Expectations, Stagnation and Fiscal Policy: a Nonlinear Analysis

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Abstract

Stagnation and fiscal policy are examined in a nonlinear stochastic New-Keynesian model with adaptive learning. There are three steady states. The steady state targeted by policy is locally but not globally stable under learning. A severe pessimistic expectations shock can trap the economy in a stagnation regime, underpinned by a low-level steady state, with falling inflation and output. A large fiscal stimulus may be needed to avoid or emerge from stagnation, and the impacts of forward guidance, credit frictions, central bank credibility and policy delay are studied. Our model encompasses a wide range of outcomes arising from pessimistic expectations shocks.


Key words: Stagnation Trap, Expectations, Fiscal Policy, Adaptive Learning, New-Keynesian Model.

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1 Introduction

The sluggish macroeconomic performance of advanced market economies in the years following the Great Recession has raised interest in the possibility of the economy becoming stuck for long periods in a distinct stagnation regime associated with the zero lower bound (ZLB) for the policy interest rate. The global COVID-19 pandemic has also raised longer-term concerns about stagnation. One possible explanation for a stagnation regime is that it is caused by a widespread lack of confidence on the part of economic agents. Specifically, the economy can become confined to a region with low output, deflation or below-target inflation, and interest rates constrained by the ZLB.

The recent pattern within many economies of extended periods of below-target inflation rates, negative output gap and near-zero policy interest rates, can be seen in the two panels of Figure 1, showing quarterly data from 2002Q1 to 2021Q2, of the US, Japan and the Euro area. The left panel gives a scatter plot of (core) inflation vs. the policy interest rate, for each country/area, as originally done in Bullard (2010) for Japan and US data and extended by Honkapohja (2016) to include Euro area data, in the context of the Fisher equation and an interest-rate policy rule. These equations identify the two steady states emphasized by rational expectations (RE): the targeted steady state corresponding to a two-percent inflation target and an unintended steady state with mild deflation at near-zero net interest rates.

The second panel plot, for each country/area gives the data on the output gap vs. the policy interest rate. This panel makes it evident that policy interest rates near or at the ZLB are frequently associated with negative output gaps. Since the unintended RE deflation steady state has a negligible output gap in standard New Keynesian models, it is clearly challenging to interpret coordination on this steady state as the main focus for explaining macroeconomic outcomes at the ZLB.

We develop an extension of a standard new Keynesian (NK) model in which there exists a stagnation regime – a region of pessimistic expectations anchored by a stagnation steady-state. Our analysis goes beyond RE by assuming that economic agents make forecasts using adaptive learning (AL). We show that in the stagnation region expectations become trapped, with the stagnation steady state acting as an attractor, preventing a return to the targeted steady state. Existence of this stagnation regime is consistent with the observation above that, under the ZLB constraint, real economic performance of the US, Japanese and the euro area economies appears to be clearly worse than in the earlier period before the ZLB became binding.

\footnote{For alternative explanations of long-lasting stagnation see, e.g., Summers (2013), Teulings and Baldwin (2014), Eggertsson, Mehrotra, and Robbins (2019) and Benigno and Fornaro (2018).}
Our approach centers squarely on the role of expectations. In line with the AL literature our agents are assumed to be boundedly rational: expectations are formed using statistical models that have the potential to converge to RE, but which can also sometimes follow trajectories away from the targeted steady state. Much of the RE literature focuses on managing an economy subject to large finite-duration, exogenous discount-rate or financial shocks, the stochastic properties of which are known, whereas our story stresses the role of pessimistic expectational overhang, continuing after the cessation of fundamental shocks, which can prevent the economy from returning to the targeted steady state.\footnote{See Section 7 for more detailed discussion of the literature.}

As in the RE literature, the ZLB plays a key role in our model, but our focus under AL is on local and global stability, not on indeterminacy or on self-fulfilling rational “sunspot” equilibria. In Section 2 we employ the basic Rotemberg adjustment-cost version of the NK model with AL, extended to include partial substitutability between private and public consumption. Section 3 develops the central result that this version of the benchmark NK model has three steady states, two of which are locally stable under AL: the targeted steady state and a subsistence-level “stagnation” steady state. The third (“unintended”) indeterminate steady state remains of interest because it lies on the edge of the domain of attraction of the targeted steady state: for a range of pessimistic expectations, the economy is drawn toward it before veering either to the targeted steady state or into the stagnation regime.

Figure 1: Left panel – inflation vs. policy interest rate. Right panel – output gap vs. policy interest rate. See Online Appendix F for details.
After establishing these central features of the economy under AL, we consider fiscal policy in the face of an adverse expectation shock. The analysis is carried out in a stochastic nonlinear economy in Section 4. At each point in time, aggregate output, consumption and inflation arise as the temporary equilibrium implied by exogenous shocks and agents’ decision rules. The latter in turn depend on point expectations of future variables obtained from forecast rules based on observed shocks, with coefficients updated over time using recursive least-squares learning.

The starting point for our approach is that low output and inflation during a period of adverse exogenous shocks, may have made agents pessimistic about the future. These pessimistic expectations may continue for a time after the shocks have ceased, and the subsequent dynamics can depend sensitively on the position of these expectations relative to the domain of attraction of the targeted steady state. A key policy question in this case is whether fiscal policy can prevent stagnation and return the economy to the targeted steady state.

Section 5 turns to policy, focusing on situations in which output expectations are sufficiently pessimistic that with high likelihood they would lead, under unchanged policy, to the economy becoming trapped in the stagnation regime. We provide numerical results for the success of a fiscal stimulus, of stated magnitude and duration, in moving the economy to a path converging to the targeted steady state. The impact of fiscal policy is highly nonlinear: for a given duration, a small stimulus may be unsuccessful, while a larger temporary stimulus can be effective in returning the economy to the targeted steady state. Success is stochastic since convergence to the targeted steady state depends in part on the sequence of stochastic shocks. The probability of success, i.e. avoiding stagnation, depends on the magnitude and length of fiscal stimulus.

Section 6 considers the implications of important extensions: (i) Combining expansionary fiscal policy with forward guidance in monetary policy can be beneficial; (ii) Policy delays reduce the efficacy of fiscal policy; (iii) With financial frictions the stagnation regime can include points with normal output expectations and positive but low inflation expectations; (iv) A higher inflation target enlarges the domain of attraction of the targeted steady state; (v) The likelihood of stagnation is reduced if the inflation target has substantial credibility. This section also illustrates the potential of our model to fit observed data using scatterplots from simulations.

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3 A similar point arises in connection with the large negative productivity, labor supply and sectoral shocks since 2020 due to the coronavirus pandemic. The course of the economy will depend heavily on the course of the “intrinsic” virus shocks. However, even after these shocks have receded, there may well be a pessimistic overhang of the type considered in this paper.
A discussion of related literature is set out in Section 7. Section 8 concludes. The Online Appendices contain numerous technical details and further results.

2 The Model

Our model is a generalization of Benhabib, Evans, and Honkapohja (2014). There is a continuum of identical household-producers $i \in [0, 1]$. Agent $i$ maximizes utility subject to flow budget and production function constraints:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log(c_{t,i} + \xi g_t) + \varkappa \log \left( \frac{M_{t-1,i}}{P_t} \right) - (1 + \varepsilon)^{-1} h_{t,i}^{\varepsilon} - \Phi \left( \frac{P_{t,i}}{P_{t-1,i}} \right) \right\}$$

s.t. $c_{t,i} + m_{t,i} + b_{t,i} + \Upsilon_{t,i} = m_{t-1,i} \pi_t^{-1} + R_{t-1} \pi_t^{-1} b_{t-1,i} + \frac{P_{t,i} y_{t,i}}{P_t}$ and $y_{t,i} = A_t h_{t,i}^{\alpha}$.

Here $0 < \alpha, \beta < 1$. $c_{t,i}$ is the consumption aggregator consumed by $i$, $M_{t,i}$ and $m_{t,i} = M_{t,i}/P_t$ denote nominal and real money balances, $h_{t,i}$ is the labor input into production of good variety $i$ and $b_{t,i}$ denotes the real quantity of risk-free one-period nominal bonds held by the agent $i$ at the end of period $t$. $g_t$ is exogenous government spending per capita, $\Upsilon_{t,i}$ is the lump-sum tax collected by the government, $R_{t-1}$ is the nominal interest rate factor between $t-1$ and $t$, $P_{t,i}$ is the price of consumption good $i$, $y_{t,i}$ is output of good $i$, $P_t$ is the aggregate price level, and the inflation rate is $\pi_t = P_t/P_{t-1}$. $\Phi \geq 0$ captures a convex pricing friction, with $\Phi(\pi^*) = 0$, where $\pi^*$ is the inflation rate targeted by policymakers. $A_t$ is a productivity shock to all firms with mean $\bar{A} > 0$. The household is subject to the usual “no Ponzi game” condition.

Utility of consumption includes both private consumption $c_{t,i}$ and public consumption $g_t > 0$ of the goods aggregator, with relative weight parameter $0 < \xi \leq 1$ capturing the degree of substitution between private and public consumption as in Christiano and Eichenbaum (1992). Under standard policy $g_t = \bar{g} > 0$. Note that $\Phi(.)$ gives the (utility) cost of adjusting prices, which arises if agent $i$ changes prices at a different rate from the central bank inflation target. We use the utility Rotemberg formulation, with household-producers, rather than either an output cost version or the Calvo model of price stickiness, because this enables us to study global dynamics in the nonlinear system. The parametric form of $\Phi$ is discussed below.

The consumption aggregator takes the usual CES form, with elasticity of substitution between two goods $\nu_t > 1$, where $\nu_t$ is a stationary AR(1) process. Output is differentiated and firms operate under monopolistic competition. Each household-
firm faces a downward-sloping demand curve,

$$P_{t,i} = \left( \frac{y_{t,i}}{y_t} \right)^{-1/\nu_t} P_t,$$

where $P_t = \left[ \int_0^1 P_{t,i}^{1-\nu_t} \, di \right]^{1/(1-\nu_t)} \cdot \tag{2}$$

Finally, the government faces the usual flow budget constraint – see Appendix A – and households are assumed Ricardian in the sense that they expect the government’s intertemporal budget constraint to be satisfied. In particular, the tax implications for an increase in government spending would be fully anticipated by the households.

In line with the AL literature, our approach consists of three key pieces:

- Specification of agent decision rules, for consumption and price setting, conditional on current and expected future variables.
- Temporary equilibrium equations for a representative agent (RA) economy, based on aggregation and market clearing, given monetary and fiscal policy.
- Updating of agent forecast rule parameters using statistical learning.

The equilibrium path is then determined recursively. This general set-up essentially implements the temporary equilibrium concept, introduced by Hicks (1939) and the Stockholm school of economic thought, within a dynamic setting in which expectations are updated over time in accordance with the AL approach. In the context of infinite-horizon agents solving dynamic optimization problems, our approach can be viewed as a version of the “anticipated utility” approach formulated by Kreps (1998), discussed in Sargent (1999) and Cogley and Sargent (2008).

We now turn to the formal description of the model. Appendix A gives the details. The decision rule of agent $i$ for consumption $c_{t,i}$ is obtained by combining their iterated consumption Euler equations, under subjective expectations, with the household’s Ricardian perceived intertemporal budget constraint.\footnote{Non-Ricardian households were considered in Benhabib, Evans, and Honkapohja (2014).} An additional bounded-rationality assumption is imposed concerning expressions with conditional expectations of nonlinear functions of future random variables. Even if agents knew the required joint probability distributions this would be a difficult calculation, and since the distributions are unknown, they would need to be estimated. We make the assumption, which we view as realistic, that agents instead use point expectations, treating the conditional expectation of a nonlinear function of random variables as equal to the nonlinear function of the conditional expectations. Put differently, they
act as if all conditional probability density of each random variable were concentrated at its expected value. This assumption is natural because it can plausibly be implemented by agents to approximate optimal decision-making.

Using superscript $e$ to denote subjective expectations, and letting $\Xi_{t,i} = P_{t,i}/P_t$, it is shown in Appendix A that consumption is given by

$$c_{t,i} = (1 - \beta) \left[ \Xi_{t,i} y_{t,i} - g_t (1 + \xi \beta / (1 - \beta)) \right] + (1 - \beta) \sum_{s=1}^{\infty} \left( D_{t,t+s}^e \right)^{-1} \left[ \Xi_{t+s,i} y_{t+s,i}^e - g_{t+s,i}^e (1 - \xi) \right],$$

if this is nonnegative, else $c_{t,i} = 0$. Here $D_{t,t+s}^e = \prod_{j=1}^{s} r_{t+j}^e$, for $r_{t+j} \equiv R_{t+j-1}/\pi_{t+j}$, are the perceived discount factors. This decision rule depends on forecasts of future incomes $\Xi_{t+s,i} y_{t+s,i}^e$, government consumption $g_{t+s,i}^e$, and discount factors $D_{t,t+s}^e$.

The agent’s production and pricing decisions are governed by the pricing Euler equation. For the adjustment cost function $\Phi(P_{t,j}/P_{t-1,j})$ we use the Linex function given in Appendix A. This form makes deflation more costly, which is often regarded as more plausible, and provides a flexible way to capture downward price rigidity. Appendix A shows that iterating the Euler equation for price setting and assuming finite-horizon anticipated-utility optimizers, but in contrast to them our agents do not assume that their future pricing decisions $P_{t+s,i}$, for example, will be consistent with what would be their optimal choices under current expectations of the variables exogenous to their decision-making, including future aggregate inflation and aggregate output. Instead we assume agents use AL based on observed data to forecast $\Xi_{t+s,i}$, $y_{t+s,i}$ and $(c_{t+s,i} + \xi g_{t+s})^{-1}$, an assumption we view as plausible and

\[ \Phi'(\pi_{t,i}) \pi_{t,i} = \zeta_{t,i} + \sum_{s=1}^{\infty} \beta^s \zeta_{t+s,i}^e, \text{ where} \]

\[ \zeta_{t,i} = \nu_t \alpha^{-1} (y_{t,i}/A_t)^{(1+\xi)/\alpha} - (\nu_t - 1) (c_{t,i} + \xi g_t)^{-1} y_t \Xi_{t,i}^{1-\nu_t} \text{ and } \Xi_{t,i} = P_{t,i}/P_t. \]

Decision rules (3) and (4) require agents to make forecasts of various future variables, and to proceed further we make an additional bounded-rationality assumption below. To forecast $\zeta_{t,s,i}^e$ agent $i$ needs to forecast future exogenous variables $\nu_{t+s}$, $A_{t+s}$, $g_{t+s}$, aggregate output $y_{t+s}$, the discount factor $D_{t,t+s}$, but also the agent’s relative price $\Xi_{t+s,i} = P_{t+s,i}/P_{t+s}$, market demand $y_{t+s,i}$, and marginal utility $(c_{t+s,i} + \xi g_{t+s})^{-1}$. Thus formally (3) and (4) are conditional decision-rules.

