sigma b, \sigma) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \begin{pmatrix} \beta - 1 & 0 \\ 0 & \sigma(\beta\rho - 1) \end{pmatrix} = \begin{pmatrix} \beta - 1 & 0 \\ 0 & \beta\rho - 1 \end{pmatrix},
\end{aligned}$$

which yields the result. ■

In the remaining sections of this Appendix, we general the model and conduct the associated bifurcation analysis. While the details of the arguments are model-specific, the proof strategy remains the same throughout. The arguments given below will be considerably more brief than provided in the proof of Theorem 1, and we will reference this proof when steps are skipped.

Proof Theorem 4. Again, we begin with derivatives.

Derivatives of $\tilde{F}\left(y_t, a, b, \xi, \eta_t^{\xi}\right) \equiv \int_{\Omega} F\left(y_t, a + b\lambda(\xi)\eta_t^{\xi} + b\varepsilon(\omega)\right) d\mu(\omega)$

$$\tilde{F}_y = \int_{\Omega} F_1 d\mu(\omega) = F_1 \tag{40a}$$

$$\tilde{F}_a = \int_{\Omega} F_2 d\mu(\omega) = F_2 \tag{40b}$$

$$\tilde{F}_b = \int_{\Omega} F_2(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega)) d\mu(\omega) = \lambda(\xi)\eta^{\xi} F_2 \tag{40c}$$

$$\tilde{F}_{\xi} = \int_{\Omega} F_2 b\eta^{\xi} d\mu(\omega) = 0 \tag{40d}$$

$$\tilde{F}_{yy} = \int_{\Omega} F_{11} d\mu(\omega) = F_{11} \tag{40e}$$

$$\tilde{F}_{ya} = \int_{\Omega} F_{12} d\mu(\omega) = F_{12} \tag{40f}$$

$$\tilde{F}_{yb} = \int_{\Omega} \lambda(\xi)\eta^{\xi} F_{12} d\mu(\omega) = \beta^{-1}\eta^{\xi} F_{12} \tag{40g}$$

$$\tilde{F}_{y\xi} = \int_{\Omega} F_{11} b\eta^{\xi} d\mu(\omega) = 0 \tag{40h}$$

$$\tilde{F}_{aa} = \int_{\Omega} F_{22} d\mu(\omega) = F_{22} \tag{40i}$$

$$\tilde{F}_{ab} = \int_{\Omega} \lambda(\xi)\eta^{\xi} F_{22} d\mu(\omega) = \beta^{-1}\eta^{\xi} F_{22} \tag{40j}$$

$$\tilde{F}_{a\xi} = \int_{\Omega} F_{22} b\eta^{\xi} d\mu(\omega) = 0 \tag{40k}$$

$$\tilde{F}_{bb} = \int_{\Omega} F_{22}\left(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega)\right)^2 d\mu(\omega) = \left(\beta^{-2}\left(\eta^{\xi}\right)^2 + \sigma_{\varepsilon}^2\right) F_{22} \tag{40l}$$

$$\tilde{F}_{\xi\xi} = \int_{\Omega} F_{22}(b\eta^{\xi})^2 d\mu(\omega) = 0 \tag{40m}$$

$$\tilde{F}_{b\xi} = \int_{\Omega} (\eta^{\xi} F_2 + b\eta^{\xi} F_{22}) d\mu(\omega) = \beta\eta^{\xi} \tag{40n}$$

$$\tilde{F}_{bbb} = \int_{\Omega} F_{222}(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega))^3 d\mu(\omega) = F_{222}((\beta^{-1}\eta^{\xi})^3 + 3\beta^{-1}\eta^{\xi}\sigma_{\varepsilon}^2) \tag{40o}$$

$$\tilde{F}_{ybb} = \int_{\Omega} F_{122}\left(\lambda(\xi)\eta^{\xi} + \varepsilon(\omega)\right)^2 d\mu(\omega) = \left(\beta^{-2}\left(\eta^{\xi}\right)^2 + \sigma_{\varepsilon}^2\right) F_{122} \tag{40p}$$

Derivatives of $\tilde{F}\left(\hat{F}\left(a,b,\xi,\eta^{\xi}\right),a,b,\xi,\eta^{\xi}\right)=0$

$$\hat{F}_a = -\frac{\tilde{F}_a}{\tilde{F}_y} = \beta \tag{41a}$$

$$\hat{F}_b = -\frac{\tilde{F}_b}{\tilde{F}_y} = \eta^{\xi} \tag{41b}$$

$$\hat{F}_{\xi} = -\frac{\tilde{F}_{\xi}}{\tilde{F}_y} = 0 \tag{41c}$$

$$\hat{F}_{aa} = -\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{aa}+2\hat{F}_a\tilde{F}_{ya}+\hat{F}_a^2\tilde{F}_{yy}\right) = -\left(\frac{\beta F_{112}}{F_1}\right)\eta^{\xi} - \frac{2\beta F_{12}+F_{22}}{F_1} \equiv \Phi_{aa}^1\cdot\eta^{\xi}+\Phi_{aa}^0 \tag{41d}$$

$$\hat{F}_{ab} = -\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{ab}+\hat{F}_b\tilde{F}_{ya}+\hat{F}_a\tilde{F}_{yb}+\hat{F}_a\hat{F}_b\tilde{F}_{yy}\right) = -\left(\frac{F_{112}}{F_1}\right)\left(\eta^{\xi}\right)^2 - \left(\frac{2\beta F_{12}+F_{22}}{\beta F_1}\right)\eta^{\xi} \equiv \Phi_{ab}^2\cdot\left(\eta^{\xi}\right)^2+\Phi_{ab}^1\cdot\eta^{\xi} \tag{41e}$$

$$\hat{F}_{bb} = -\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{bb}+2\hat{F}_b\tilde{F}_{yb}+\hat{F}_b^2\tilde{F}_{yy}\right) = -\left(\frac{F_{112}}{\beta F_1}\right)\left(\eta^{\xi}\right)^3 - \left(\frac{2\beta F_{12}+F_{22}}{\beta^2 F_1}\right)\left(\eta^{\xi}\right)^2 - \frac{F_{22}\sigma_{\varepsilon}^2}{F_1} \tag{41f}$$

$$\equiv \Phi_{bb}^3\cdot\left(\eta^{\xi}\right)^3+\Phi_{bb}^2\cdot\left(\eta^{\xi}\right)^2+\Phi_{bb}^0 \tag{41g}$$

$$\hat{F}_{b\xi} = -\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{b\xi}+\hat{F}_{\xi}\tilde{F}_{yb}+\hat{F}_b\tilde{F}_{y\xi}+\hat{F}_b\hat{F}_{\xi}\tilde{F}_{yy}\right) = \beta\eta^{\xi} \tag{41h}$$

$$\hat{F}_{\xi\xi} = 0 \tag{41i}$$

$$\hat{F}_{bbb} = -\left(\tilde{F}_y\right)^{-1}\left(\tilde{F}_{bbb}+3\hat{F}_{bb}\tilde{F}_{yb}+3\hat{F}_{bb}\hat{F}_b\tilde{F}_{yy}+3\hat{F}_b\tilde{F}_{ybb}+3\hat{F}_b^2\tilde{F}_{yyb}+\hat{F}_b^3\tilde{F}_{yyy}\right) \tag{41j}$$

$$= \left(\frac{3F_{112}^2}{\beta^2F_1^2}\right)\left(\eta^{\xi}\right)^5 + \left(\frac{9\beta F_{12}F_{112}+3F_{22}F_{112}}{\beta^3F_1^2}\right)\left(\eta^{\xi}\right)^4 + \left(\frac{\beta^3 6\beta F_{12}^2 - F_1F_{111} - 3\beta F_1F_{122} + 3F_{12}F_{22} - F_1F_{222}}{\beta^3F_1^2}\right)\left(\eta^{\xi}\right)^3$$

$$+\left(\frac{3\beta^2F_{22}F_{112}\sigma_{\varepsilon}^2 - 3\beta^3F_1F_{11}}{\beta^3F_1^2}\right)\left(\eta^{\xi}\right)^2 + \left(\frac{3\beta^2F_{12}F_{22}\sigma_{\varepsilon}^2 - 3\beta^3F_1F_{122}\sigma_{\varepsilon}^2 - 3\beta^2F_1F_{222}\sigma_{\varepsilon}^2}{\beta^3F_1^2}\right)\eta^{\xi} \tag{41k}$$

$$\equiv \Phi_{bbb}^5\cdot\left(\eta^{\xi}\right)^5+\Phi_{bbb}^4\cdot\left(\eta^{\xi}\right)^4+\Phi_{bbb}^3\cdot\left(\eta^{\xi}\right)^3+\Phi_{bbb}^2\cdot\left(\eta^{\xi}\right)^2+\Phi_{bbb}^1\cdot\eta^{\xi}+\Phi_{bbb}^0$$