Our approach is to assume that agents use these conditional decision rules supplemented by forecasts of future variables, including some that they themselves will be setting. Thus we share with Eusepi and Preston (2010) the assumption that agents are infinite-horizon anticipated-utility optimizers, but in contrast to them our agents do not assume that their future pricing decisions $P_{t+s,i}$, for example, will be consistent with what would be their optimal choices under current expectations of the variables exogenous to their decision-making, including future aggregate inflation and aggregate output. Instead we assume agents use AL based on observed data to forecast $\Xi_{t+s,i}$, $y_{t+s,i}$ and $(c_{t+s,i} + \xi g_{t+s})^{-1}$, an assumption we view as plausible and
which also simplifies our model and makes possible a nonlinear global analysis.\(^5\)

In line with the anticipated utility approach agents update forecasts over time but do not explicitly take into account that their forecasting model parameters will change over time. This is a boundedly rational decision-making approach widely used in the AL literature; see Cogley and Sargent (2008) and Sargent (2008).

Because of our representative agent framework, in which all agents behave identically, in temporary equilibrium \(\Sigma_{t,i} = 1\) for all agents \(i\) at all times \(t\). Under AL, agents will therefore learn over time that \(\Xi_{t+s,i}^{e} \rightarrow 1\) with probability one. Similarly, \(y_{t,i} = y_{t}\) and \(c_{t,i} = c_{t}\), so that under AL we would have \(y_{t+s,i}^{e} \rightarrow y_{t+s}^{e}\) and \(c_{t+s,i}^{e} \rightarrow c_{t+s}^{e}\). Although we could allow for initial out-of-equilibrium expectations for these variables, this would add little to our analysis. Thus we now assume \(c_{t+s,i}^{e} = c_{t+s}\). Furthermore, from market clearing \(y_{t} = c_{t} + g_{t}\), and \(c_{t} + \xi g_{t} = y_{t} - (1 - \xi)g_{t}\), so we can assume \(c_{t+s,i}^{e} + \xi g_{t+s} = y_{t+s}^{e} - (1 - \xi)g_{t+s}^{e}\).

We can now list the representative-agent temporary equilibrium equations:

\[
y_t = \max \left\{ g_t, (1 - \xi)g_t + (\beta^{-1} - 1) \left[ \sum_{s=1}^{\infty} (D_{t,t+s}^e)^{-1} (y_{t+s}^e - (1 - \xi)g_{t+s}^e) \right] \right\}
\tag{5}
\]

\[
\Phi'(\pi_t)\pi_t = \zeta_t + \sum_{s=1}^{\infty} \beta^s \zeta_{t+s}^e, \quad \text{where}
\tag{6}
\]

\[
\zeta_{t+s}^e = \alpha^{-1} \nu_{t+s}^e \left( y_{t+s}^e / A_{t+s}^e \right)^{1+\varepsilon} - (\nu_{t+s} - 1) y_{t+s}^e (y_{t+s}^e - (1 - \xi)g_{t+s}^e)^{-1} \quad \text{for} \quad s \geq 0.
\tag{7}
\]

There remains only to specify the policy variables and to discuss \(D_{t,t+s}^e\). In normal times \(g_t = \bar{g}\) is fixed. When active fiscal policy is used, it will follow an announced exogenous path. Monetary policy follows the forward-looking interest rate rule\(^6\)

\[
R_t = R(\pi_{t+1}^e, y_{t+1}^e) = 1 + (R^* - 1) \left( \pi_{t+1}^e / \pi^* \right)^{BR^*/(R^*-1)} (y_{t+1}^e/y^*)^{\phi_y},
\tag{8}
\]

where \(B > 1\) and \(\phi_y \geq 0\). Here \(R^* = \beta^{-1} \pi^*\) and \(y^*\) is the target level of output, assumed equal to the output level at the nonstochastic targeted steady state. Note \(R_t \geq 1\), i.e. the \(R_t\) satisfies the ZLB for net interest rates. Agents are assumed to

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\(^5\)The assumption that agents forecast some future variables that are under their control has also been used by Eusepi and Preston (2012) and Woodford (2013).

\(^6\)We have also considered contemporaneous rules \(R_t = R(\pi_t, y_t)\) with a similar functional form, and the main results appear unchanged. The rule (8) is formally and computationally simpler to implement. One interpretation of (8) is that the policy rate reacts to private-sector expectations.
Given the exogenous variables $A_t$, $\nu_t$, $g_t$ and expectations $\{y_{t+s}^e\}$, $\{\pi_{t+s}^e\}$, $\{A_{t+s}^e\}$, $\{\nu_{t+s}^e\}$, $\{g_{t+s}^e\}$, equations (5) - (9) determine the temporary equilibrium.

\section{Steady States and Learning Dynamics}

To examine our model under AL, we begin with a nonstochastic setting. The model has three interior perfect-foresight steady states. Under AL expectations are revised over time and the learning dynamics can be studied. AL rules are particularly simple in the nonstochastic case; thus, formal results for local stability can be obtained and the global dynamics characterized. We then extend AL to the stochastic model and study the learning dynamics numerically.

\subsection{Steady states and learning in nonstochastic case}

The nonstochastic model sets $\nu_t = \nu > 1$, $A_t = A$ and $g_t = \bar{g}$. Adaptive learning in nonstochastic models usually study agents attempting to learn a steady state using expectations based on long-run averages. Introducing the notation $y_{t+s}^e = y_{t}^e$, and $\pi_{t+s}^e = \pi_{t}^e$ for time $t$ expectations of future values for all horizons $s > 0$, AL takes the simple form

$$y_{t}^e = y_{t-1}^e + \omega(y_{t-1} - y_{t-1}^e)$$

$$\pi_{t}^e = \pi_{t-1}^e + \omega(\pi_{t-1} - \pi_{t-1}^e),$$

where $0 < \omega < 1$ is the learning “gain” parameter.\footnote{In the stochastic case, “decreasing gains” $\omega_t$ are sometimes used, in which $\omega_t$ is proportional to $t^{-1}$. See Appendix B for the recursive least squares equations.} The focus is usually on small and thus examines local stability of steady states for sufficiently small $\omega > 0$.

Rules of this form are often called “steady-state” learning, because they are simple adaptive rules that can converge to perfect foresight steady states.

For this setting, the temporary equilibrium equations simplify to\footnote{We remark that for a range of values of $\pi_{t}^e$, $y_{t}^e$ it is possible that $R(\pi_{t}^e, y_{t}^e) < \pi_{t}^e$. This issue is discussed further in Section 4 but it does not arise for local stability under steady state learning.}

$$y_{t} = \max \{\bar{g}, \bar{g}(1 - \xi) + (\beta^{-1} - 1) [(y_{t}^e - \bar{g}(1 - \xi))\pi_{t}^e / (R(\pi_{t}^e, y_{t}^e) - \pi_{t}^e)]\}$$

$$r_{t+j}^e \equiv R(\pi_{t+j}^e, y_{t+j}^e) / \pi_{t+j}^e, \text{ and } D_{t,t+s}^e = \prod_{j=1}^{s} r_{t+j}^e.$$
\[
\pi_t = Q^{-1} \left[ \left( \frac{\nu}{\alpha} \right) (y_t/A)^{(1+\epsilon)/\alpha} - (\nu - 1) y_t (1 - \xi \bar{g})^{-1} + \right. \\
\beta (1 - \beta)^{-1} \left[ \left( \frac{\nu}{\alpha} \right) (y_t^e/A)^{(1+\epsilon)/\alpha} - (\nu - 1) y_t^e (1 - \xi \bar{g})^{-1} \right] \right], \text{ or} \\
y_t = G_2(\pi_t^e, y_t^e) \quad \text{and} \quad \pi_t = G_1(y_t, y_t^e)
\]

in general notation. Here \( Q(\pi) \equiv \Phi'(\pi) \pi \).

In a perfect-foresight steady state, with \( y_t = y_t^e = y \) and \( \pi_t = \pi_t^e = \pi \), the temporary equilibrium equation for \( \pi_t \) implies

\[
(1 - \beta)\Phi'(\pi)\pi = (\nu/\alpha) (y/A)^{(1+\epsilon)/\alpha} - (\nu - 1) y \times (y - (1 - \xi \bar{g})^{-1}). \tag{11}
\]

The consumption Euler equation implies \( \beta^{-1} = r = R/\pi \), provided \( c > 0 \). The steady state targeted by monetary policy is at \( \pi = \pi^* \) with a corresponding output level \( y^* > \bar{g} \) given by (11). This is the value \( y^* \) used in (8), together with \( R^* = \pi^* / \beta \).

Smooth interest-rate rules that obey the Taylor principle, \( (d/d\pi^e)R(\pi^*, y^*) > \beta^{-1} \), imply a second steady state \( (\pi_L, y_L) \) with \( \pi_L < \pi^* \). In this “unintended” steady state \( R = \pi_L / \beta \) and \( y_L > \bar{g} \) is determined by (11). If the ZLB were strictly binding at \( \pi = \pi_L \), so that \( R = 1 \), then we would have \( \pi_L = \beta \), i.e. there would be a net deflation rate of \( 1 - \beta \). Under our calibration of (8) \( 1 > \pi_L > \beta \) with \( \pi_L \approx \beta \).

In our model there is also generally a third “stagnation” steady state, at \( y = \bar{g} \), \( c = 0 \) and deflation. The corresponding inflation rate \( \pi_S < \pi_L \) is determined from (11) by

\[
(1 - \beta)\Phi'(\pi_S)\pi_S = (\nu/\alpha) (\bar{g}/A)^{(1+\epsilon)/\alpha} - (\nu - 1) \xi^{-1}.
\]

The condition for existence of the stagnation steady state is \( \bar{g}/A < (\alpha (\nu - 1) / \nu \xi)^{(\alpha/(1+\epsilon))} \)

as \( \Phi'(\pi) \pi < 0 \) if and only if \( \pi < \pi^* \). For the calibration below, the condition \( \bar{g} < 1.338 \), approximately, is required.

In the stagnation steady state the consumption Euler equation, and the Fisher equation, are not satisfied with equality, since households are at the corner solution to \( c \geq 0 \). The nominal interest rate \( R \geq 1 \) is very close to the ZLB, i.e. \( R \approx 1 \), so the real interest rate is high, \( r \approx 1/\pi_S \). However, households cannot increase their saving because their income net of taxes is zero. They are required to pay their taxes, levied by the government to finance the production of public consumption goods \( \bar{g} \). Households use their labor to produce and sell sufficient goods to cover these taxes. They could increase their income further by increasing their labor and production, but the needed reduction in prices to sell the output would, due to the Rotemberg pricing friction, cause greater disutility. Household-producers are at a corner solution with private consumption zero but with positive public consumption and marginal utility bounded above zero.

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We emphasize that the stagnation steady state is extreme, and there is no suggestion that the economy has been or is likely to be in this steady state. Its importance and role is that it is a well-defined steady state that, as we will see, acts as an attractor outside the domain of attraction of the targeted steady state.

We now turn to stability of the three steady states $\pi^*, \pi^L$ and $\pi_S$ under AL. E-stability, defined in terms of the ordinary differential equation (ODE) given below, is known to be the condition for local convergence of steady state learning to a (steady-state) fixed point. In general, for a vector of learning parameters $\theta$, the E-stability ODE is $d\theta/d\tau = T(\theta) - \theta$, where $T(\theta)$ gives the corresponding actual temporary equilibrium outcome parameters corresponding to given perceived law of motion parameters $\theta$. Here $\tau$ denotes “notional” time, which can, however, be linked to real time $t$. From the above temporary equilibrium equations we obtain the E-stability differential equations:

$$
\frac{d\pi^e}{d\tau} = F_{\pi}(\pi^e, y^e) \equiv G_1(G_2(\pi^e, y^e), y^e) - \pi^e, \text{ and} \tag{12}
$$

$$
\frac{dy^e}{d\tau} = F_{y}(\pi^e, y^e) \equiv G_2(\pi^e, y^e) - y^e, \tag{13}
$$

so in our model $\theta = (\pi^e, y^e)^T$ and $T(\theta) = [F_{\pi}(\pi^e, y^e), F_{y}(\pi^e, y^e)]^T$. We have the following results, which are proved in Appendix E:

**Proposition 1.** (a) (i) The targeted steady state at $(\pi^*, y^*)$ is E-stable provided $\phi_y$ is not too large. (ii) The steady state $(\pi_L, y_L)$ is not E-stable if $\phi_y$ is not too large. (iii) The steady state $(\pi_S, y_S)$ is E-stable.

(b) Hence, provided $\phi_y$ is not too large and for all $\omega > 0$ sufficiently small, under the learning rule (10), the steady state $(\pi_L, y_L)$ is not locally stable and the steady states $(\pi^*, y^*)$ and $(\pi_S, y_S)$ are locally stable.

The condition $\phi_y$ not too large is standard and known to be necessary, with forward-looking interest rate rules, to avoid indeterminacy of the targeted steady state.

With temporary equilibrium of the nonlinear system fully specified, we next extend our analysis numerically to look at the global system under learning.

### 3.2 Global analysis of E-stability dynamics

For the nonstochastic system, the dynamics of the differential equations (12)-(13) give the global learning dynamics under (10) corresponding to small learning gain $\omega > 0$. In the current Section we assume government spending is constant, i.e. $\gamma = \bar{\gamma}$. The numerical values of the parameters are typical and correspond to a quarterly calibration given in Appendices C and D.
Figure 2 provides a sketch of the global E-stability dynamics that includes all three steady states: the targeted steady state \((\pi^*, y^*)\), the unintended liquidity trap steady state \((\pi_L, y_L)\), and the boundary stagnation steady state \((\pi_S, y_S = \bar{g})\).\(^9\) Steady states \((\pi^*, y^*)\) and \((\pi_L, y_L)\) have been widely discussed in the RE literature.