Derivatives of $T^a(a,b,\xi) = \int_{\Omega}\hat{F}(a,b,\xi,\eta^{\xi}(\omega))d\mu(\omega)$

$$T_a^a = \int_{\Omega}\hat{F}_a d\mu(\omega) = \beta \tag{42a}$$

$$T_b^a = \int_{\Omega}\hat{F}_b d\mu(\omega) = \int_{\Omega}\eta^{\xi}(\omega)d\mu(\omega) = 0 \tag{42b}$$

$$T_{\xi}^a = \int_{\Omega}\hat{F}_{\xi} d\mu(\omega) = 0 \tag{42c}$$

$$T_{aa}^a = \int_{\Omega}\hat{F}_{aa} d\mu(\omega) = \int_{\Omega}\left(\Phi_{aa}^1\cdot\eta^{\xi}(\omega)+\Phi_{aa}^0\right)d\mu(\omega) = \Phi_{aa}^0 \tag{42d}$$

$$T_{ab}^a = \int_{\Omega}\hat{F}_{ab} d\mu(\omega) = \int_{\Omega}\left(\Phi_{ab}^2\cdot\left(\eta^{\xi}(\omega)\right)^2+\Phi_{ab}^1\cdot\eta^{\xi}\omega)\right)d\mu(\omega) = \Phi_{ab}^2\cdot\sigma_{\eta^{\xi}}^2 \tag{42e}$$

$$T_{bb}^a = \int_{\Omega}\hat{F}_{bb} d\mu(\omega) = \int_{\Omega}\left(\Phi_{bb}^3\cdot\left(\eta^{\xi}(\omega)\right)^3+\Phi_{bb}^2\cdot\left(\eta^{\xi}(\omega)\right)^2+\Phi_{bb}^0\right)d\mu(\omega) = \Phi_{bb}^3\cdot\mu_3^{\eta^{\xi}}+\Phi_{bb}^2\cdot\sigma_{\eta^{\xi}}^2+\Phi_{bb}^0 \tag{42f}$$

$$T_{\xi b}^a = \int_{\Omega}\hat{F}_{\xi b} d\mu(\omega) = 0 \tag{42g}$$

Derivatives of $T^b(a,b,\xi) = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \hat{F}(a,b,\xi,\eta^\xi(\omega))d\mu(\omega)$

$$T^b_a = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega) = 0 \tag{43a}$$

$$T^b_b = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega (\eta^\xi(\omega))^2 d\mu(\omega) = 1 \tag{43b}$$

$$T^b_\xi = \left(\sigma^2_{\eta^\xi}\right)^{-2}\left[\sigma^2_{\eta^\xi}\int_\Omega\left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right)d\mu(\omega) - \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega \eta^\xi(\omega)\hat{F}d\mu(\omega)\right] = 0 \tag{43c}$$

$$T^b_{bb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{bb} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\left(\Phi^3_{bb}\cdot\left(\eta^\xi(\omega)\right)^3 + \Phi^2_{bb}\cdot\left(\eta^\xi(\omega)\right)^2 + \Phi^0_{bb}\right)d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\sigma^4_{\eta^\xi} + \Phi^2_{bb}\mu_3^{\eta^\xi}\right) \tag{43d}$$

$$T^b_{ab} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\left(\Phi^2_{ab}\cdot\left(\eta^\xi(\omega)\right)^2 + \Phi^1_{ab}\cdot\eta^\xi(\omega)\right)d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^2_{ab}\mu_3^{\eta^\xi} + \Phi^1_{ab}\sigma^2_{\eta^\xi}\right) \tag{43e}$$

$$T^b_{aa} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{aa} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\left(\Phi^1_{aa}\cdot\eta^\xi(\omega) + \Phi^0_{aa}\right)d\mu(\omega) = \Phi^1_{aa} \tag{43f}$$

$$\begin{aligned}T^b_{\xi\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-4}\Bigg\{&(\sigma^2_\eta)^2\Bigg[\sigma^2_{\eta^\xi}\int_\Omega\left(2\hat{F}_\xi\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \hat{F}\frac{\partial^2}{(\partial\xi)^2}\eta^\xi(\omega) + \hat{F}_{\xi\xi}\eta^\xi(\omega)\right)d\mu(\omega)\\
&+\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega\left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right)d\mu - \int_\Omega\frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}d\mu(\omega)\\
&-\int_\Omega\eta^\xi(\omega)\hat{F}_\xi\frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}d\mu(\omega) - \int_\Omega\eta^\xi(\omega)\hat{F}\frac{\partial^2}{(\partial\xi)^2}\sigma^2_{\eta^\xi}d\mu(\omega)\Bigg]\\
&-\sigma^2_{\eta^\xi}\int_\Omega\left(\hat{F}\frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right)\frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega) + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega\eta^\xi(\omega)\hat{F}\frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega)\Bigg\} = 0\end{aligned} \tag{43g}$$

$$T^b_{b\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2}\left[\sigma^2_{\eta^\xi}\int_\Omega\left(\frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F}_b + \eta^\xi(\omega)\hat{F}_{b\xi}\right)d\mu(\omega) - \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi}\int_\Omega\eta^\xi(\omega)\hat{F}_b d\mu(\omega)\right] = \beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{43h}$$

$$T^b_{bbb} = \tfrac{1}{\sigma^2_{\eta^\xi}}\int_\Omega \hat{F}_{bbb}\eta^\xi(\omega)d\mu(\omega) \tag{43i}$$

$$= \tfrac{1}{\sigma^2_{\eta^\xi}}\int_\Omega\left(\Phi^5_{bbb}\cdot\left(\eta^\xi(\omega)\right)^5 + \Phi^4_{bbb}\cdot\left(\eta^\xi(\omega)\right)^4 + \Phi^3_{bbb}\cdot\left(\eta^\xi(\omega)\right)^3 + \Phi^2_{bbb}\cdot\left(\eta^\xi(\omega)\right)^2 + \Phi^1_{bbb}\cdot\eta^\xi(\omega) + \Phi^0_{bbb}\right)\eta^\xi(\omega)d\mu(\omega) \tag{43j}$$

$$= \tfrac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^5_{bbb}\cdot\sigma^6_{\eta^\xi} + \Phi^4_{bbb}\cdot\sigma^5_{\eta^\xi} + \Phi^3_{bbb}\cdot\mu_4^\xi + \Phi^2_{bbb}\cdot\mu_3^{\eta^\xi} + \Phi^1_{bbb}\cdot\sigma^2_{\eta^\xi}\right) \tag{43k}$$

We now turn to the body of the argument, which, as before, requires bifurcation analysis of the system (31). The center manifold may be characterized locally as a C^4 function: $a = h(b,\xi)$, which satisfies the following functional equation:

$$(\beta-1)h(b,\xi) + f(h(b,\xi),b,\xi) = h_b(b,\xi)g(h(b,\xi),b,\xi).$$

Working as before, we find that

$$h_{bb} = \tfrac{1}{1-\beta}f_{bb} = \tfrac{1}{1-\beta}T^a_{bb} = \tfrac{1}{1-\beta}\left(\Phi^3_{bb}\cdot\mu_3^{\eta^\xi} + \Phi^2_{bb}\cdot\sigma^2_{\eta^\xi} + \Phi^0_{bb}\right).$$

The projected system is given by

$$\dot{b} = g(h(b,\xi),b,\xi) \equiv G(b,\xi). \tag{44}$$

To conduct bifurcation analysis, the higher-order derivatives of G are needed. That $G(0,0) = 0$ is immediate. Since $g = T^b - b$ we have that

$$g_{aa} = T^b_{aa} = \Phi^1_{aa} \tag{45a}$$

$$g_{ab} = T^b_{ab} = \frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^2_{ab}\mu_3^{\eta^\xi} + \Phi^1_{ab}\sigma^2_{\eta^\xi}\right) \tag{45b}$$

$$g_{bb} = T^b_{bb} = \frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\sigma^4_{\eta^\xi} + \Phi^2_{bb}\mu_3^{\eta^\xi}\right) \tag{45c}$$

$$g_{b\xi} = T^b_{b\xi} = \beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{45d}$$

$$g_{bbb} = T^b_{bbb} = \frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^5_{bbb}\cdot\sigma^6_{\eta^\xi} + \Phi^4_{bbb}\cdot\sigma^5_{\eta^\xi} + \Phi^3_{bbb}\cdot\mu_4^{\xi} + \Phi^2_{bbb}\cdot\mu_3^{\eta^\xi} + \Phi^1_{bbb}\cdot\sigma^2_{\eta^\xi}\right). \tag{45e}$$

Using our information about h, we compute

$$G_b = g_a\cdot h_b + g_b = 0 \tag{46a}$$

$$G_\xi = g_a\cdot h_\xi + g_\xi = 0 \tag{46b}$$

$$G_{bb} = g_a\cdot h_{bb} + h_b\cdot(g_{aa}\cdot h_b + g_{ab}) + g_{ab}\cdot h_b + g_{bb} = g_{bb} = \frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\sigma^4_{\eta^\xi} + \Phi^2_{bb}\mu_3^{\eta^\xi}\right) \tag{46c}$$

$$G_{b\xi} = h_b\cdot\frac{\partial}{\partial\xi}g_a + g_a\cdot h_{b\xi} + g_{ba}\cdot h_\xi + g_{b\xi} = \beta\left(\frac{\beta^2-2}{\beta^2-1}\right) \tag{46d}$$

$$\begin{aligned}G_{bbb} &= g_a h_{bbb} + 2h_{bb}\cdot(g_{aa}\cdot h_b + g_{ab}) + h_b\cdot\tfrac{\partial}{\partial b}(g_{aa}\cdot h_b + g_{ab})\\ &\quad + \; g_{ab}\cdot h_{bb} + h_b\cdot\tfrac{\partial}{\partial b}\, g_{ab} + h_b\cdot g_{bba} + g_{bbb} = 3h_{bb}\cdot g_{ab} + g_{bbb} \qquad (46e)\\ &= \frac{1}{(1-\beta)\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\cdot\mu_3^{\eta^\xi} + \Phi^2_{bb}\cdot\sigma^2_{\eta^\xi} + \Phi^0_{bb}\right)\left(\Phi^2_{ab}\mu_3^{\eta^\xi} + \Phi^1_{ab}\sigma^2_{\eta^\xi}\right)\\ &\quad + \frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^5_{bbb}\cdot\mu_6^{\eta^\xi} + \Phi^4_{bbb}\cdot\mu_5^{\eta^\xi} + \Phi^3_{bbb}\cdot\mu_4^{\eta^\xi} + \Phi^2_{bbb}\cdot\mu_3^{\eta^\xi} + \Phi^1_{bbb}\cdot\sigma^2_{\eta^\xi}\right) \equiv \mathscr{I}\mathscr{C} \qquad (46f)\end{aligned}$$

where, in each computation, the second equality follows from the work just above and that h and g have no first order terms.

Since $G = G_b = G_\xi = 0$, and $G_{b\xi}$ is generically non-zero, the type of bifurcation experienced by the projected system depends on whether $G_{bb} = 0$. Noting that $G_{bb} = \frac{1}{\sigma^2_{\eta^\xi}}\left(\Phi^3_{bb}\mu_4^{\eta^\xi} + \Phi^2_{bb}\mu_3^{\eta^\xi}\right)$ and that $\Phi^3_{bb} = -\frac{F_{112}}{\beta F_1}$ and $\Phi^2_{bb} = \frac{2\beta F_{12}+F_{22}}{\beta^2 F_1}$, we have two cases:

<u>Case 1:</u> $(2\beta F_{12} + F_{22})\mu_3^{\varepsilon} \neq \beta F_{112}$;

In this case we have $G_b = 0$, $G_\xi = 0$, $G_{b\xi} \neq 0$, and $G_{bb} \neq 0$; thus the projected system undergoes a transcritical bifurcation as ξ crosses zero, indicating the emergence of two additional fixed points.

<u>Case 2:</u> $(2\beta F_{12} + F_{22})\mu_3^{\varepsilon} = \beta F_{112}$ and $\mathscr{I}\mathscr{C} \neq 0$.

Since G_b, G_ξ and $G_{bb} = 0$, and $G_{b\xi} \neq 0$ the system undergoes a pitchfork bifurcation as ξ crosses zero provided that $G_{bbb} \neq 0$, thus $\mathscr{I}\mathscr{C}$ must be non-zero.

The remainder of the proof is completed as before, with stability in Case 1 requiring that $\mathscr{IC}$ be negative. ■

Proof Theorem 5 In what follows, unless otherwise specified, derivatives are evaluated at

$$(a,b,c,\xi) = (a^*, b^*, 0, 0).$$

Recall that in the body we stated that by choosing $|\sigma|$ small we may assume that $DF_\star \approx DF_\star(0,0)$ for $\star = y, v, yy$, etc. To see this, first observe that if $(\hat{a}(\sigma), \hat{b}(\sigma))'$ is the fixed point of the map $\hat{T}$ (see proof of Theorem 3) then $\lim_{\sigma\to 0}(\hat{a}(\sigma), \hat{b}(\sigma)) = (0,0)$, and since $a^*(\sigma) = \hat{a}(\sigma)$, we may assume $|a^*(\sigma)|$ is small. Also, since $b^*(\sigma) = \frac{1}{\sigma}\hat{b}(\sigma)$, we may assume $|\sigma b^*(\sigma)|$ is small. Since $v_t = \sigma\hat{v}_t$ follows that for small $|\sigma|$,

$$DF_\star \equiv DF_\star(a^*(\sigma) + b^*(\sigma)\rho v_t + b^*(\sigma)\sigma\zeta_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega)) \approx DF_\star(0,0) \tag{47}$$

for $\star = y, v, yy$, etc.

Turning now the the main argument, the proof follows the same structure as the proof of Theorem 1, and because of this, we will be considerably more brief. Again, we require a host of derivatives.