Figure 2 illustrates that the steady state at \(\pi^*\) is locally stable under learning dynamics, while the one at \(\pi_L\) is locally unstable under learning; these observations are well known, see e.g. Benhabib, Evans, and Honkapohja (2014).\(^{10}\) At the third, stagnation, steady state, output \(y = \bar{g}\) is at the minimal level, with households

\(^9\)Under our calibration, \(\pi^* = 1.005, y^* = 1.00003\) and \(R^* = \beta^{-1}\pi^* \approx 1.01515\). At the unintended steady state, \(\pi_L = 0.996393, y_L = 0.999862\) and \(R_L = \beta^{-1}\pi_L \approx 1.00646\). At the stagnation steady state, \(\pi_S = 0.647161, y_S = 0.2\) and \(R_S \approx 1\).

\(^{10}\)Quantitatively \(y_L\) is only slightly smaller than \(y^*\). This result is not sensitive to \(\xi\), the degree of substitutability between private and public consumption. Ceteris paribus, increases in \(\xi\) lead, via the labor-leisure choice, to approximately equal decreases in \(y^*\) and \(y_L\).
receiving only $\bar{g}$ as subsistence consumption (private consumption is zero), and there is rapid deflation and a high real interest rate. $(\pi_S, \bar{g})$ is locally stable, and more specifically is a sink with dynamics nearby that are not oscillatory. Interestingly, above the targeted steady state the bound $y_t \geq \bar{g}$ is also binding for sufficiently high values of $\pi_t^e$. This is because the real interest rate $R(\pi_t^e, y_t^e)/\pi_t^e$ then becomes very high (due to the Taylor rule) reducing $c_t$ to zero.$^{11}$

Noting the saddle-point nature of the unstable middle steady state $(\pi_L, y_L)$ in Figure 2, it is possible to construct the domain of attraction for the locally stable targeted steady state under the E-stability dynamics.$^{12}$ It will be convenient henceforth to refer to the targeted steady state domain of attraction as the DOA. In Figure 3 (left panel) the DOA is the “liver-shaped” region bounded by the thick solid (blue) curve with a narrow tail toward the north-west and is shaded (yellow) in the figure.$^{13}$ The targeted steady state is at $\pi^* = 1.005$ and $y^* = 1.00003$ and is shown by the star in Figure 3 (left panel). For any expectations $(\pi^e, y^e)$ inside the DOA, in the non-stochastic case under consideration, the economy will converge under learning to the targeted steady state, whereas it will diverge to the stagnation steady state from all points outside this domain.

In other words, under imperfect knowledge, there is a real possibility that after significant shocks, leading to an adverse shift in expectations $(\pi^e, y^e)$, the economy can move into, and become trapped in, a region leading to stagnation under unchanged monetary and fiscal policy.$^{14}$ It is convenient to refer to the part of the DOA of the stagnation steady state in which $\pi^e < \pi_L$ and $d\pi^e/d\tau < 0$ as the stagnation “regime” or “region” or as the “stagnation trap.” (The related term “deflation trap” is also used in the literature.) The underlying forces are, first, that the interest rate is at or near the ZLB and, second, that with output low, inflation and expected inflation are falling. Consequently, expected real interest rates are high and increasing. This in turn leads to lower demand and output leading to self-reinforcing stagnation dynamics. For future reference, when $\pi^e = \pi^*$ the lower boundary of the DOA is approximately $y^e = 0.98792$.

Figure 3 (right panel) shows that the model has sensitive dependence on initial

$^{11}$In Figure 2 this phenomenon would appear in the curve $d\pi^e/d\tau = 0$ which gradually turns near-horizontal large $\pi^e > \pi^*$.

$^{12}$We note one issue for global E-stability dynamics, which is that $R(\pi^e, y^e) < \pi^e$ for some configurations $(\pi^*, y^*)$. This issue does not arise in Figure 3, but would arise if $y^e < 0.9$ and $\pi^e \approx 1$. This point is addressed in Section 4 in the context of global numerical simulations.

$^{13}$The DOA extends beyond the range shown in the figure but becomes increasingly narrow.

$^{14}$Parameter values of the monetary policy rule matter for the size of the DOA, e.g. for $\phi_y = 0$ the DOA is smaller than in our base case. The preference parameter $\xi$ affects steady-state outputs $y^*$ and $y_L$ but the impact on the size of the DOA is small.
conditions in a relevant area of the state space. Consider time paths of the economy from a starting point at $\pi_0 = \pi^*$ and $y_0^e$ slightly below or slightly above the boundary of DOA $y^e = 0.98792$. The dotted-dashed (purple) curve shows the time path from an initial value $y_0^e$ slightly below 0.98792 while the dashed (orange) time path corresponds to $y_0^e$ slightly above 0.98792. The two time paths are very close to each other until they get near the middle steady state $(\pi_L, y_L)$. They then evolve in very different ways: one path moving deep into the stagnation region, and the other path eventually converging to the targeted steady state in dampening oscillations.

Figure 3: The left panel shows the domain of attraction (DOA) of the targeted steady state $(\pi^*, y^*)$. The domain of attraction is shaded (yellow) in the figure and the target steady state is indicated by the star. The right panel shows the sensitive dependence of dynamics on initial conditions. The arrows show the direction of movement of the variables.

This sensitivity to initial conditions is local to the boundary of the DOA, but it occurs in a critical area and complicates decision-making for policymakers. Looking at the illustration in Figure 3 (right panel), it can be difficult to know, for some time, whether or not aggressive policies need to be, or retrospectively should have been, followed. For both paths shown, over an extended stretch of time, $y^e$ is low but improving and $\pi^e$ is below target and falling, with interest rates (not shown) near the ZLB, as the unstable middle steady state $(\pi_L, y_L)$ is approached. Only then, after a possibly extended period near $(\pi_L, y_L)$, does it become evident whether the economy will recover or will instead deteriorate and move deep into the stagnation region.

The preceding discussion suggests that there will be some challenges in the design of fiscal and monetary policy. As we will see in Section 5, if the economy is outside the DOA and fiscal policy is used to try to direct the economy to the targeted steady
state it will be important to choose the magnitude and length of the fiscal stimulus carefully. Finally, as also discussed in detail in Section 5, an aggressive policy change is required if expectations are quite pessimistic.

4 Extension to the stochastic economy

We now turn to the model under AL when the economy is subject to stochastic shocks. We use the recursive least-square learning approach to expectation formation as developed in Bray and Savin (1986), Marcet and Sargent (1989) and Evans and Honkapohja (2001). Under this approach agents forecast like econometricians, regressing variables to be forecasted on observed explanatory variables, updating the forecast rule coefficients as new data become available.

The productivity shocks \( A_t \) and mark-up shocks \( \nu_t \) are assumed to be independent of each other and to take the form

\[
\ln(A_t/\bar{A}) = \rho_A \ln(A_{t-1}/\bar{A}) + \ln(\varepsilon_{A,t+1}) \quad \text{and} \quad \ln(\nu_t/\bar{\nu}) = \rho_\nu \ln(\nu_{t-1}/\bar{\nu}) + \ln(\varepsilon_{\nu,t+1}),
\]

where \( 0 \leq \rho_A, \rho_\nu < 1 \), and where \( \ln(\varepsilon_{A,t+1}) \sim \mathcal{NI}(0, \sigma_A^2) \) and \( \ln(\varepsilon_{\nu,t+1}) \sim \mathcal{NI}(0, \sigma_\nu^2) \). We assume \( A_t, \nu_t \) are observable, and, for convenience, the parameters \( \bar{A}, \bar{\nu}, \rho_A, \rho_\nu \) are assumed to be known to agents. (If the parameters were unknown it would be straightforward for agents to use consistent estimates of them). We assume the forecast rules include linear dependence on the observable \( A_t \) and \( \nu_t \). Alternative assumptions could be entertained at the cost of further analytical complexity.

Specifically, agents have a perceived law of motion (PLM) taking the form

\[
\ln(y_t) = f_y + d_{yA} \ln(\bar{A}_t) + d_{y\nu} \ln(\bar{\nu}_t) + \eta_{yt}
\]

\[
\ln(\pi_t) = f_\pi + d_{\pi A} \ln(\bar{A}_t) + d_{\pi\nu} \ln(\bar{\nu}_t) + \eta_{\pi t},
\]

where \( \eta_{yt}, \eta_{\pi t} \) are perceived white noise shocks, where \( \bar{A}_t \equiv A_t/\bar{A} \) and \( \bar{\nu}_t = \nu_t/\bar{\nu} \). To form forecasts \( y_{t+s}^e \) and \( \pi_{t+s}^e \) at time \( t \) agents estimate the parameters of the PLM using data up to period \( t-1 \) and iterate the estimated PLM forward to period \( t+s \).

The PLMs are estimated using constant-gain recursive least squares, see Appendix B for formal details. Letting \( f_y, d_{yA}, d_{y\nu}, f_\pi, d_{\pi A}, d_{\pi\nu} \) now denote the time \( t \) values of the parameter estimates, expectations of output and inflation \( s \) steps ahead, based on the observed exogenous shocks \( \bar{A}_t \) and \( \bar{\nu}_t \), are given by

\[
y_{t+s}^e = e^{f_y} \bar{A}_t^{d_{yA}} \bar{\nu}_t^{d_{y\nu}}
\]

\[
\pi_{t+s}^e = e^{f_\pi} \bar{A}_t^{d_{\pi A}} \bar{\nu}_t^{d_{\pi\nu}},
\]
With these expectations, the temporary equilibrium at time $t$ is given by (5) - (9), subject to the modification described in the beginning of Section 4.1.

The dynamic path under AL is then specified recursively. At the beginning of time $t+1$ estimates of $(f_y, d_{yA}, d_{yu}, f_{\pi}, d_{\pi A}, d_{\pi u})$ are updated to include the time $t$ data point using the recursive least-squares (RLS) equations. Then, after the time $t+1$ exogenous random variables are drawn, the temporary equilibrium equations determine $y_{t+1}, c_{t+1}, \pi_{t+1}$ and $R_{t+1}$. Given initial conditions and continuing in this way generates a time path of temporary equilibria $\{y_t, c_t, \pi_t, R_t\}_{t=0}^{\infty}$ for the economy under AL. For further details see Appendix B.

### 4.1 Simulation Results

For our numerical analysis we conduct stochastic simulations over long periods of time and one must allow for trajectories that can go very far from steady states. Consequently two modifications in our simulations are made. First, it is assumed that after $T$ periods the transitory stochastic component of output and inflation forecasts can be ignored by agents, i.e. we set $y_{t+s} = e^{f_s}$ and $\pi_{t+s} = e^{\pi}$ for $s \geq T$. This is a convenient way of speeding up computations. In the simulations we set $T = 28$. Second, agents are assumed to believe that after $T_1$ periods the real interest rate reverts to its steady-state value $\beta^{-1}$, i.e. $r_{t+s} = \beta^{-1}$ for $s \geq T_1$. Some assumption like this is needed for examining global dynamics since there are some regions of the expectational parameter space for which the expected real interest rate factor would be less than one, implying undefined consumption.

In our benchmark simulations we set $T_1 = 20$, i.e. at each time $t$ agents believe real interest rates will return to their steady-state value after five years.\(^\text{15}\) Thus $T_1$ (and $T$) are rolling windows. While an assumption like this is needed for technical reasons, it can also be viewed as making a substantive assumption about expectations: agents believe that periods of persistently high or low real interest rates will end after five years.\(^\text{16}\) One could, of course, use higher values for $T$ and $T_1$.

Before turning to numerical results we discuss the role of the gain sequence $\omega_t$. Consider first the decreasing gain case in which $\omega_t \rightarrow 0$, as $t \rightarrow \infty$. As with the nonstochastic case, if the variances of stochastic shocks are not too large, and with additional plausible assumptions, we can expect there to be fixed forecast parameters $\tilde{\phi}_y, \tilde{\phi}_\pi$ that correspond to an equilibrium near the targeted steady state. The

\(^{15}\text{This is consistent with expected long real rates varying over horizons longer than five years.}\)

\(^{16}\text{Of course, monetary policy can in principle commit to a path of future nominal interest rates over a much longer period. In Section 6 we explore the impact of credible forward guidance by the Central Bank about future nominal rates.}\)
resulting equilibrium, usually called a “restricted perceptions equilibrium” (RPE), is a generalization of a rational expectations equilibrium (REE): the forecast coefficients $\tilde{\phi}$ are minimum mean squared error within the restricted class of linear forecast models used by agents, though in principle better nonlinear forecast rules may exist.\textsuperscript{17} The RPE also differs from the REE due to our boundedly optimal agents’ use of point-expectations in their forecasting. However, the RPE can be viewed as an approximation to the REE centered at the targeted steady state.\textsuperscript{18}

The \textit{E-stability principle} states that, in the decreasing gain case, with suitable additional assumptions, this RPE will be locally stable under RLS learning, so that for initial expectations near the RPE parameters $\tilde{\phi}_y, \tilde{\phi}_\pi$ we will have $\phi_{yt} \to \tilde{\phi}_y$ and $\phi_{\pi t} \to \tilde{\phi}_\pi$. Similarly, we can expect an RPE at the stagnation steady state to be locally stable but for the middle steady state $(\pi_L, y_L)$ to be locally unstable under RLS learning. Thus in the stochastic model, local stability of the equilibrium paths under RLS learning is inherited from E-stability of the steady states.

In practice, in applied macro models, a constant gain $\omega_t = \omega$ with $0 < \omega < 1$, is almost invariably assumed. This allows agents to track structural change and changes in policy, but also results in “perpetual learning dynamics” around an REE or RPE. An advantage of this in empirical applications is that the learning dynamics are part of a stationary system.\textsuperscript{19} In our numerical simulations constant gain learning is employed. Some theoretical stochastic approximation results are available for constant-gain learning in the stochastic model in the limiting case $\omega > 0$ sufficiently small,\textsuperscript{20} based on an ODE approximation to the RLS system.