Derivatives of $\hat{F} = \int_\Omega F(a + b\rho v_t + b\sigma\zeta_{t+1}(\omega) + c\lambda(\xi)\eta_t^\xi + c\varepsilon_{t+1}(\omega), \rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega)$

$$\hat{F}_a = \int_\Omega DF_y \cdot d\mu(\omega) = DF_y \tag{48a}$$

$$\hat{F}_b = \int_\Omega DF_y \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) = DF_y \cdot \rho v_t \tag{48b}$$

$$\hat{F}_c = \int_\Omega DF_y \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))d\mu(\omega) = DF_y \cdot \lambda(\xi)\eta_t^\xi \tag{48c}$$

$$\hat{F}_\xi = \int_\Omega DF_y \cdot c\eta_t^\xi d\mu(\omega) = 0 \tag{48d}$$

$$\hat{F}_{aa} = \int_\Omega DF_{yy} \cdot d\mu(\omega) = DF_{yy} \tag{48e}$$

$$\hat{F}_{ab} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))d\mu(\omega) = DF_{yy} \cdot \rho v_t \tag{48f}$$

$$\hat{F}_{ac} = \int_\Omega DF_{yy} \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))d\mu(\omega) = DF_{yy} \cdot \lambda(\xi)\eta_t^\xi \tag{48g}$$

$$\hat{F}_{a\xi} = \int_\Omega DF_{yy} \cdot c\eta_t^\xi d\mu(\omega) = 0 \tag{48h}$$

$$\hat{F}_{bb} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))^2 d\mu(\omega) = DF_{yy} \cdot (\rho^2 v_t^2 + \sigma^2\sigma_\zeta^2) \tag{48i}$$

$$\hat{F}_{bc} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))(\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))\mu(\omega) = DF_{yy} \cdot \lambda(\xi)\rho\eta_t^\xi v_t \tag{48j}$$

$$\hat{F}_{b\xi} = \int_\Omega DF_{yy} \cdot (\rho v_t + \sigma\zeta_{t+1}(\omega))c\eta_t^\xi d\mu(\omega) = 0 \tag{48k}$$

$$\hat{F}_{cc} = \int_\Omega DF_{yy} \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))^2 d\mu(\omega) = DF_{yy} \cdot \left(\lambda(\xi)^2\left(\eta_t^\xi\right)^2 + \sigma_\varepsilon^2\right) \tag{48l}$$

$$\hat{F}_{c\xi} = \int_\Omega \left(DF_{yy} \cdot c\eta_t^\xi\left(\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega)\right) + DF_y \cdot \eta_t^\xi\right)d\mu(\omega) = DF_y \cdot \eta_t^\xi \tag{48m}$$

$$\hat{F}_{\xi\xi} = \int_\Omega \left(DF_{yy} \cdot \left(c\eta_t^\xi\right)^2 + DF_y \cdot c \cdot \tfrac{\partial}{\partial\xi}\eta_t^\xi\right)d\mu(\omega) = 0 \tag{48n}$$

$$\hat{F}_{ccc} = \int_\Omega DF_{yyy} \cdot (\lambda(\xi)\eta_t^\xi + \varepsilon_{t+1}(\omega))^3 d\mu(\omega) = DF_{yyy} \cdot \left(\lambda(\xi)^3\left(\eta_t^\xi\right)^3 + 3\lambda(\xi)\sigma_\varepsilon^2\eta_t^\xi + \mu_4^\varepsilon\right) \tag{48o}$$

Derivatives of $T^a = \int_\Omega \hat{F}(a,b,c,\xi,v(\omega),\eta^\xi(\omega))d\mu(\omega)$

$$T^a_a = \int_\Omega \hat{F}_a \cdot d\mu(\omega) = DF_y \tag{49a}$$

$$T^a_b = \int_\Omega \hat{F}_b \cdot d\mu(\omega) = DF_y \cdot \rho \int_\Omega v(\omega)d\mu(\omega) = 0 \tag{49b}$$

$$T^a_c = \int_\Omega \hat{F}_c \cdot d\mu(\omega) = DF_y \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{49c}$$

$$T^a_\xi = \int_\Omega \hat{F}_\xi \cdot d\mu(\omega) = 0 \tag{49d}$$

$$T^a_{aa} = \int_\Omega \hat{F}_{aa} \cdot d\mu(\omega) = DF_{yy} \tag{49e}$$

$$T^a_{ab} = \int_\Omega \hat{F}_{ab} \cdot d\mu(\omega) = DF_{yy} \cdot \rho \int_\Omega v(\omega)d\mu(\omega) = 0 \tag{49f}$$

$$T^a_{ac} = \int_\Omega \hat{F}_{ac} \cdot d\mu(\omega) = DF_{yy} \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{49g}$$

$$T^a_{a\xi} = \int_\Omega \hat{F}_{a\xi} \cdot d\mu(\omega) = 0 \tag{49h}$$

$$T^a_{bb} = \int_\Omega \hat{F}_{bb} \cdot d\mu(\omega) = DF_{yy} \int_\Omega \left(\rho^2 v(\omega)^2 + \sigma^2_\zeta\right) d\mu(\omega) = DF_{yy} \cdot \sigma^2_v \tag{49i}$$

$$T^a_{bc} = \int_\Omega \hat{F}_{bc} \cdot d\mu(\omega) = DF_{yy} \cdot \rho \cdot \lambda(\xi) \int_\Omega v(\omega)\eta^\xi(\omega)d\mu(\omega) = 0 \tag{49j}$$

$$T^a_{b\xi} = \int_\Omega \hat{F}_{b\xi} d\mu(\omega) = 0 \tag{49k}$$

$$T^a_{cc} = \int_\Omega \hat{F}_{cc} \cdot d\mu(\omega) = DF_{yy} \int_\Omega \left(\lambda(\xi)^2 \eta^\xi(\omega)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF_{yy} \cdot \sigma^2_{\eta^\xi} \tag{49l}$$

$$T^a_{c\xi} = \int_\Omega \hat{F}_{c\xi} \cdot d\mu(\omega) = DF_y \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{49m}$$

Derivatives of $T^b = \frac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}(a,b,c,\xi,v(\omega),\eta^\xi(\omega))d\mu(\omega)$

$$T_a^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_a d\mu(\omega) = 0 \tag{50a}$$

$$T_b^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_y\cdot\rho\int_\Omega v(\omega)^2 d\mu(\omega) = DF_y\cdot\rho \tag{50b}$$

$$T_c^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_c d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_y\cdot\lambda(\xi)\int_\Omega v(\omega)\eta^\xi(\omega)d\mu(\omega) = 0 \tag{50c}$$

$$T_\xi^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_\xi d\mu(\omega) = 0 \tag{50d}$$

$$T_{aa}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{aa} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\int_\Omega v(\omega)d\mu(\omega) = 0 \tag{50e}$$

$$T_{ab}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\cdot\rho\int_\Omega v(\omega)^2 d\mu(\omega) = \rho\cdot DF_{yy} \tag{50f}$$

$$T_{ac}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{ac} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\cdot\lambda(\xi)\int_\Omega v(\omega)\eta^\xi(\omega)d\mu(\omega) = 0 \tag{50g}$$

$$T_{a\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{a\xi} d\mu(\omega) = 0 \tag{50h}$$

$$T_{bb}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{bb} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\int_\Omega v(\omega)\left(\rho^2 v(\omega)+\sigma^2\sigma_\zeta^2\right)d\mu(\omega) = DF_{yy}\cdot\rho^2\cdot\left(\frac{\mu_3^v}{\sigma_v^2}\right) \tag{50i}$$

$$T_{bc}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{bc} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\cdot\lambda(\xi)\cdot\rho\int_\Omega v(\omega)^2\eta^\xi(\omega)d\mu(\omega) = 0 \tag{50j}$$

$$T_{b\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{b\xi} d\mu(\omega) = 0 \tag{50k}$$

$$T_{cc}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{cc} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_{yy}\int_\Omega v(\omega)\left(\lambda(\xi)^2\eta^\xi(\omega)^2+\sigma_\varepsilon^2\right)d\mu(\omega) = 0 \tag{50l}$$

$$T_{c\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{c\xi} d\mu(\omega) = \tfrac{1}{\sigma_v^2}\cdot DF_y\int_\Omega v(\omega)\cdot\eta^\xi(\omega)d\mu(\omega) = 0 \tag{50m}$$

$$T_{\xi\xi}^b = \tfrac{1}{\sigma_v^2}\int_\Omega v(\omega)\hat{F}_{\xi\xi} d\mu(\omega) = 0 \tag{50n}$$