In the current setting the E-stability principle confirms local stability of both the targeted and the stagnation steady state, and local instability of the middle steady state.\textsuperscript{21} However, as was already noted, with constant gain learning perpetual fluctuations remain e.g. near the targeted steady state RPE. The forecast rule parameters $\phi_{yt}, \phi_{\pi t}$ have means near their RPE values and variances approximately proportional to the gain $\omega$. Stochastic approximation results based on the ODE approximation

\textsuperscript{17}For discussion of RPE see, e.g. Evans and Honkapohja (2001), Ch. 13, and Branch (2006). For applications in nonlinear models see Evans and McGough (2020b), Evans and McGough (2020a).

\textsuperscript{18}For $A_t, \nu_t$ with finite support, REE and RPE coincide as $\sigma_A, \sigma_\nu \to 0$ and $\rho_A, \rho_\nu \to 0$.


\textsuperscript{20}See e.g. Evans and Honkapohja (2001), Section 7.4 of Ch. 7 and Ch. 14, Cho, Williams, and Sargent (2002), Evans and Honkapohja (2009), and Williams (2019).

\textsuperscript{21}The intercepts of the expectations functions govern the evolving means of $y_t$ and $\pi_t$ in the sequence of temporary equilibria, so that the preceding E-stability analysis remains central to the model’s dynamics.
to the updating equations (see Appendix B) can be used to compute the “mean
dynamics” globally, but in practice it is convenient to study the dynamics directly
using stochastic simulations.

We are particularly interested in how the size of an initial pessimistic expectations
shock affects whether the economy returns to the targeted steady state RPE or
whether it is pushed into the stagnation regime along a path toward the stagnation
steady state. To study this using simulations of our calibrated model, we consider the
impact over time of an unmodelled adverse shock to output expectations $y_0^e$, such as
might have occurred following the 2007-8 financial crisis, lowering agents’ estimates
of future output and incomes.

Assume the economy is initially in the targeted steady state (with $y^e = y^s = 1.00003 \approx 1$) when a shock to expectations occurs. Because our model is now
stochastic, we anticipate that, at least for a range of initial $y_0^e$, whether the economy
returns to the targeted steady state will itself be a stochastic event. Under AL
dynamics the gain parameter must be specified and in our numerical simulations
we set this to $\omega = 0.01$. For long-horizon models, because of the high sensitivity
of temporary equilibrium output and inflation to long-run expectations, the gain is
typically set somewhat lower. However, in the presence of a large shock to the
economy, and with a possible change in policy, a higher gain is warranted to track
the evolving data.

Consider first a small negative shock to $y^e$ which is inside the DOA in Figure
3. A shock of 0.2 percent to steady-state output expectations (or its present value
equivalent), with $\pi^e = \pi^s$ unchanged, shifts output expectations to $y_0^e = \lambda y^s$, where
$\lambda = 0.998$. In this case the economy will converge back with very high probability to
the targeted steady state. This is as expected since this shock places expectations
substantially inside the DOA: the lower boundary of the DOA is approximately
$y^e = 0.98792$ when $\pi^e = \pi^s$.

Larger adverse shocks to $y_0^e$ lead to an increasing likelihood of failure to return
to the targeted steady state under unchanged policy. The key results are shown in
Table 1. For $\lambda = 0.9975$ the probability of convergence to the targeted steady state
is 69% percent and for $\lambda = 0.99745$ this probability is only 15%. Thus for a range
of expectation shocks the dynamics of the economy can depend sensitively on the
sequence of exogenous random shocks affecting output and inflation.

The numerical results seen in Table 1 show that failure to converge to the targeted
steady state arises even for pessimistic output expectations well inside the theoretical
E-stability DOA shown in Figure 3. The discrepancy for initial expectations inside
the nonstochastic DOA in Figure 3 arises for several reasons. First, our assumption

---

$^{22}$See. e.g. Eusepi and Preston (2011).
$T_1 = 20$ has a sizable effect. Additional simulations show that at $\pi^e = \pi^*$ the lower boundary $y^e$ to the numerical stochastic DOA of the targeted steady state falls as $T_1$ increases. The intuition for this is as follows. With $\pi^e = \pi^*$ and $y^e < y^*$, expected nominal and real interest rates are lower, raising demand and output. This stabilizing effect of monetary policy, in the face of pessimistic output expectations, is blunted, however, because we impose that expected real interest rates are expected to return to the steady-state value after a finite number of periods $T_1$. Qualitative results are not affected by the precise choice of $T_1$.

There are two other factors that arise from our stochastic set-up. In the non-stochastic model generating Figure 3 there are only two parameters, corresponding to the intercepts of the RLS system given in Appendix B. In our stochastic set-up there are six parameters in $\phi_y, \phi_{\pi}$, as well as additional parameters in the estimated second-moment matrix $\mathcal{R}$. The global ODE approximation to the RLS algorithm thus differs from the E-stability dynamics shown in Figure 3. In addition, with constant gain learning the mean dynamics corresponding to the ODE are only a good approximation for $\omega > 0$ very close to zero and can differ significantly for values even as small as $\omega = 0.01$. The combination of constant $\omega = 0.01$ and stochastic intrinsic shocks leads to sufficient variation in $(y_t, \pi_t)$ over time so that expectations are more frequently pushed into unstable trajectories.

The key numerical findings are clearly consistent with our general theoretical results. The targeted steady state is locally stable under LS learning, but it is not globally stable. For sufficiently pessimistic initial output expectation shocks, i.e. $0 < \lambda < 1$ sufficiently low, the proportion of trajectories that converge to the targeted steady state is near zero. In our stochastic set-up this arises for $\lambda \leq 0.9974$. The numerical results are shown in Table 1.

<table>
<thead>
<tr>
<th>Initial expectation</th>
<th>$P$(target)</th>
<th>$P$(stagn.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_0^e/y^* = 0.9980$</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>$y_0^e/y^* = 0.9975$</td>
<td>69</td>
<td>31</td>
</tr>
<tr>
<td>$y_0^e/y^* = 0.99745$</td>
<td>15</td>
<td>85</td>
</tr>
<tr>
<td>$y_0^e/y^* = 0.99742$</td>
<td>1</td>
<td>99</td>
</tr>
<tr>
<td>$y_0^e/y^* = 0.9974$</td>
<td>0</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 1: Percentage convergence to target or stagnation under unchanged policy, 100 replications, gain $= 0.01$

These results illustrate that with constant fiscal policy in the stochastic model there are situations where the long-run outcome may be either the targeted steady
state or stagnation depending on the realization of the exogenous random shocks $\xi_t$ and $\nu_t$. Relatedly, stochastic simulations can deliver a cloud of points that reflect features of the data shown in Figure 1, see Section 6.6 for details. We also see that, on a formal level, the global E-stability analysis of Section 3.2, based on nonstochastic one-parameter PLMs, provides key, though approximate, results concerning convergence of real-time constant-gain RLS learning in the stochastic model.

To understand the magnitudes of the expectation shock given in Table 1, it is helpful to consider a reinterpretation of the role of $y^e$ in the temporary equilibrium model. For the consumption function (3) in the representative agent case with $\Xi_t \equiv \tau$, temporary equilibrium output $y_t$, depends to first-order on the present value of $\{y_{t+s}\}_{s=1}^{\infty}$ of the sequence of output expectations. We have interpreted steady state learning as agents acting as if $y_{t+s}^e = y_t^e$ for all horizons $s = 1, 2, 3, \ldots$. However, this is behaviorally equivalent to assuming that agents have an expected output profile with the same present value. Further discussion is at the end of Appendix B.

In interpreting these results it is important to bear in mind that we are employing a benchmark New Keynesian model without capital and without additional frictions often employed in serious empirical DSGE models, such as indexation, habit persistence and adjustment costs for capital. Extensions like these, which introduce inertia into the dynamics, would possibly enlarge the DOA, without, however, altering the qualitative features of our model in which there are three steady states, including a locally stable targeted steady state and a stagnation region.

5 Fiscal Policy

We turn now to fiscal policy. A growing literature has been reconsidering the effects of fiscal policy in light of the relatively large fiscal stimuli adopted in various countries in the aftermath of the Great Recession. For example, Christiano, Eichenbaum, and Rebelo (2011), Corsetti, Kuester, Meier, and Muller (2010) and Woodford (2011) demonstrate the effectiveness of fiscal policy in models with monetary policy when the ZLB on the interest rate is reached. For a contrary view see Mertens and Ravn (2014). Most of this literature makes the RE assumption. The AL literature has shown that quite different results can arise both in NK and Real Business Cycle models; see Evans, Guse, and Honkapohja (2008), Benhabib, Evans, and Honkapohja (2014), Mitra, Evans, and Honkapohja (2013), Gasteiger and Zhang (2014) and Mitra, Evans, and Honkapohja (2019).

We examine fiscal policy under AL using the long-horizon anticipated-utility approach advocated by Preston (2005) and Eusepi and Preston (2010), and extended for policy changes in Evans, Honkapohja, and Mitra (2009) and Mitra, Evans, and
Honkapohja (2013). We consider an economy in which expectations are pessimistic relative to the targeted steady state and in which the path of the economy adjusts through learning. For concreteness this is modeled as a negative shock to output expectations $y^e$ (other shocks could be studied). We direct our attention to negative expectation shocks sufficiently large so that without policy change the path of the economy would with high probability fail to return to the targeted steady state and would instead be trapped in the stagnation region.\textsuperscript{23}

Because Ricardian households are assumed, we examine the impact of changes in the level of government purchases and focus on temporary increases in the level of government spending on goods and services.\textsuperscript{24} When there is a change in fiscal policy, agents take account of the tax effects of the announced path of policy. Given the Ricardian assumption, balanced budget increases in spending can be assumed so that the path of taxes matches the path of government spending.

Evidently, fiscal policy needs to be tuned to the size of the exogenous expectations shock. We consider the case where at $t = 1$ the government announces an increase in government spending for $T_p$ periods, i.e.

$$g_t = \gamma_t = \begin{cases} \bar{g}', & t = 1, \ldots, T_p \\ \bar{g}, & t \geq T_p + 1, \end{cases}$$

where $\bar{g}' > \bar{g}$. Thus government spending and taxes are changed in period $t = 1$ and this change is reversed at a later period $T_p + 1$. We assume that the announcement is fully credible and the policy is implemented as announced. These assumptions could, of course, be relaxed at the cost of added complexity in the analysis.

Using stochastic simulations we study the evolution of the economy, under AL, after a pessimistic shock and examine the potential role for fiscal policy to prevent stagnation or ameliorate bad outcomes.\textsuperscript{25} The focus is whether fiscal policy can alter the dynamic path so that there is instead convergence to the targeted steady state. The impact of fiscal policy may depend critically on the size and length of fiscal policy. In addition, the sequence of random shocks $A_t$ and $\nu_t$ have an impact on the success of fiscal policy.

\textsuperscript{23} Analogous simulations could of course be done when the $y^e_0$ shock is smaller and there is eventual convergence to targeted steady state without change in policy. The question of interest would then be whether fiscal policy can speed up the recovery back to the targeted steady state.

\textsuperscript{24} In further work it would be of interest to introduce alternative fiscal frameworks with distortionary taxes and/or public debt.

\textsuperscript{25} We emphasize that these simulation results are designed to be illustrative, i.e. to exhibit the range of possible results that can be obtained in our model. Using the model to fit actual historical episodes is reserved for future research.
As a first illustration consider the case \( y^e = 0.997 \times y^* \) which, based on Table 1, is big enough shock to result in convergence to the stagnation steady state approximately 100\% of the time in our calibrated stochastic model.\(^{26}\) As a specific illustration we set \( T_p = 4 \), i.e. a one-year fiscal package and a range of government spending increases from \( \bar{\gamma} = 0.2 \) to 0.4. The simulation is replicated 100 times and with length 500 periods. Table 2 gives the following results.\(^{27}\)

<table>
<thead>
<tr>
<th>( T_p ) ( \bar{\gamma} )</th>
<th>0.2</th>
<th>0.225</th>
<th>0.25</th>
<th>0.275</th>
<th>0.3</th>
<th>0.325</th>
<th>0.35</th>
<th>0.375</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>95</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>67</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Percentage of simulations in which fiscal policy successfully results in convergence to the targeted steady state starting from \( y^e_0 = 0.997 \times y^* \).

Evidently, temporary increases in \( g \) are effective in raising output. Small temporary increases in \( g \) may lead only to temporary increases in \( y \), but larger temporary increases in \( g \) can shift the economy back to a path converging to the targeted steady state. In the latter situation policy results in a permanent increase in output relative to the paths that would be followed without the fiscal stimulus.

Another observation is that if probability of convergence to target steady state is between 0 and 1, the sequence of serially correlated random productivity and mark-up shocks can matter: for a fiscal policy that is usually successful, a particularly unfavorable sequence of shocks can adversely affect expectations enough to prevent the policy from working.

It is also seen that a fiscal stimulus that is too large or too long can be counterproductive. In Table 2, an increase from \( \bar{\gamma} = 0.375 \) to 0.4 reduces the effectiveness of the stimulus greatly from 67\% to 1\%.

The case \( y^e = 0.997 \times y^* \) is systemically examined in Table A.1 in Appendix G which shows the probability of success for a range of both \( g \) and \( T_p \). A stimulus for too long can reduce the effectiveness of fiscal policy. This is a reflection of the negative effect on consumption of the tax burden associated with higher government spending, which is assumed correctly foreseen by households.

We now examine the case of a very large expectations shock \( y^e_0 = 0.991 \times y^* \), which corresponds to an expected two-year recession of 11.7\% of GDP. Following this shock a temporary fiscal stimulus is applied with government spending increased from \( \bar{\gamma} = 0.2 \) to \( \bar{\gamma}' = 0.3, \ldots, 0.7 \) for \( T_p = 1, \ldots, 6 \) quarters. Table 3 shows the probability

\(^{26}\)Using the present value interpretation of the expectation shock given at the end of the preceding section, the shock \( y^e = 0.997 \times y^* \) corresponds to an expected two-year recession of 3.9\% of GDP.

\(^{27}\)Extended version of Table 2 is given as Table A.1 in Online Appendix G.
(in percentages) of cases where the policy is successful. For $T_p = 1, 2$ the probabilities are zero over this range of $\bar{\gamma}$. The success probabilities are generally lower than those for the case shown Table 2 and Table A.1 of Appendix G. Also values of $\bar{\gamma}$ need to be significantly larger than those in Table A.1 in order to be successful. However, there are still policies with a high degree of success: the highest success rate shown in Table 3 is 94%.

<table>
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<th>0.35</th>
<th>0.4</th>
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Table 3: Percentage of simulations in which fiscal policy successfully results in convergence to the targeted steady state starting from very pessimistic output expectations $y_0^0 = 0.991 \times y^*$. Based on 100 replications in each cell.

We find from these results that a sufficiently large stimulus of appropriate duration can have a high probability of extracting the economy back to convergence to target even if pessimistic output expectations are deep inside the stagnation region of the stochastic model. However, it should be emphasized that a higher probability of avoiding the stagnation regime can be achieved, with a much smaller stimulus, if the policy is implemented when expectations are less pessimistic. This suggests that following a large adverse shock to expectations, in which there is major risk of the economy descending into the stagnation regime, a fiscal stimulus should be implemented as early as possible. This is discussed below in Section 6.2.

An interesting observation in Table 3 and also other tables is that the cases with relatively high success probability lie in a “corridor” taking the form of a “thick diagonal” from South-West to North-East. There is a negative trade-off between magnitude and length of stimulus.

The detailed quantitative results also depend on $\xi$, the degree of substitutability between private and public consumption. It can be seen from equation (5) that the impact output multiplier $\partial y_t / \partial d g_t = 1 - \xi > 0$ depends negatively on $\xi$. This is consistent with Ercolani and Azevedo (2019). We nonetheless obtain huge output multipliers if an appropriately aggressive fiscal stimulus is used when expectations are pessimistic. This arises because the increases in output and inflation resulting from

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28This accords with testimony by Lawrence Summers to the Joint Economic Committee hearing on 16 January 2008, that fiscal “stimulus program should be timely, targeted and temporary.”
the fiscal stimulus lead over time, through AL, to upward revisions in expectations sufficient to eventually return the economy to the targeted steady state.

A general implication of our fiscal policy results, which is evident but worth emphasizing, is that the size and impulse response profile of the government spending multiplier depends sensitively on both the current state of expectations, when the policy is initiated, and nonlinearly on the size and duration of the spending increase.

6 Extensions

We consider several extensions about designing policies to avoid stagnation and discuss features of simulated data of the model.

6.1 Including forward guidance in monetary policy

In the preceding Section it was seen that fiscal policy is not always successful, in the sense of guaranteeing that the economy escapes the stagnation regime, and the probability of success becomes lower with more pessimism. It therefore makes sense to ask whether supplementary unconventional monetary policy can help.

The current framework is well suited to analyze forward guidance, which of course has been one form of unconventional monetary policy that central banks used during and following the Great Recession.\textsuperscript{29} We model this as a commitment by the central bank to keep the policy interest rate at the ZLB for the first $T_m$ periods after the expectation shock occurs. With forward guidance the interest rate rule (8) becomes

$$ R_t = \begin{cases} 
1, & t = 1, \ldots, T_m \\
R(\pi_{t+1}^e, y_{t+1}^e), & t \geq T_m + 1.
\end{cases} $$

We consider output expectation shocks even more pessimistic than used in Table 3. We first set $y_0^e = 0.985 \times y^*$, which corresponds to an expected two-year recession of 19.5\% of GDP in terms of the computations mentioned at the end of Section 4 and discussed at the end of Appendix B. Without a change in policy, the economy always, in our simulations, converges toward the stagnation steady state. We explore the effectiveness of various settings of temporary fiscal stimulus $\bar{g}_1$, $T_p$ combined with forward guidance $T_m$.

If only forward-guidance monetary policy is used, without including fiscal stimulus, the probability of convergence to the targeted steady state is zero for $T_m \leq 10$

\begin{footnotesize}
\textsuperscript{29}Without extensions the model is not suited to analyzing other forms of unconventional policies, such as large scale asset purchases.
\end{footnotesize}
or $T_m \geq 15$, and is positive only for $T_m = 11$ (43%), $T_m = 12$ (25%), $T_m = 13$ (11%) and $T_m = 14$ (1%). If instead only fiscal stimulus is employed, the probability of convergence to the targeted steady state is close to less than 10% except for some specific policy settings, and is above 50% in just a few cases: $\tilde{g}_1 = 0.75$ with $T_p = 5$ (60%), $\tilde{g}_1 = 0.8$ with $T_p = 5$ (53%) and $\tilde{g}_1 = 0.9$ with $T_p = 4$ (55%).

<table>
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Table 4: Percentage of convergence to target for different specifications of combined policy based on 100 replications for $y_0^c = 0.985 \times y^*$, $T_m = 6$.

Better outcomes can be achieved by combining fiscal policy and forward guidance. Table 4 illustrates the results from detailed analysis of the case with $y_0^c = 0.985 \times y^*$ in which forward guidance setting $T_m = 6$ is combined with different fiscal stimulus settings. The highest probability of success (convergence to the targeted steady state) is 73% with $\tilde{g}_1 = 0.55$ with $T_p = 3$. Similar results are obtained for nearby values of $T_m$ (further results are in the tables in Appendix H). Thus for the case of severely depressed output expectations, there is a significant increase in the probability of escape from stagnation when both policies are actively employed.

<table>
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Table 5: Percentage of convergence to target for different specifications of combined policy for $y_0^c = 0.98$, $T_m = 7$ and 100 replications.

For even more pessimistic expectations, it can happen that both fiscal policy alone and forward guidance alone are ineffective in moving the economy to the targeted steady state, while combined policy can still achieve some success. As an example, consider initial output expectations $y_0^c = 0.98 \times y^*$. This corresponds to an expected two-year recession of 25.9% of GDP. In this case forward guidance alone is totally...
ineffective (for $T_m = 1, \ldots, 20$). Using fiscal policy alone is also largely ineffective: in the range $\bar{g}_1 = 0.25, \ldots, 0.90$ there are only a few cases with positive probability for convergence to target, and the highest probability is 28% for $\bar{g}_1 = 0.85$ with $T_p = 6$. However, combined policy improves the chances of converging to the targeted steady state. In Table 5 the highest probability of convergence is 45% when $\bar{g}_1 = 0.6$, $T_p = 5$ and $T_m = 7$. (More results are given in Appendices I and J.)

The results of this section show that, for very pessimistic output expectations, adding forward guidance to fiscal policy can substantially improve the chances of converging to the targeted steady state, at least for the wide range of fiscal policies we considered. A different approach might be to use an even larger fiscal stimulus for which there is some improvement but the results are not very encouraging and would require implausibly large increases in $\bar{g}$.

Our framework also has implications that contrast with the literature. Under RE forward guidance of future low interest rates is very effective – so effective that these implications have been called the “forward guidance puzzle.” A large literature has shown that for recessions RE overstates the extent to which forward guidance – of near zero interest rates for an extended period – will stimulate GDP, relative to what is found under a range of bounded rationality assumptions. See, for example Cole (2021), Garcia-Schmidt and Woodford (2019) and Eusepi, Gibbs, and Preston (2021). These papers have focused on linearized New Keynesian models. In our nonlinear framework we find that, following a large negative expectations shock, forward guidance can be unable to return the economy to the targeted steady state unless it is complemented by a fiscal stimulus.

### 6.2 Delays in policy

In Section 5 we suggested that in the face of a large pessimistic expectations shock it may be important to implement a fiscal stimulus quickly. We here briefly illustrate the effect of policy delays. For a given output expectations shock $\hat{y}_0$ we consider the effect on the probability of success of a delay by $T_s$ periods. We restrict attention to the case, examined earlier in Table 3, of a large pessimistic initial output expectations shock $\hat{y}_0 = 0.991 \times y^*$, and we now assume policy is executed with a delay of 4 periods (one year). Table 6 reports the relevant part of the table, i.e. ranges $T_p = 3, \ldots, 6$ and $\bar{g}' = 0.45, \ldots, 0.7$ based on 100 replications.\(^{30}\)

\(^{30}\)The policy thus starts in period 5 and ends in period $5 + T_p$.\)
<table>
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Table 6: Percentage of simulations in which fiscal policy $\bar{g}'$ for $T_p$ periods and delay of $T_s = 4$ periods results in convergence to target from pessimistic output expectations $y^c_0 = 0.991 \times y^*$. Based on 100 replications with length 500.

It is seen that the percentage of success with delay is generally lower than the corresponding percentage when there is no delay. Most noticeably, if we compare no-delay fiscal policies, with the highest chance of success, with six-period delayed policies with the highest probability of success, the probability of success falls from 94% to 53%. The reason for this is that during the period of delay output expectations deteriorate further and inflation expectations also begin to decline.

### 6.3 Credit frictions and calibration of the discount factor

In Section 3.1 we noted that at the low steady state ($\pi_L, y_L$) the (gross) policy interest rate is approximately equal to one while the (gross) inflation rate is approximately equal to $\beta$. From Figures 2 and 3 it is evident that $\pi_L$ plays a key role in the expectation dynamics since the unstable steady state ($\pi_L, y_L$) is on the edge of the DOA of the targeted steady state and for $\pi^e < \pi_L$ and $y^e < y_L \approx y^*$ the economy lies within the stagnation trap. The appropriate calibration of the discount factor $\beta$ is thus worth discussing. Our numerical results have used the quarterly calibration of $\beta = 0.99$ i.e. a quarterly deflation rate at $\pi_L$ of 1%.

While $\beta = 0.99$ is fairly standard, there are good reasons to consider alternative, higher, values. The historical average realized net real interest rate on US Treasuries bills is not more than 1% per annum. In an economy without growth this corresponds to a discount factor of about $\beta = 0.9975$.\(^{31}\) The critical inflation rate at the edge of the stagnation trap at $y^e < y_L$ is then an annual deflation rate of 1%.

A second factor that can lead to a higher level of the critical inflation rate is the existence of credit frictions. Various models have been proposed that generate a spread between different interest rates on loans. A prominent example within a NK setting is described in Curdia and Woodford (2010) and developed at length in

\(^{31}\)We note Eggertsson (2010) uses a calibration of $\beta = 0.997$ in a model of the US economy during the Great Depression. In its trough deflation reached 10% per year.
Curdia and Woodford (2015). Their framework posits a heterogeneous agents set-up with two types of household, at any given time, experiencing different realizations of taste shocks. This leads to lending from agents who are currently more patient to those who are currently more impatient. Frictions in the financial intermediation sector result in a borrowing rate above the lending rate.

Embedding a heterogeneous agents framework into our model is beyond the scope of the current paper. However, it is natural to incorporate a shortcut, motivated by Woodford (2011), which is to assume that the market interest rate relevant in household Euler equations for the “intertemporal allocation of expenditure is not the same as the central bank’s policy rate” (Woodford, p. 16). Woodford (2011) and Curdia and Woodford (2015) focus on the implications of the time variation in this spread, while for our purposes the key implication is a positive steady state spread \( \varphi = R - i > 0 \), where \( i \) is the policy rate and \( R \) is the interest rate relevant for household decision-making. The benchmark calibration in Curdia and Woodford (2015) corresponds to a value \( \varphi = 0.0025 \), i.e. to 1% per annum.

With credit frictions we assume a spread \( \varphi > 0 \) between the market rate \( R_t \) and the policy rate \( R_t - \varphi \). Since the policy rate obeys the ZLB for net interest rates, the market interest rate factor relevant for the consumption Euler equation satisfies \( R_t \geq 1 + \varphi \). The interest-rate rule (8), with inflation target \( \pi^* \), is then replaced by

\[
R_t = 1 + \varphi + (R^* - (1 + \varphi)) \left( \frac{\pi_{t+1}^e}{\pi^*} \right)^{BR^*/(R^*-(1+\varphi))} \left( \frac{y_{t+1}^e}{y^*} \right)^{\phi_y}. \tag{14}
\]

The positive spread \( \varphi \) increases the low steady-state inflation rate to \( \pi_L \approx \beta(1 + \varphi) \). This has a number of implications, one of which is particularly relevant for policy: if \( \beta(1 + \varphi) > 1 \) then it is possible to have \( 1 < \pi_L < \pi^* \), so that the critical inflation rate at \( (\pi_L, y_L) \) is a zero or low positive inflation rate, rather than a deflation rate.

The central consequences of a credit spread can be seen by comparing the domains of attraction of \( \pi^* \) with and without a credit spread. Figure 4 illustrates the DOA of the targeted steady state for the model with a high subjective discount rate and a positive credit spread. The DOA is now significantly smaller than that in the basic model.\(^{32}\) At \( \pi^e = \pi^* \) the value of \( y^e \) at the low boundary of the DOA is approximately \( y^e = 0.9986 \), much higher than the corresponding value in Figure 3. Similarly at \( y^e = y^* \), the value of \( \pi^e \) at the low boundary of the DOA is \( \pi_L \) and now corresponds to positive net inflation. Thus the impact of a higher discount factor

\(^{32}\)The truncation of expected real interest rates to a finite horizon in consumer optimization is employed because a wide state space is needed for the analysis. Here we set \( T_1 \) to a fairly high value, \( T_1 = 100 \), in order to reduce its numerical impact. See Section 4 for discussion of \( T_1 \). Using finite \( T_1 \) reduces somewhat the domain of attraction.
and a positive credit spread is to reduce the size of the DOA of \((y^*, \pi^*)\), making the targeted steady state less robustly stable.

Figure 4: Domain of attraction with credit spread and \(\pi^* = 1.005\). 

\[ \beta = 0.9975, \varphi = 0.0025, T_1 = 100. \]

Thus the qualitative aspects of dynamics shown in Figures 2 and 3 remain unchanged. However, taking into account credit frictions, expected inflation rates significantly below the central bank target, even if positive, increase the possibility of a path toward stagnation and the possible need for aggressive policy. We next explore the implications of a higher inflation target which may well be a way to increase the robustness of standard monetary policy.

6.4 Higher Inflation Target

Adopting a higher inflation target became a popular though controversial subject in the policy discussion during the Great Recession. See for example the influential paper Blanchard, Dell Ariccia, and Mauro (2010). The implications of a higher inflation target in our setup can be examined most readily using the nonstochastic model of Sections 2 and 3. Figure 5 compares the results for an inflation target of \(\pi^* = 1.005\), i.e. two percent annually (left panel), versus \(\pi^* = 1.01\), i.e. four percent.
annually (right panel). The policy with higher inflation target appears clearly effective in the sense that the DOA of the targeted steady state is substantially larger with the higher target. It is possible to compute numerically the area of the DOA – see Appendix K for details. Comparing $\pi^* = 1.005$ to 1.01 the DOA increases approximately 2.4 fold. Our finding that the DOA increases with the magnitude of the inflation target holds for alternative model specifications (i) in which the cost of price adjustment in utility function (1) takes the form $\Phi \left( \frac{P_{t,1}}{P_{t-1,1}} - 1 \right)$, and (ii) the model with credit spread.