Derivatives of $T^c = \left(\sigma^2_{\eta^\xi}\right)^{-1} \int_\Omega \eta^\xi(\omega)\hat{F}(a,b,c,\xi,v(\omega),\eta^\xi(\omega))d\mu(\omega)$

$$T^c_a = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega) = 0 \tag{51a}$$

$$T^c_b = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_y \int_\Omega \eta^\xi(\omega)v(\omega)d\mu(\omega) = 0 \tag{51b}$$

$$T^c_c = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_c d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_y \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = 1 \tag{51c}$$

$$T^c_\xi = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F} \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F} d\mu(\omega)\right) = 0 \tag{51d}$$

$$T^c_{aa} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{aa} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} DF_{yy} \cdot \int_\Omega \eta^\xi(\omega)d\mu(\omega) = 0 \tag{51e}$$

$$T^c_{ab} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ab} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \cdot \rho \int_\Omega \eta^\xi(\omega)v(\omega)d\mu(\omega) = 0 \tag{51f}$$

$$T^c_{ac} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ac} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \cdot \lambda(\xi) \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = DF_{yy} \cdot \lambda(\xi) \tag{51g}$$

$$T^c_{a\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}_a \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_{a\xi}\right) d\mu(\omega) + \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}_a d\mu(\omega)\right) = 0 \tag{51h}$$

$$T^c_{bb} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{bb} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \int_\Omega \eta^\xi(\omega) \left(\rho^2 v(\omega) + \sigma^2\sigma^2_\zeta\right) d\mu(\omega) = 0 \tag{51i}$$

$$T^c_{bc} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{bc} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yy} \cdot \lambda(\xi) \cdot \rho \int_\Omega \eta^\xi(\omega)^2 v(\omega)d\mu(\omega) = 0 \tag{51j}$$

$$T^c_{b\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}_b \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_{b\xi}\right) d\mu(\omega) + \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}_b d\mu(\omega)\right) = 0 \tag{51k}$$

$$T^c_{cc} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{cc} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} DF_{yy} \int_\Omega \eta^\xi(\omega) \left(\lambda(\xi)^2\eta^\xi(\omega)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF_{yy} \cdot \lambda(\xi)^2 \left(\frac{\mu_3^{\eta^\xi}}{\sigma^2_{\eta^\xi}}\right) \tag{51l}$$

$$T^c_{c\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F}_c \tfrac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_{c\xi}\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}_c d\mu(\omega)\right) = DF_y \left(\frac{DF_y^2 - 2}{DF_y^2 - 1}\right) \tag{51m}$$

$$\begin{aligned} T^c_{\xi\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-4} \Bigg\{ \left(\sigma^2_{\eta^\xi}\right)^2 \Bigg[& \sigma^2_{\eta^\xi} \int_\Omega \left(2\hat{F}_\xi \frac{\partial}{\partial\xi}\eta^\xi(\omega) + \hat{F} \frac{\partial^2}{(\partial\xi)^2}\eta^\xi(\omega) + \hat{F}_{\xi\xi}\eta^\xi(\omega)\right) d\mu(\omega) \\ & + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F} \frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) d\mu - \int_\Omega \frac{\partial}{\partial\xi}\eta^\xi(\omega)\hat{F} \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} d\mu(\omega) \\ & - \int_\Omega \eta^\xi(\omega)\hat{F}_\xi \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} d\mu(\omega) - \int_\Omega \eta^\xi(\omega)\hat{F} \frac{\partial^2}{(\partial\xi)^2}\sigma^2_{\eta^\xi} d\mu(\omega) \Bigg] \\ & - \sigma^2_{\eta^\xi} \int_\Omega \left(\hat{F} \frac{\partial}{\partial\xi}\eta^\xi(\omega) + \eta^\xi(\omega)\hat{F}_\xi\right) \frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega) + \frac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F} \frac{\partial}{\partial\xi}(\sigma^2_{\eta^\xi})^2 d\mu(\omega) \Bigg\} = 0 \end{aligned} \tag{51n}$$

$$T^c_{ccc} = \tfrac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}_{ccc} d\mu(\omega) = \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yyy} \int_\Omega \left(\lambda(\xi)^3\eta^\xi(\omega)^4 + 3\lambda(\xi)\sigma^2_\varepsilon\eta^\xi(\omega)^2\right) d\mu(\omega) \tag{51o}$$

$$= \tfrac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yyy} \left(\lambda(\xi)^3\mu_4^\xi + \lambda(\xi)\sigma^2_{\eta^\xi}\sigma^2_\varepsilon\right) \tag{51p}$$

The computations (51d), (51h), (51k) require that at $c = 0$, $\hat{F}$ and its first partials are independent of η, and that $\int_\Omega \frac{\partial}{\partial\xi}\eta(\omega)d\mu(\omega) = 0$, which follows from equation (30). Also, (51m) follows from the same argument as (29h).

We turn now to the bifurcation analysis. Change coordinates: $\alpha = a - a^*, \gamma = b - b^*$, and consider the dynamic system

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\gamma} \\ \dot{c} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} T^a(\alpha+a^*,\gamma+b^*,c,\xi) \\ T^b(\alpha+a^*,\gamma+b^*,c,\xi) \\ T^c(\alpha+a^*,\gamma+b^*,c,\xi) \\ 0 \end{pmatrix} - \begin{pmatrix} \alpha+a^* \\ \gamma+b^* \\ c \\ 0 \end{pmatrix} \equiv H(\alpha,\gamma,c,\xi),$$

noting that the origin is a rest point. Following the usual proof strategy, write

$$H(\alpha,\gamma,c,\xi) = \begin{pmatrix} T^a_a & T^a_b, & T^a_c & T^a_\xi \\ T^b_a & T^b_b, & T^b_c & T^b_\xi \\ T^c_a & T^c_b, & T^c_c & T^c_\xi \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} f^1(\alpha,\gamma,c,\xi) \\ f^2(\alpha,\gamma,c,\xi) \\ g(\alpha,\gamma,c,\xi) \\ 0 \end{pmatrix},$$

where f^i and g are $\mathscr{O}\left(\|(a,b,c,\xi)\|^2\right)$. By appealing to our previous computations, we find that

$$\begin{pmatrix} T^a_a & T^a_b, & T^a_c & T^a_\xi \\ T^b_a & T^b_b, & T^b_c & T^b_\xi \\ T^c_a & T^c_b, & T^c_c & T^c_\xi \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} DF_y-1 & 0 & 0 & 0 \\ 0 & \rho DF_y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} f^1(\alpha,\gamma,c,\xi) &= T^a(\alpha+a^*,\gamma+b^*,c,\xi) - DF_y\cdot\alpha - a^* \\ f^2(\alpha,\gamma,c,\xi) &= T^b(\alpha+a^*,\gamma+b^*,c,\xi) - \rho DF_y\cdot\gamma - b^* \\ g(\alpha,\gamma,c,\xi) &= T^c(\alpha+a^*,\gamma+b^*,c,\xi) - c. \end{aligned}$$

The center manifold is parameterized by $\alpha = h^\alpha(c,\xi)$ and $\gamma = h^\gamma(c,\xi)$; and, using invariance, these parameterizations satisfy the following functional equations:

$$L^\alpha(c,\xi) \equiv (DF_y-1)h^\alpha + f^1(h^\alpha,h^\gamma,c,\xi) = h^\alpha_c\cdot g(h^\alpha,h^\gamma,c,\xi) \equiv R^\alpha(c,\xi) \quad (52)$$

$$L^\gamma(c,\xi) \equiv (\rho DF_y-1)h^\gamma + f^2(h^\alpha,h^\gamma,c,\xi) = h^\gamma_c\cdot g(h^\alpha,h^\gamma,c,\xi) \equiv R^\gamma(c,\xi). \quad (53)$$

Computing as in (32), we find that

$$\begin{aligned} h^\alpha_{cc} &= \frac{f^1_{cc}}{1-DF_y} = \frac{T^a_{cc}}{1-DF_y} = \left(\frac{DF_{yy}}{1-DF_y}\right)\sigma^2_{\eta^\xi} \\ h^\gamma_{cc} &= \frac{f^2_{cc}}{1-\rho DF_y} = \frac{T^b_{cc}}{1-\rho DF_y} = 0, \end{aligned}$$

and, as before, these are the only partials we require.