Figure 5: Domain of attraction for different inflation targets $\pi^* = 1.005$ (left panel) and 1.01 (right panel) in quarterly values.

These results may appear surprising in light of other results in the literature. Ascari, Florio, and Gobbi (2017) consider a linearized NK model with Calvo pricing frictions and allowing for trend inflation. They find that with a higher inflation target the set of interest rate policy parameters giving E-stability is smaller when the inflation target is higher. However, this is a conceptually different exercise from the one examined here. Figure 5 considers an interest-rate rule that is unchanged except for having a higher inflation target and considers the size of the stability

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Footnote 33: Figure 5 is computed using truncation parameter $T_1 = 100$. An analogous result holds for the untruncated model ($T_1 = \infty$), provided we restrict $\pi^*$ to values for which convergence issues associated with negative (net) real interest rate do not arise (for the current calibration this constraint is $\pi^*$ not greater than 3.5 percent annually).
region with respect to perturbations in expectations \((\pi^e, y^e)\) away from the targeted steady state. This important question can only be addressed in a nonlinear setup.

Using a linearized model, Branch and Evans (2017) consider AL rules that allow for autoregressive or VAR dynamics in forecasting inflation and output, and find that an unlucky series of shocks can lead to unstable “escape paths.” They emphasize in particular that an increase in the inflation target must be done carefully to avoid de-anchoring of inflation expectations under AL.

While we use a Rotemberg instead of a Calvo pricing friction, we think that the intuition for our finding arises from our nonlinear set-up in which the unintended steady state \(\pi_L\), which does not vary with \(\pi^*\), is positioned on the edge of the DOA for the targeted steady state. An increase in \(\pi^*\) leads to a greater separation of \(\pi^*\) from \(\pi_L\) and thence to a larger DOA.

6.5 Blended expectations

Inflation targeting has been practiced by a substantial number of Central Banks since it was formally adopted by New Zealand and Canada in 1990 and 1991. In our numerical calibrations we have used a target of 2%, which, for example was formally adopted by the Bank of England in 2003. The target of 2% in the US was formally announced by the Federal Reserve in January 2012, bringing it in line with a number of other countries, but this was preceded by a period in which 2% was believed to be the Fed’s informal target. One of the main reasons given for having an explicit inflation target is that this can anchor expectations, so that expected inflation is less sensitive to observed inflation rates or exogenous shocks.

It is certainly possible that having an explicit inflation target helps anchor expectations, e.g. see Gurkaynak, Swanson, and Levin (2010). Against this Branch and Evans (2017) have argued that policymakers should take into account that expectations can become de-anchored by observed economic data. In this section we take a balanced approach to this issue by considering “blended expectations,” in which inflation expectations are a weighted average of the forecasts arising from our AL rules and the inflation target set by the central bank. Thus we now set

\[
\pi_t^e = \varpi \hat{\pi}_t^e + (1 - \varpi)\pi^*, \text{ for } 0 < \varpi < 1, \tag{15}
\]

where \(\varpi\) is the weight placed on the AL forecast \(\hat{\pi}_t^e\) and \(1 - \varpi\) is the weight on the central bank inflation target.

We now look at global E-stability dynamics with blended expectations in comparison with the benchmark case given in Section 3.2. Temporary equilibrium, given expectations \((\pi_t^e, y_t^e)\), continues to be given by \(\pi_t = G_1(y_t, y_t^e)\) and \(y_t = G_2(\pi_t^e, y_t^e)\),

31
and the E-stability differential equations are now \( \frac{d \pi^e}{d \tau} = F_y(\pi^e, y^e) \) and \( \frac{d \tilde{\pi}^e}{d \tau} = F_\pi(\pi^e, y^e) \). From (15) we have \( \frac{d \tilde{\pi}^e}{d \tau} = \varpi^{-1}(d \pi^e/d\tau) \), so that in terms of blended expectations the E-stability equations are

\[
\frac{d y^e}{d \tau} = F_y(\pi^e, y^e) \quad \text{and} \quad \frac{d \tilde{\pi}^e}{d \tau} = \varpi F_\pi(\pi^e, y^e).
\]

These considerations imply that the earlier analysis is unchanged if the relevant state space is thought to be in terms of blended \( \pi^e \) where the ODE for \( \pi^e \) is the usual ODE for inflation expectations with the right-hand side multiplied by the weight \( \varpi \). (Note that the state space is the usual one when \( \varpi = 1 \).) Changes in \( \varpi \) correspond to changes in the adjustment speed of inflation expectations \( \pi^e \), so that smaller \( \varpi \) means lower value for derivative and slower adjustment. The steady states and their E-stability properties are clearly unchanged, so we have the result:\(^{34}\)

**Proposition 2.** (i) The targeted steady state is E-stable provided \( \phi_y \) is not too large. (ii) The steady state \( (\pi_L, y_L) \) is not E-stable provided \( \phi_y \) is not too large. (iii) The steady state \( (\pi_S, y_S) \) is E-stable.

\(^{34}\)The proof is a straightforward modification of the proof of Proposition 1.

Figure 6: The left panel shows the domain of attraction of the target steady state with \( \varpi = 0.8 \). The right panel shows the domain of attraction of the target steady state with \( \varpi = 0.5 \).
Looking at the global picture, the qualitative dynamics for different \( \varpi \in (0,1) \) are unchanged relative to those in Figure 3, which corresponds to \( \varpi = 1 \). There is, however, a major quantitative change: the DOA becomes larger when the weight \( 1 - \varpi \) on the fixed central bank forecast \( \pi^* \) is larger. See the two panels in Figure 6 which should be compared to Figure 3.

One way to see the quantitative significance of the value of \( \varpi \) is to consider the value of \( y^e \) which is on the lower boundary of the DOA when \( \pi^e = \pi^* = 1.005 \). In the left panel of Figure 6 with \( \varpi = 0.8 \), the corresponding value is \( y^e \approx 0.985 \), whereas in the right panel of Figure 6, with \( \varpi = 0.5 \), this value falls to \( y^e \approx 0.975 \). The enlargement of the DOA is also very visible for high values \( \pi^e > \pi^* \). In terms of areas the DOA with \( \varpi = 0.5 \) is about 2.9 times the magnitude in the case \( \varpi = 0.8 \). These results suggest that with more anchored inflation expectations fiscal stimulus is needed over a smaller range of pessimistic inflation expectations.

This result has a natural interpretation. \( 1 - \varpi \) can be viewed as a measure of the Central Bank’s credibility in being able to deliver inflation rates in line with its announced target. For \( 1 - \varpi \) large, \( \pi^e \) will stay near \( \pi^* \) even if econometric forecasts based on recent past data give, say, a much lower forecast. This credibility increases the robustness of the targeted steady state by increasing its DOA under AL.

Again, the qualitative aspects of dynamics shown in Figure 3 remain unchanged in the two panels of Figure 6 – the possibility of a stagnation trap remains for \( y^e, \pi^e \) sufficiently pessimistic. A natural extension of the blended expectations approach is reinforcement learning, in which the weight \( \varpi \) is made time-varying with \( \varpi \), evolving based on the relative accuracy of the two forecast rules.\textsuperscript{35} Reinforcement learning would limit the degree to which credibility could be maintained if inflation were persistently different from the target. Nonetheless, it is clear that a credible inflation target makes the targeted steady state more robust to expectation shocks under AL.

### 6.6 Illustrative scatterplots

In the Introduction we noted that in post 2000 data, in addition to a constellation of points centered around the steady state targeted by monetary policy, and another bunching of points with low inflation and interest rates, there has also been an association between very low interest rates and negative output gaps. The combination in our model of a locally stable targeted steady state and a stagnation regime has the potential to explain these features of the data, which we illustrate using simulations of a modified version of our model.

\textsuperscript{35}For an application of this approach see Honkapohja and Mitra (2020).
As emphasized at the end of Section 4.1, to obtain a tractable global stochastic nonlinear setup with AL dynamics, our model has focused on the standard basic New Keynesian setup without capital and without other frictions usually introduced in empirical models. Our approach has the advantage of showing that the central qualitative dynamics we identify arise from a standard basic setup; however, an implication is that output and inflation respond immediately and strongly to changes in expectations. To more realistically correspond to historical data we moderate this sensitivity to expectations by altering the tails of the pricing friction to reduce the range of inflation.\footnote{Recall interest and inflation rates are measured as quarterly factors. The modified pricing friction does not affect \((\pi^*, y^*)\) and \((\pi_L, y_L)\), or local dynamics under learning, and easily accommodates the range of US inflation over the last 100 years. See Appendix F for simulation details.}

Figure 7 shows scatterplot results combining simulations of 80 periods each for three different starting points for expectations. Two of the simulations start with \((y^e, \pi^e)\) near \((y^*, \pi^*)\): in one initial \(y^e\) is somewhat below \(y^*\) and in the other initial \(\pi^e\) is somewhat below \(\pi^*\). Both of these initial \((y^e, \pi^e)\) are within the DOA, and their data clouds, generated under unchanged policy, are centered on the targeted steady state. A third simulation starts near \((y_L, \pi_L)\), just outside the DOA. For this simulation expectations gradually become more pessimistic, and both output and inflation decline over time, leading to negative output gaps reaching 3%. After an initial delay, monetary policy reduces the net interest rate to an effective lower bound of 0.8% per year and, with forward guidance, holds it there for 14 quarters. With an additional short delay there is a large fiscal stimulus for 18 quarters that overlaps with the forward guidance. These combined measures increase output and inflation substantially, eventually returning the economy to the targeted steady state.

Figure 7: Scatterplots of data from three simulations of 80 periods, showing inflation vs. interest rate (left) and % output gap vs. interest rate (right).
Figure 7 exhibits two qualitative features emphasized in the introduction – clouds of points surrounding the targeted steady state and a region of points with low interest rates and negative output gaps, consistent with a stagnation trap that is eventually overcome by active macroeconomic policy.\textsuperscript{37}

7 Discussion of Related Literature

Within the context of standard NK models and rational expectations (RE), the implications of the ZLB have been considered from several angles. One natural approach is to examine exogenous shocks to demand that push the economy to the ZLB. Exogenous discount rate or, more plausibly, credit-spread shocks have been emphasized by Eggertsson and Woodford (2003), Christiano, Eichenbaum, and Rebelo (2011), Corsetti, Kuester, Meier, and Muller (2010) and Woodford (2011). These shocks are often assumed to follow a two-state Markov process in which the credit-spread shock disappears each period with a fixed probability, with aggregate output and inflation recovering as soon as the exogenous shock stops operating.

While this approach has been fruitful in suggesting suitable policy responses to such shocks, it has several somewhat unattractive features. It relies heavily on the persistence of a shock that evaporates according to an exogenous process, with recession ending as soon as the exogenous negative shocks cease. Furthermore, this approach does not do justice to an independent role for pessimistic expectations.

Another approach, emphasized by Benhabib, Schmitt-Grohe, and Uribe (2001), focuses squarely on the existence of multiple rational expectations equilibria (REE). Under an interest-rate rule that follows the Taylor principle at the targeted steady state, there is a second, unintended and indeterminate, steady state at a low inflation, or modest deflation, rate. However, while in this steady state the policy interest rate is at or near the ZLB, the level of aggregate output is only very slightly below that of the targeted steady state. In addition, as we have emphasized and was also stressed in Evans, Guse, and Honkapohja (2008) and Benhabib, Evans, and Honkapohja (2014), this unintended low-inflation steady state is not stable under AL.

In the context of this approach, Benhabib, Schmitt-Grohe, and Uribe (2002) and chapter 4, pp. 316-17, of Woodford (2003) suggest a strong fiscal expansion that violates the transversality condition of RE equilibrium so that the indeterminate

\textsuperscript{37} The points with low interest rates and high outputs arise from expansionary policies used to push the economy out of recession and toward the targeted steady state. A more graduated fiscal stimulus, and additional frictions, would smooth this trajectory.
steady state ceases to exist. The economy then coordinates on the remaining non-explosive RE equilibrium.\textsuperscript{38}

A related approach relies on sunspot equilibria based on the targeted and unintended steady states. This can either be a stationary 2-state sunspot equilibrium, as in Aruoba, Cuba-Borda, and Schorfheide (2018) or a 2-state sunspot equilibrium with an absorbing state at the targeted steady state, as in Mertens and Ravn (2014). This approach does give full weight to self-fulfilling expectations; furthermore the state corresponding to deflation has lower output due not to a fundamental shock, but to a pure confidence shock. However, in addition to the practical question of exactly which sunspot variable coordinates expectations, there is also the issue of stability under learning.\textsuperscript{39} Furthermore, the associated recessions have relatively small magnitude: in the illustrations given in Mertens and Ravn (2014) the impact on output in the sunspot state is $-1.6\%$.

Regime shifts arising from policy shocks that follow a 2-state Markov chain also generate REE Markov chains, and these models have some affinity with models of sunspot equilibria. Bianchi and Melosi (2017) and Bianchi and Melosi (2019) introduce regime switching of monetary-led and fiscally-led policy mixes into an NK model under RE.\textsuperscript{40} These papers show how some policy regimes can contribute to long-run stability by mitigating occasional recessions subject to the ZLB, and how conflicts between monetary-led and fiscally-led policy coordination may lead to adverse outcomes that can be resolved by appropriate sequences of policy regimes.