Projected onto the center, the dynamics take the form

$$\dot{c} = g(h^\alpha(c,\xi),h^\gamma(c,\xi),c,\xi) \equiv G(c,\xi).$$

Computing as in (46), we find $G_\star = 0$ and

$$G_{cc} = g_{cc} = T^c_{cc} = DF_{yy} \cdot DF_y^{-2} \left(\frac{\mu_3^{\eta^\xi}}{\sigma^2_{\eta^\xi}} \right) \tag{54a}$$

$$G_{c\xi} = g_{c\xi} = T^c_{c\xi} = DF_y \left(\frac{DF_y^2 - 2}{DF_y^2 - 1} \right) \tag{54b}$$

$$G_{ccc} = 3(h^\alpha_{cc} \cdot g_{ac} + h^\gamma_{cc} \cdot g_{bc}) + g_{ccc} = 3h^\alpha_{cc} \cdot T^c_{ac} + T^c_{ccc} = \frac{3DF_{yy}}{DF_y} \left(\frac{DF_{yy}}{1-DF_y} \right) \sigma^2_{\eta^\xi} + T^c_{ccc} \tag{54c}$$

$$= \frac{3DF_{yy}}{DF_y} \left(\frac{DF_{yy}}{1-DF_y} \right) \sigma^2_{\eta^\xi} + \frac{1}{\sigma^2_{\eta^\xi}} \cdot DF_{yyy} \left(DF_y^{-3} \mu_3^\varepsilon + DF_y^{-1} \sigma^2_{\eta^\xi} \sigma^2_\varepsilon \right). \tag{54d}$$

The proofs of existence and stability are complete arguing as in the proof of Theorem 1. ■

Proof of Theorem 6. First, we require some derivatives. As notation, write

$$T = \begin{pmatrix} T^a \\ T^b \end{pmatrix} \text{ and } DF = \left(DF^i_{y_j} \right).$$

We compute as follows:

Derivatives of $\hat{F}(a,b,\xi,\eta_t^\xi) = \int_\Omega (F^i(a_1 + b_1\lambda(\xi)\eta_t^\xi + b_1\varepsilon_{t+1}(\omega), a_2 + b_2\lambda(\xi)\eta_t^\xi + b_2\varepsilon_{t+1}(\omega)) d\mu(\omega)$

$$\hat{F}^i_{a_j} = \int_\Omega DF^i_{y_j} \cdot d\mu(\omega) = DF^i_{y_j} \tag{55a}$$

$$\hat{F}^i_{b_j} = \int_\Omega DF^i_{y_j} \cdot \left(\lambda(\xi) \cdot \eta_t^\xi + \varepsilon_{t+1}(\omega) \right) d\mu(\omega) = DF^i_{y_j} \cdot \lambda(\xi) \cdot \eta_t^\xi \tag{55b}$$

$$\hat{F}^i_\xi = \int_\Omega \left(\sum_{j=1}^n DF^i_{y_j} \cdot b_j \right) \eta_t^\xi d\mu(\omega) = 0 \tag{55c}$$

$$\hat{F}^i_{a_j a_k} = \int_\Omega DF^i_{y_j y_k} \cdot d\mu(\omega) = DF^i_{y_j y_k} \tag{55d}$$

$$\hat{F}^i_{a_j b_k} = \int_\Omega DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \eta_t^\xi d\mu(\omega) = DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \eta_t^\xi \tag{55e}$$

$$\hat{F}^i_{a_j \xi} = \int_\Omega \left(\sum_{k=1}^n DF^i_{y_j y_k} \cdot b_k \right) \eta_t^\xi d\mu(\omega) = 0 \tag{55f}$$

$$\hat{F}^i_{b_j b_k} = \int_\Omega DF^i_{y_j y_k} \cdot \left(\lambda(\xi) \cdot \eta_t^\xi + \varepsilon_{t+1}(\omega) \right)^2 d\mu(\omega) = DF^i_{y_j y_k} \left(\left(\lambda(\xi) \cdot \eta_t^\xi \right)^2 + \sigma^2_\varepsilon \right) \tag{55g}$$

$$\hat{F}^i_{b_j \xi} = \int_\Omega \left(\left(\sum_{k=1}^n DF^i_{y_j y_k} \cdot b_k \right) \eta_t^\xi + DF^i_{y_j} \cdot \eta_t^\xi \right) d\mu(\omega) = DF^i_{y_j} \cdot \eta_t^\xi \tag{55h}$$

Derivatives of $T^{a_i} = \int_\Omega \hat{F}^i(a,b,\xi,\eta^\xi(\omega))d\mu(\omega)$

$$T^{a_i}_{a_j} = \int_\Omega \hat{F}^i_{a_j} d\mu(\omega) = DF^i_{y_j} \tag{56a}$$
$$T^{a_i}_{b_j} = \int_\Omega \hat{F}^i_{b_j} d\mu(\omega) = \int_\Omega DF^i_{y_j} \cdot \lambda(\xi) \cdot \eta^\xi(\omega) d\mu(\omega) = 0 \tag{56b}$$
$$T^{a_i}_{\xi} = \int_\Omega \hat{F}^i_{\xi} d\mu(\omega) = 0 \tag{56c}$$
$$T^{a_i}_{a_j a_k} = \int_\Omega \hat{F}^i_{a_j a_k} d\mu(\omega) = DF^i_{y_j y_k} \tag{56d}$$
$$T^{a_i}_{a_j b_k} = \int_\Omega \hat{F}^i_{a_j b_k} d\mu(\omega) = \int_\Omega DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \eta^\xi(\omega) d\mu(\omega) = 0 \tag{56e}$$
$$T^{a_i}_{a_j \xi} = \int_\Omega \hat{F}^i_{a_j \xi} d\mu(\omega) = 0 \tag{56f}$$
$$T^{a_i}_{b_j b_k} = \int_\Omega \hat{F}^i_{b_j b_k} d\mu(\omega) = \int_\Omega DF^i_{y_j y_k} \left(\lambda(\xi)^2 \cdot \eta^\xi(\omega)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF^i_{y_j y_k} \cdot \sigma^2_{\eta^\xi} \tag{56g}$$
$$T^{a_i}_{b_j \xi} = \int_\Omega \hat{F}^i_{b_j \xi} d\mu(\omega) = \int_\Omega DF^i_{y_j} \cdot \eta^\xi(\omega) d\mu(\omega) = 0 \tag{56h}$$