There is also a substantial related literature focusing on “sentiments” or “confidence.” The term “sentiment” has been used in various ways in both RE and AL set-ups. Under RE it can be viewed as similar to an exogenous sunspot variable on which agents coordinate. Angeletos and La’O (2013) shows that if communication frictions between traders are introduced into a standard unique equilibrium model, sunspot-like extrinsic shifts in expectations can be self-fulfilling.\textsuperscript{41} For a recent example in a HANK model, with labor frictions and two steady states, see Lagerborg, Pappa, and Ravn (2021).\textsuperscript{42}

\begin{itemize}
\item \textsuperscript{38}This approach is potentially problematic as it relies on complete trust of RE asymptotics.
\item \textsuperscript{39}Two-state sunspot equilibria are not locally stable under learning when they are near two steady states, one of which is not locally stable under learning as in this case; e.g. see Evans and Honkapohja (2001), Chapter 12.
\item \textsuperscript{40}The regimes correspond to active money/passive fiscal and passive money/active fiscal policy combination in Leeper (1991).
\item \textsuperscript{41}In an RE set-up Benhabib, Wang and Wen (2015), relying on signal extraction problems, show that sentiment can lead to solutions far away from the usual RE solution.
\item \textsuperscript{42}In the context of an endogenous growth model with multiple steady states, Evans, Honkapohja, and Romer (1998) show that stochastic growth cycles are stable under AL.
\end{itemize}
In the AL literature, sentiment has been viewed as an extension in which serially correlated expectation shocks are added to AL forecasts to capture exogenous waves of optimism or pessimism. With this approach, using an estimated NK model with AL that includes survey data on expectations, Milani (2011) argues that sentiment driven by psychological factors, can explain a significant portion of business cycle fluctuations.\footnote{Evans and Honkapohja (2003), Section 4 considered optimal monetary policy under AL when expectations were affected by additional optimistic or pessimistic shocks.} Milani (2017) argues that sentiments are particularly important in explaining fluctuations in business investment, and Cole and Milani (2021) provide an extension to heterogeneous expectations. Also closely related to sentiments are the “exuberance” equilibria analyzed by Bullard, Evans, and Honkapohja (2008), Bullard, Evans, and Honkapohja (2009).

In contrast to these approaches the current paper does not require extrinsic variables to drive expectations. Within a familiar nonlinear global NK set-up, expectational dynamics are driven by recent observations of inflation and output. The existence of multiple steady states, with two distinct regions of expectational dynamics, implies a major role for macroeconomic policy in preventing the economy becoming trapped in a stagnation regime.

Schmitt-Grohe and Uribe (2017) and Eggertsson, Mehrotra, and Robbins (2019) develop models of secular stagnation with nominal wage stickiness. The Schmitt-Grohe and Uribe (2017) model has representative agents and downward wage rigidity taking the form of a lower bound on nominal wage growth, where the lower bound is negatively related to the unemployment rate. This setup yields two perfect foresight steady states: the targeted steady state and a steady state with involuntary unemployment and binding downward wage rigidity. From our perspective a major concern is that their recommended Fisherian monetary policy is premised on RE and is subject to the AL critique discussed in Howitt (1992) and Evans and McGough (2018).

Eggertsson, Mehrotra, and Robbins (2019) develop a perfect foresight/RE analysis with overlapping generations, downward sticky nominal wages and collateral constraints. The basic model assumes households with three-period lifetimes.\footnote{There is also a quantitative 56-period version of the model. Gibbs (2018) shows E-stability of the stagnation equilibrium in the model where agents live for three periods.} The model can have a locally determinate stagnation steady state with zero inflation target and negative real interest rate. The latter in turn requires a sufficiently tight collateral constraint that is well below aggregate output at full employment.

In contrast to this literature, our framework uses a benchmark representative-agent New Keynesian model which is the basis for more elaborate macroeconomic
models most frequently used for monetary and fiscal policy analysis. The nonlinear model, relying on a Rotemberg pricing friction is well known to have two steady states. We have established that when government spending is a partial substitute for private consumption, there is a third stagnation steady state, which is locally stable (under AL), in addition to the targeted and unintended steady states.\footnote{In Evans, Guse, and Honkapohja (2008) and Benhabib, Evans, and Honkapohja (2014) AL was introduced into the NK model with two steady states and deflation region arising from the ZLB. However, the destination of the possible deflationary paths was not resolved.}

8 Conclusions

Sluggish real economic performance and a long-lasting ZLB has made the possibility of secular stagnation a prominent topic of economic discussion. Japan has mostly been in this situation for over 20 years and the western economies US, Euro Area and UK for much of the post-2007 period. Our first objective in this paper was to extend a standard NK model in a way that exhibits stagnation as a well-defined regime for the economy, which is present despite the existence of a locally stable regime that includes the steady state targeted by policymakers.

Our model abstracts from exogenous technological progress, as well as population growth, so the stagnation region should be viewed as a “trap” in which economic activity, under normal policy, would tend to remain and fall further below potential output, with declining inflation and inflation expectations. The stagnation region contains a well-defined steady state – a theoretical lower bound on economic activity accompanied by rapid deflation – which acts as an attractor within the stagnation region in the absence of a strong policy response.

A second objective of our paper has been to consider the potential for fiscal policy to avoid or extract the economy from the stagnation region in a setting with boundedly rational, adaptive-learning agents. In this setting a fiscal stimulus, an appropriately chosen increase in government spending, can push the economy out of the stagnation trap. Simulations indicate that effectiveness of a fiscal stimulus (and thus the probability for escaping stagnation) depends not only on the size and length of the policy but also on the realized random sequence of exogenous random productivity and mark-up shocks. Important extensions and alternative policies were also discussed.

The results obtained in the paper are all based on the basic standard New Keynesian model and simple extensions. Particular crises, such as that due to the ongoing covid-19 shock, can require substantial extensions of the model to incorporate specific
key aspects of the crisis. However, the central features of our model will continue to be relevant. After the exogenous shocks have dissipated, there can be an expectational overhang due to economic experience during the crisis. If output and inflation expectations lie outside the domain of attraction of the targeted steady state, then extraordinary macroeconomic policies may be required.

References


Online Appendix

A Derivations of model equations, section 2

**Consumption decisions:** The consumption Euler equation is

\[
(c_{t,i} + \xi g_t)^{-1} = \beta R_t \hat{E}_{t,i} \left( \pi_{t+1}^{-1} (c_{t+1,i} + \xi g_{t+1})^{-1} \right),
\]

provided \(c_{t,i} > 0\). The household’s consumption decision rule is obtained by combining iterations of this with the household intertemporal budget constraint and its perceived intertemporal budget constraint for the government.

Ricardian households are assumed to internalize the intertemporal budget constraint (IBC) of the government. The flow budget constraint of the government is

\[
b_t = m_t + \Upsilon_t = g_t + m_{t-1} \pi_t^{-1} + r_t b_{t-1},
\]

where we now write \(r_t = R_{t-1} \pi_t^{-1}\). Setting \(\Delta_t = g_t - \Upsilon_t - m_t + m_{t-1} \pi_t^{-1}\) we have

\[
b_t = \Delta_t + r_t b_{t-1}.
\]

Note that \(\Upsilon_t + m_t - m_{t-1} \pi_t^{-1}\) is total tax revenue, equal to the sum of lump-sum taxes and seigniorage.

Substituting in recursively we obtain

\[
0 = r_t b_{t-1} + \sum_{j=1}^{s} D_{t,t+j}^{-1} \Delta_{t+j} + \Delta_t - D_{t,t+s}^{-1} b_{t+s}
\]

where \(D_{t,t+s}^{-1} = 1\). Imposing \(\lim_{s \to \infty} D_{t,t+s}^{-1} b_{t+s} = 0\) gives the IBC of the government,

\[
0 = r_t b_{t-1} + \sum_{j=0}^{\infty} D_{t,t+j}^{-1} \Delta_{t+j},
\]

where for convenience we set \(D_{t,t} = 1\).

For households the flow budget constraint

\[
c_{t,i} + m_{t,i} + b_{t,i} + \Upsilon_{t,i} = m_{t-1,i} \pi_t^{-1} + R_{t-1} \pi_{t}^{-1} b_{t-1,i} + (P_{t,i}/P_t) y_{t,i}
\]
can be written as
\[ b_{t,i} = \Lambda_{t,i} + r_t b_{t-1,i}, \]
where \( \Lambda_{t,i} = \frac{P_{t,i}}{P_t} y_{t,i} - \Upsilon_{t,i} - c_{t,i} - m_{t,i} + m_{t-1,i} \pi_t^{-1}. \)

Hence \( 0 = r_t b_{t-1,i} + \sum_{j=0}^{s} D_{t,t+j} \Lambda_{t+j,i} - D_{t,t+s} b_{t+s,i} \) and imposing \( \lim_{s \to \infty} D_{t,t+s} b_{t+s,i} = 0 \) gives the household IBC
\[ 0 = r_t b_{t-1,i} + \sum_{j=0}^{\infty} D_{t,t+j} \Lambda_{t+j,i}. \]

We have representative agents and assume they believe future lump-sum taxes and seigniorage revenue provided to the government will be identical across agents, so that
\[ \Upsilon_{t,i} - m_{t,i} + m_{t-1,i} \pi_t^{-1} = \Upsilon_t - m_t + m_{t-1} \pi_t^{-1} \]
and
\[ \Lambda_{t,i} = \frac{P_{t,i}}{P_t} y_{t,i} - \Upsilon_t - c_{t,i} - m_t + m_{t-1} \pi_t^{-1}. \]
It follows that
\[ \Lambda_{t+j,i} = \frac{P_{t+j,i}}{P_{t+j}} y_{t+j,i} - c_{t+j,i} - g_{t+j} + \Delta_{t+j}. \]

Incorporating the government IBC into the household IBC yields the Ricardian household IBC, which we assume holds in expectation, and with point expectations becomes
\[ 0 = \sum_{j=0}^{\infty} D_{t,t+j} e^{-1} \left( \frac{P_{t+j,i}}{P_{t+j}} y_{t+j,i}^e - c_{t+j,i}^e - g_{t+j}^e \right). \]

Finally, to obtain the household consumption function we make use of their consumption Euler equation
\[ (c_{t,i} + \xi g_t)^{-1} = \beta \hat{E}_{t,i} \left( r_{t+1}(c_{t+1,i} + \xi g_{t+1})^{-1} \right). \]

Iterating and assuming point expectations gives
\[ c_{t+j,i}^e = -\xi g_{t+j,i}^e + \beta^e \left( D_{t,t+j}^e \right) (c_{t,i} + \xi g_t). \]
Substituting for \( c_{t+j,i}^e \) in the household IBC and solving for \( c_t \) gives the consumption function
\[
(1 - \beta) \left[ \frac{P_{t,i}}{P_t} y_{t,i} - g_t \left( 1 + \frac{\xi \beta}{1 - \beta} \right) \right] + \\
(1 - \beta) \sum_{s=1}^{\infty} \left( D_{t,t+s,i}^e \right)^{-1} \left[ \left( \frac{P_{t+s,i}}{P_{t+s}} \right)^e y_{t+s,i}^e - g_{t+s,i}^e (1 - \xi) \right].
\]

Imposing the non-negativity constraint \( c_{t,i} \geq 0 \) gives (3) in the main text.
Impose now the representative agent assumption, \( c_{t,i} = c_t, \ y_{t,i} = y_t, \ \Xi_{t,i} = \frac{P_{t,i}}{P_t} = 1, \ y_{t+s,i}^c = y_{t+s}^c \) and \( g_{t+s,i}^c = g_{t+s}^c \). Assuming also agents have learned that \( \Xi_{t,i} = 1 \) we have \( \Xi_{t,s}^c = (\frac{P_{t+s,i}}{P_{t+s}})^c = 1 \). The market clearing equation \( y_t = c + g_t \) then yields

\[
y_t = \max \left\{ g_t, \beta (1 - \xi) g_t + (1 - \beta) \left[ y_t + \sum_{s=1}^{\infty} (D_{t,t+s}^c)^{-1} \left( y_{t+s}^c - (1 - \xi) g_{t+s}^c \right) \right] \right\}.
\]

Solving for \( y_t \) gives the temporary equilibrium output equation (5) in the main text.

**Remark:** In the Ricardian case, with monetary policy specified as an interest-rate rule, it is unnecessary to track money supply and demand. However, it is straightforward to show that with our utility function real money demand satisfies \( \mu_{t,i} = \chi (1 - R_t^{-1})^{-1} c_{t,i} \). The cashless limit corresponds to \( \chi \to 0 \).

**Production decisions:** The adjustment cost function \( \Phi(\frac{P_{t,i}}{P_{t-1,i}}) \) is the Linex function (see Kim and Ruge-Murcia (2009)), centered on \( \bar{\pi} \), given by

\[
\Phi(\frac{P_{t,i}}{P_{t-1,i}}) \equiv \frac{\phi}{\psi^2} \left[ \exp(-\psi(\frac{P_{t,i}}{P_{t-1,i}} - \bar{\pi})) + \psi(\frac{P_{t,i}}{P_{t-1,i}} - \bar{\pi}) - 1 \right],
\]

where \( \phi > 0 \), and we assume the case \( \psi > 0 \), consistent with asymmetric adjustment costs. The function \( \Phi'(\pi)\pi = (\phi/\psi) \pi (\exp(-\psi(\pi - \bar{\pi})) + 1) \) is monotonically increasing above a critical value \( \bar{\pi} \), given by the condition \( \frac{d}{d\pi} \Phi'(\pi) \pi = 0 \). We restrict attention to regions for which \( \pi > \bar{\pi} \). We compute the derivative

\[
\frac{d}{dP_{t,i}} \left[ \Phi \left( \frac{P_{t,i}}{P_{t-1,i}} \right) \right] = \frac{\phi}{\psi} P_{t-1,i} \left[ - \exp(-\psi(\frac{P_{t,i}}{P_{t-1,i}} - \bar{\pi})) + 1 \right].
\]

Note that \( \Phi'(\pi) = \frac{\phi}{\psi} (\exp(-\psi(\pi - \bar{\pi})) + 1) \), so

\[
\frac{d}{dP_{t,i}} \left[ \Phi \left( \frac{P_{t,i}}{P_{t-1,i}} \right) \right] = P_{t-1,i} \Phi'(\frac{P_{t,i}}{P_{t-1,i}}).
\]