Derivatives of $T^{b_i} = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i(a,b,\xi,\eta^\xi(\omega))d\mu(\omega)$

$$T^{b_i}_{a_j} = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{a_j} d\mu(\omega) = 0 \tag{57a}$$
$$T^{b_i}_{b_j} = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{b_j} d\mu(\omega) = \frac{1}{\sigma^2_{\eta^\xi}} \cdot DF^i_{y_j} \cdot \lambda(\xi) \cdot \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = DF^i_{y_j} \cdot \lambda(\xi) \tag{57b}$$
$$T^{b_i}_{\xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\eta^\xi(\omega)\hat{F}^i_\xi + \hat{F}^i \tfrac{\partial}{\partial\xi}\eta^\xi(\omega)\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}^i d\mu(\omega)\right) = 0 \tag{57c}$$
$$T^{b_i}_{a_j a_k} = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{a_j a_k} d\mu(\omega) = 0 \tag{57d}$$
$$T^{b_i}_{a_j b_k} = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{a_j b_k} d\mu(\omega) = \frac{1}{\sigma^2_{\eta^\xi}} \cdot DF^i_{y_j y_k} \cdot \lambda(\xi) \cdot \int_\Omega \eta^\xi(\omega)^2 d\mu(\omega) = DF^i_{y_j y_k} \cdot \lambda(\xi) \tag{57e}$$
$$T^{b_i}_{b_j b_k} = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{b_j b_k} d\mu(\omega) = \frac{1}{\sigma^2_{\eta^\xi}} \int_\Omega \eta^\xi(\omega) DF^i_{y_j y_k} \left(\left(\lambda(\xi)\cdot\eta^\xi(\omega)\right)^2 + \sigma^2_\varepsilon\right) d\mu(\omega) = DF^i_{y_j y_k} \cdot \lambda(\xi)^2 \left(\frac{\mu^{\eta^\xi}_3}{\sigma^2_{\eta^\xi}}\right) \tag{57f}$$
$$T^{b_i}_{b_j \xi} = \left(\sigma^2_{\eta^\xi}\right)^{-2} \left(\sigma^2_{\eta^\xi} \int_\Omega \left(\eta^\xi(\omega)\hat{F}^i_{b_j\xi} + \hat{F}^i_{b_j} \tfrac{\partial}{\partial\xi}\eta^\xi(\omega)\right) d\mu(\omega) - \tfrac{\partial}{\partial\xi}\sigma^2_{\eta^\xi} \int_\Omega \eta^\xi(\omega)\hat{F}^i_{b_j} d\mu(\omega)\right) = DF^i_{y_j}\left(\frac{1-2\lambda(\xi)^2}{1-\lambda(\xi)^2}\right) \tag{57g}$$

The above computations show that $DT = DF \oplus \lambda(\xi)DF$. Next, let $\hat{S} = S \oplus S$, $\theta = (a', b')'$ and $\phi = \hat{S}^{-1}\theta$, and consider the dynamic system

$$\dot{\phi} = \hat{S}^{-1}T(\hat{S}\phi, \xi) - \phi = \hat{H}(\phi, \xi), \tag{58}$$

which is topologically equivalent to the E-stability differential equation of our economic model, except now, to first order, the dynamics are decoupled. In particular, after adjoining ξ as usual, we may write the dynamic system (58) as

$$\begin{pmatrix} \dot{\phi}_1 \\ \vdots \\ \dot{\phi}_n \\ \dot{\phi}_{n+1} \\ \vdots \\ \dot{\phi}_{2n-1} \\ \dot{\phi}_{2n} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} \beta_1 - 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta_n - 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & \frac{\beta_1}{\beta_n} - 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{\beta_{n-1}}{\beta_n} - 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \\ \phi_{n+1} \\ \vdots \\ \phi_{2n-1} \\ \phi_{2n} \\ \xi \end{pmatrix} + \begin{pmatrix} f^1(\phi,\xi) \\ \vdots \\ f^n(\phi,\xi) \\ f^{n+1}(\phi,\xi) \\ \vdots \\ f^{2n-1}(\phi,\xi) \\ g(\phi,\xi) \\ 0 \end{pmatrix}, \tag{59}$$

where f^i and g comprise higher-order terms.

The center manifold is parameterized by $\phi_i = h^i(\phi_{2n},\xi)$ for $i = 1,\dots 2n-1$. Invariance provides the following functional equations in ϕ_{2n} and ξ:

$$h^i_{\phi_{2n}} \cdot g = f^i - D\hat{H}_{ii} \cdot h^i. \tag{60}$$

These may be used to compute a second-order approximation to the h^i. Finally, the projected dynamics are given by

$$\dot{\phi}_{2n} = g\left(h^1(\phi_{2n},\xi),\dots,h^{2n-1}(\phi_{2n},\xi),\phi_{2n},\xi\right) \equiv G(\phi_{2n},\xi).$$

We now turn to bifurcation analysis of $\dot{\phi}_{2n} = G(\phi_{2n},\xi)$.

Note that G is second order: $G = G_\star = 0$. Thus, to show that a transcritical bifurcation occurs it suffices to show that $G_{\phi_{2n}\phi_{2n}}$ and $G_{\phi_{2n}\xi}$ are non-zero. Using $h^i = h^i_\star = 0$ we find that

$$G_{\phi_{2n}\phi_{2n}} = g_{\phi_{2n}\phi_{2n}} \text{ and } G_{\phi_{2n}\xi} = g_{\phi_{2n}\xi},$$

just as in previous arguments.

Recalling that $S^{-1} = (S^{ij})$ we find

$$g(*,\phi_{2n},\xi) = \sum_{i=1}^{n} S^{ni} \cdot T^{b_i}(*,b_1(\phi_{2n}),\dots,b_n(\phi_{2n}),\xi),$$

where $b_i(\phi_{2n}) = * + S_{in} \cdot \phi_{2n}$, and here and below an "$*$" captures terms that are not relevant to the local argument. We compute

$$
\begin{aligned}
g_{\phi_{2n}} &= \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot T^{b_i}_{b_j} \\
g_{\phi_{2n}\phi_{2n}} &= \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot \sum_{k=1}^{n} S_{kn} \cdot T^{b_i}_{b_j b_k} = \lambda(\xi)^2 \left(\frac{\mu_3^{\varepsilon}}{\sigma^2_{\eta^{\xi}}} \right) \sum_{i=1}^{n} S^{ni} \left(S_n' \cdot D^2 F^i \cdot S_n \right) \\
g_{\phi_{2n}\xi} &= \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot T^{b_i}_{b_j \xi} = \left(\frac{1-2\lambda(\xi)^2}{1-\lambda(\xi)^2} \right) \sum_{i=1}^{n} S^{ni} \cdot \sum_{j=1}^{n} S_{jn} \cdot DF^i_j \\
&= \left(\frac{1-2\lambda(\xi)^2}{1-\lambda(\xi)^2} \right) \sum_{i=1}^{n} S^{ni} \cdot DF^i \cdot S_n = \left(\frac{1-2\lambda(\xi)^2}{1-\lambda(\xi)^2} \right) \beta_n .
\end{aligned}
$$

Existence is now established as in case 1 of the proof of Theorem 1, and stability follows from the topological equivalence of (58) with the E-stability ode, together with the fact that, under the assumptions, the non-zero eigenvalues of $D\hat{H}$ are negative. ■

Details of the NK example

A unit mass of households indexed as $\omega \in \Omega$ maximizes discounted expected utility, where the utility flow is given by

$$
\frac{1}{1-\sigma}\left(c_t(\omega)^{1-\sigma} - 1\right) + \log\left(\frac{m_{t-1}(\omega)}{\pi_t(\omega)}\right) - \frac{h_t(\omega)^{1+\chi}}{1+\chi} - \frac{\gamma}{2}\left(\frac{p_t(\omega)}{p_{t-1}(\omega)} - \pi^*\right)^2 ,
$$

where $\sigma, \chi, \gamma > 0$. Here household ω's consumption index is

$$
c_t(\omega) = \left(\int c_t(\omega, \varpi)^{\frac{\nu-1}{\nu}} d\varpi \right)^{\frac{\nu}{1-\nu}} ,
$$

with $c_t(\omega, \varpi)$ denoting the consumption by household ω of good ϖ. Household ω produces the quantity $y_t(\omega)$ of good ω using labor $h_t(\omega)$ via the technology $y_t(\omega) = h_t(\omega)^{\alpha}$, for $0 < \alpha < 1$. The household then sells this good at price $p_t(\omega)$ under conditions of monopolistic competition against the demand curve $p(\omega) = p \cdot (y(\omega)/y)^{-\frac{1}{\nu}}$, for $\nu > 1$, where $p^{1-\nu} = \int p(\omega)^{1-\nu} d\omega$ and $y^{\frac{\nu-1}{\nu}} = \int y(\omega)^{\frac{\nu-1}{\nu}} d\omega$ are the usual CES aggregates. Also, $m_{t-1}(\omega)$ is real money holdings at $t-1$ and $\pi_t(\omega) = p_t(\omega)/p_{t-1}(\omega)$ is the household-specific inflation rate. The first three terms of the utility flow are standard. The fourth term involving $p_t(\omega)/p_{t-1}(\omega)$ reflects the internalized cost of price adjustment, with π^* corresponding to the inflation target.

The budget constraint of the household is

$$
c_t(\omega) + m_t(\omega) + b_t(\omega) + \mathscr{T}_t = \frac{m_{t-1}(\omega)}{\pi_t} + \frac{R_{t-1}}{\pi_t} b_{t-1}(\omega) + \frac{p_t(\omega)}{p_t} y_t(\omega),
$$

where $b_{t-1}(\omega)$ is the real bond holdings at the start of period t ,R_{t-1} is the nominal interest rate factor, $\pi_t = p_t/p_{t-1}$ is the aggregate inflation factor, and $\mathscr{T}_t$ is lump-sum taxes net of transfers (including real monetary injections).

The government prints money and levies lump sum taxes. The government's flow constraint is given by

$$m_t^s + \mathscr{T}_t = \frac{m_{t-1}^s}{\pi_t}. \tag{61}$$

Through transfers, the government may choose m_t^s to implement its instrument rule.

Aggregate market-clearing conditions are given by

$$y_t = \int c_t(\omega)d\omega, \;\; m_t^s = \int m_t(\omega)d\omega, \;\text{ and }\; 0 = \int b_t(\omega)d\omega.$$

These equations, coupled with homogeneity and the agents' first-order conditions, yield the non-linear three-equation model, reproduced here for reference:

$$y_t^{-\sigma} = \beta \cdot R_t E_t \pi_{t+1}^{-1} y_{t+1}^{-\sigma} \tag{62}$$

$$\gamma \cdot \pi_t(\pi_t - \pi^*) = \beta \cdot \gamma \cdot E_t \pi_{t+1}(\pi_{t+1} - \pi^*) + (\nu/\alpha)\, y_t^{\frac{1+\chi}{\alpha}} + (1-\nu) y_t^{1-\sigma} \tag{63}$$

$$R_t = R^* \left(E_t\left(\frac{\pi_{t+1}}{\pi^*}\right)\right)^{\alpha_\pi \cdot \pi^*} \left(E_t\left(\frac{y_{t+1}}{y^*}\right)\right)^{\alpha_y \cdot y^*} e^{v_t} \tag{64}$$

$$v_t = \rho v_{t-1} + u_t. \tag{65}$$

To apply the MSV-principle, we must linearize our model. Recall that we set the inflation target to one, so that $R^* = \beta^{-1}$. Linearizing the model about the steady state, we obtain

$$\begin{aligned} dy_t &= E_t dy_{t+1} + (y^*/\sigma)\left(\beta dR_t - E_t d\pi_{t+1}\right) \\ d\pi_t &= \beta E_t d\pi_{t+1} + \kappa dy_t \\ dR_t &= (\alpha_\pi/\beta)\, E_t d\pi_{t+1} + (\alpha_y/\beta)\, E_t dy_{t+1} + R^* dv_t, \end{aligned}$$

where

$$\begin{aligned} y^* &= \left(\frac{\alpha(\nu-1)}{\nu}\right)^{\frac{\alpha}{1+\chi+\alpha(\sigma-1)}} \\ \kappa &= \frac{1}{\gamma}\left(\frac{(1+\chi)\nu}{\alpha^2}\,(y^*)^{\frac{1+\chi}{\alpha}-1} + (1-\nu)(1-\sigma)\,(y^*)^{-\sigma}\right), \end{aligned}$$

and where $d\star$ represents the deviation of the variable $\star$ from its steady-state value (of course, $dv = v$). Combining the interest-rate rule with the IS equation eliminates the dependency on dR_t, so that the linearized model may be written $dx_t = FE_t dx_{t+1} + Gdv_t$ for appropriate matrices F, G, where we recall that $x = (y, \pi)'$.

The MSV solution of the linearized model is given by $dx_t = Cdv_t$, for appropriate matrix C. If both eigenvalues of F have modulus less than one then the linearized model is determinate (or, equivalently, the steady state of the nonlinear model) is locally determinate; in this case the MSV solution is the unique non-explosive REE of the linearized model. If one or both eigenvalues of F lies outside the unit circle the linearized model is indeterminate. Finally, regardless of the linearized model's determinacy status, it is well-known and straightforward to show that the MSV solution is E-stable provided the eigenvalues of F have real part less than unity.

For calibration used in our example only one of the eigenvalues of F is outside the unit circle. Write $\xi_{t+1} = dx_{t+1} - E_t dx_{t+1}$ so that

$$\begin{pmatrix} dx_t \\ dv_t \end{pmatrix} = \begin{pmatrix} F^{-1} & -F^{-1}G \\ 0 & \rho \end{pmatrix} \begin{pmatrix} dx_{t-1} \\ dv_{t-1} \end{pmatrix} + \begin{pmatrix} I_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_t \\ du_t \end{pmatrix}.$$

Next, write

$$\begin{pmatrix} F^{-1} & -F^{-1}G \\ 0 & \rho \end{pmatrix} = S(\lambda_1 \oplus \lambda_2 \oplus \rho)S^{-1}$$

with $|\lambda_1| > 1 > |\lambda_2|$. Noting that the bottom row of S^{-1} is $(0,0,1)$, and changing coordinates to $(dz,dv)' = S^{-1}(dx,dv)'$, we obtain the system $z_{it} = \lambda_i z_{it-1} + \tilde{\xi}_{it}$ for $i = 1,2$, where $\tilde{\xi}_t = S^{-1}(\xi_t, du_t)'$.

Write S^{ij} as the ij-th entry of S^{-1}. Non-explosiveness requires that $z_{1t} = 0$. This restriction pins down one of the forecast errors, i.e.

$$\tilde{\xi}_{1t} = S^{11}\xi_{1t} + S^{12}\xi_{2t} + S^{13}du_t = 0.$$

If we select the forecast errors so that we also have $\tilde{\xi}_{2t} = 0$ then the associated equilibrium may be written

$$\begin{pmatrix} dy_t \\ d\pi_t \end{pmatrix} = -\begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}^{-1} \begin{pmatrix} S^{31} \\ S^{32} \end{pmatrix} dv_t,$$

which is exactly the MSV solution $dx_t = Cdv_t$ referenced above. If, on the other hand, $\tilde{\xi}_{2t} \neq 0$, then, writing $\eta_t = \lambda_2 \eta_{t-1} + \tilde{\xi}_{2t}$, the associated equilibrium becomes

$$\begin{pmatrix} dy_t \\ d\pi_t \end{pmatrix} = -\begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}^{-1} \begin{pmatrix} S^{31} \\ S^{32} \end{pmatrix} dv_t - \begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \eta_t,$$

which has the form $dx_t = Cdv_t + D\eta_t$ for appropriate matrix D. In particular, this equilibrium presents as the MSV solution plus a serially correlated sunspot with resonance frequency λ_2, as analogous to sunspot equilibria in the univariate linearized model of the main text.