The agent’s period utility is

\[
U_{t,i} = \log(c_{t,i} + \xi g_t) + \kappa \log \left( \frac{M_{t-1,i}}{P_t} \right) - (1 + \varepsilon)^{-1} h_{t,i}^c - \Phi \left( \frac{P_{t,i}}{P_{t-1,i}} - \bar{\pi} \right).
\]
and the first-order condition for optimal price setting is

\[ 0 = \frac{\partial U_{t,i}}{\partial P_{t,i}} + \beta E_{t,i} \frac{\partial U_{t+1,i}}{\partial P_{t,i}} = \frac{\nu_t}{\alpha} h_{t,i}^{\varepsilon+1} \frac{1}{P_{t,i}} - \Phi'(\pi_{t,i}) \frac{1}{P_{t-1,i}} \]

\[ + (c_{t,i} + \xi g_t)^{-1} (1 - \nu_t) y_t \left( \frac{P_{t,i}}{P_t} \right)^{-\nu_t} \frac{1}{P_t} + \beta \Phi'(\pi_{t+1,i}^e) \left( \frac{P_{t+1,i}}{P_{t,i}^2} \right)^e, \]  

where again we have used point expectations and here \( \pi_{t,i} = P_{t,i}/P_{t-1,i} \). Multiplying the right-hand side by \( P_{t,i} \) we can write this equation as

\[ \Phi'(\pi_{t,i}) \pi_{t,i} = \frac{\nu_t}{\alpha} h_{t,i}^{\varepsilon+1} + (c_{t,i} + \xi g_t)^{-1} (1 - \nu_t) y_t \left( \frac{P_{t,i}}{P_t} \right)^{1-\nu_t} + \beta \Phi'(\pi_{t+1,i}^e) \pi_{t+1,i}^e. \]  

We now discuss the properties of

\[ \Phi'(\pi)\pi = \frac{\phi}{\psi} \pi (- \exp(-\psi(\pi - \pi^*)) + 1). \]  

The function \( \Phi'(\pi)\pi \) is monotonically increasing above a critical value \( \bar{\pi} \) which is given by the condition

\[ \frac{d}{d\pi} \Phi'(\pi)\pi = 0. \]  

We compute the derivative

\[ \frac{d}{d\pi} \Phi'(\pi)\pi = \frac{\phi}{\psi} (1 - (1 - \phi\pi) \exp(-\psi(\pi - \pi^*))), \]

so the condition giving \( \bar{\pi} \) can be written as

\[ 1 = (1 - \phi\pi) \exp(-\psi(\pi - \pi^*)). \]

This equation has a unique solution \( \bar{\pi} < \phi^{-1} \). It is easily seen that (i) \( \bar{\pi} \) is increasing in \( \psi \) with \( \lim_{\psi \to \infty} \bar{\pi} = 1/\phi^{-1} \) and (ii) \( \bar{\pi} \) is decreasing in \( \phi \) with \( \lim_{\phi \to \infty} \bar{\pi} = 0 \) ceteris paribus. Throughout the paper we restrict attention to regions for which \( \pi > \bar{\pi} \). In the calibrated model we will compute \( \bar{\pi} \) to check and impose the inequality \( \pi > \bar{\pi} \) when solving for the temporary equilibrium.
Using the production function, \( \zeta_{t,i} \) in the main text is

\[
\zeta_{t,i} = \frac{\nu_t}{\alpha} \left( \frac{y_{t,i}}{A_t} \right)^{(1+\epsilon)/\alpha} - (\nu_t - 1) (c_{t,i} + \xi g_t)^{-1} y_t \left( \frac{P_{t,i}}{P_t} \right)^{1-\nu_t}
\]

Here

\[
y_{t,i} = \int_0^1 c_{t,j}(i) dj + g_t(i) = \frac{c_t(i) + g_t(i)}{A_t}
\]

is the total demand for variety \( i \).

Note that the term \( y_t \left( \frac{P_{t,i}}{P_t} \right)^{1-\nu_t} \) combines \( y_t \), which is exogenous to the firm, with the relative price \( \frac{P_{t,i}}{P_t} \), in which the aggregate price level is exogenous while \( P_{t,i} \) is a decision variable of the firm. Iterating forward we get the expression (4)

\[
\Phi' (\pi_{t,i}) \pi_{t,i} = \zeta_{t,i} + \sum_{s=1}^{\infty} \beta^s \zeta^s_{t+s,i},
\]

which is our infinite-horizon pricing decision rule. Here \( \zeta^s_{t+s,i} \) is the point expectation of

\[
\zeta_{t+s,i} = \frac{\nu_{t+s}}{\alpha} \left( \frac{y_{t+s,i}}{A_{t+s}} \right)^{(1+\epsilon)/\alpha} - (\nu_{t+s} - 1) y_{t+s} \left( \frac{P_{t+s,i}}{P_{t+s}} \right)^{1-\nu_t} \times (c_{t+s,i} + \xi g_{t+s})^{-1},
\]

where

\[
y_{t+s,i} = c_{t+s}(i) + g_{t+s}(i)
\]

is the future market demand for variety \( i \).

### B Implementation of stochastic model

From Section 2 we have the representative agent NK PC temporary equilibrium (TE) equation

\[
Q (\pi_t) = \zeta_t + \sum_{s=1}^{\infty} \beta^s \zeta^s_{t+s}, \quad \text{where} \quad Q (\pi_t) = \Phi' (\pi_t) \pi_t,
\]
with \( Q(\pi_t) > 0 \) for \( \pi_t > \tilde{\pi} \), and where
\[
\zeta_t = \frac{\nu_t}{\alpha} \left( \frac{y_t}{A_t} \right)^{(1+\varepsilon)/\alpha} - (\nu_t - 1) y_t \times \left( y_t - (1 - \xi) g_t \right)^{-1}
\]
and
\[
\zeta_{t+s}^e = \frac{\nu_t}{\alpha} \left( \frac{y_{t+s}^e}{A_{t+s}} \right)^{(1+\varepsilon)/\alpha} - (\nu_{t+s}^e - 1) y_{t+s}^e \times \left( y_{t+s}^e - (1 - \xi) g_{t+s}^e \right)^{-1},
\]
for \( s = 1, 2, 3, \ldots, T \). As we discuss below, we will set \( \zeta_{t+s}^e \) at its perceived mean value after \( T + 1 \) periods, for some (suitably large) period \( T \). It is assumed that \( \Xi_t \equiv 1 \) as discussed in Section 2. Note that \( \nu_t \) is stochastic and we assume that, faced with stochastic shocks in a nonlinear setting, agents use point expectations. We also assume that the future path of government spending is credibly announced and implemented; hence \( g_{t+s}^e = g_{t+s} \).

Also from Section 2, the aggregate demand temporary equilibrium equation (5) is obtained by combining the IH consumption function, market clearing, i.e. \( y_t = c_t + g_t \), and \( \Xi_t \equiv 1 \). This yields the TE equation for output
\[
y_t = (1 - \xi) g_t + (\beta^{-1} - 1) \sum_{s=1}^{\infty} (D_{t,t+s}^e)^{-1} \left( y_{t+s}^e - (1 - \xi) g_{t+s}^e \right),
\]
where
\[
D_{t,t+s}^e = \prod_{j=1}^{s} r_{t+j}^e \quad \text{and} \quad r_{t+j}^e = \frac{R_{t+j-1}^e}{\pi_{t+j}^e}.
\]
Here for \( j = 1 \) we have \( R_t^e = R_t \).\(^{46}\) The interest-rate rule is assumed known and is given by a forward-looking rule (8). As before, point expectations for forecasting all unknown future values is assumed.

We turn next to the data-generating process for the stochastic shocks. As assumed in Section 4, \( \ln A_t \) and \( \ln \nu_t \) are independent stationary exogenous AR(1) processes
\[
\ln \left( \frac{A_{t+1}}{\bar{A}} \right) = \rho_A \ln \left( \frac{A_t}{\bar{A}} \right) + \ln \varepsilon_{A,t+1}
\]
where \( 0 \leq \rho_A < 1 \), \( \ln \varepsilon_{A,t} \sim N(0, \sigma_A^2) \), and
\[
\ln \left( \frac{\nu_{t+1}}{\bar{\nu}} \right) = \rho_{\nu} \ln \left( \frac{\nu_t}{\bar{\nu}} \right) + \ln \varepsilon_{\nu,t+1}
\]

\(^{46}\)As we explain below we need to replace \( r_{t+j}^e = R_{t+j-1}^e/\pi_{t+j}^e \) by \( r_{t+j}^e = \beta^{-1} \) for \( j \geq T_1 \) for some positive \( T_1 \).
where $0 \leq \rho_{\nu} < 1$, $\ln \varepsilon_{\nu,t} \overset{\text{iid}}{\sim} N(0, \sigma^2_{\nu})$.

It follows that

$$\nu_{t+1}/\nu = (\nu_{t}/\nu)^{\rho_{\nu}} \varepsilon_{\nu,t+1} \quad \text{and} \quad A_{t+1}/\bar{A} = (A_t/\bar{A})^{\rho_{A}} \varepsilon_{A,t+1},$$

and that

$$\nu_{t+s}/\nu = (\nu_{t}/\nu)^{\rho_{\nu}} \prod_{j=0}^{s-1} \varepsilon_{\nu,t+j}^{\rho_{\nu}}.$$ 

Under point expectations $\ln \varepsilon_{\nu,t+j}^{e} = 0$ and $\varepsilon_{\nu,t+j}^{e} = 1$ so that

$$\nu_{t+s}^{e} = \bar{\nu} (\nu_{t}/\nu)^{\rho_{\nu}},$$

and analogously we have

$$A_{t+s}^{e} = \bar{A} (A_t/\bar{A})^{\rho_{A}}.$$ 

In Section 4, the PLMs for output and inflation use a linear forecasting rule based of the observed exogenous variables. To first-order these correspond to a stochastic REE at a steady state. Thus the perceived laws of motion are

$$\ln (y_t) = f_y + d_y A \ln (A_t/\bar{A}) + d_{y\nu} \ln (\nu_t/\bar{\nu}) + \eta_{yt}$$

$$\ln (\pi_t) = f_{\pi} + d_{\pi} A \ln (A_t/\bar{A}) + d_{\pi\nu} \ln (\nu_t/\bar{\nu}) + \eta_{\pi t},$$

where $\eta_{yt}, \eta_{\pi t}$ are perceived white noise shocks.

Under recursive least squares (RLS) learning the coefficient vectors, $\phi_y, \phi_\pi$ where $\phi_y' = (f_y, d_y A, d_{y\nu})$ and $\phi_\pi' = (f_{\pi}, d_{\pi} A, d_{\pi\nu})$, are time-varying and updated over time using recursive least squares regressions of $(\ln (y_t), \ln (\pi_t))$ on $x_t' = (1, \bar{A}_t, \bar{\nu}_t)$. The recursive updating equations, which are standard, are

$$\phi_{yt} = \phi_{yt-1} + \omega_t R^{-1}_{t} x_{t-1} (y_{t-1} - \phi_{yt-1} x_{t-1})$$

$$\phi_{\pi t} = \phi_{\pi t-1} + \omega_t R^{-1}_{t} x_{t-1} (\pi_{t-1} - \phi_{\pi t-1} x_{t-1})$$

$$R_t = R_{t-1} + \omega_t (x_{t-1} x_{t-1}' - R_{t-1}).$$

(27)

Note that $\phi_{yt}, \phi_{\pi t}$ are updated based on their most recent forecast errors. Here $R_t$ is an estimate of the second-moment matrix of regressors. RLS updating equations allow for a time-varying gain $\omega_t$. We focus on the constant gain case $\omega_t = \omega$ for $0 < \omega < 1$. In the decreasing gain case, $\omega$ is replaced by $0 < \omega_t < 1$ with $\omega_t \to 0$ at an appropriate rate, for instance at rate $t^{-1}$.

Under constant gain RLS learning the coefficients $\phi = (f_y, d_y A, d_{y\nu}, f_{\pi}, d_{\pi} A, d_{\pi\nu})$

$^{47}$See, for example, Evans and Honkapohja (2001), Chapter 2, or Evans and Honkapohja (2009).
are time-varying and updated over time using recursive least squares regressions of 
$\ln(y_t), \ln(\pi_t)$ on $(1, A_t, \bar{\nu}_t)$.

Letting $f_y, d_y, d_y', f_x, d_x, d_x'$ now denote the time $t$ values of their estimates, 
equations of output $s$ steps ahead are given by

$$y_{t+s}^e = e^{f_y} A_t e^{d_y} d_y^e - e^{d_y} d_y^e$$

for $s = 1, \ldots, T$,

where as usual point expectations are assumed. Here $T$ denotes the period after 
which agents believe that all relevant processes will have reverted to their mean 
steady-state values. Thus

$$\zeta_{t+s}^e = \bar{\zeta} \equiv \left( \frac{e^{f_y}}{A} \right)^{(1+\varepsilon)/\alpha} - (\bar{\nu} - 1) e^{f_y} \times (e^{-1} \bar{\gamma})^{-1}$$

for $s \geq T$.

Using these expectations $\pi_t$ is determined by the temporary equilibrium equation

$$Q(\pi_t) = \zeta_t + \sum_{s=1}^{T} \beta^s \zeta_{t+s} + \frac{\beta^{T+1}}{1 - \beta} \zeta.$$  \hspace{1cm} (28)

We now turn to the aggregate demand temporary equilibrium equation (5). Expectations $y_{t+j}^e$ and $\pi_{t+j}$ are given as above. For the discount factors we have

$$D_{t,t+s}^e = \prod_{j=1}^{s} \frac{R_{t+j-1}^e}{\pi_{t+j}}.$$ 

where $R_{t+j-1}^e$ is given by the forward-looking $R$-rule (8). Here $R_t = R(\pi_{t+1}, y_{t+1})$ 
with $R_{t+j-1}^e = R(\pi_{t+j}, y_{t+j}^e)$, so that

$$D_{t,t+s}^e = \prod_{j=1}^{s} \frac{R(\pi_{t+j}, y_{t+j}^e)}{\pi_{t+j}}.$$ for $s \leq T_1 - 1$.

The restriction $s \leq T_1 - 1$ is included because in order to ensure that consumption
and output is positive and finite we need discount factors \( D_{t,t+s}^e \) to be bounded above 1. This can be an issue because the interest rate \( R(\pi^e, y^e) \) can be less than \( \pi^e \) for some \( \pi^e \) between \( \pi_L \) and \( \pi^* \) and for a range of \( y^e \) if the rule also depends on \( y^e \). This difficulty is avoided by assuming that after \( T_1 \) periods the expected real interest rate factor is the steady-state value \( \beta^{-1} \). Thus we assume \( r_{t+j}^e = \beta^{-1} \) for \( j \geq T_1 \) implying

\[
D_{t,t+s}^e = \prod_{j=1}^{T_1-1} \frac{R(\pi_{t+j}^e, y_{t+j}^e)}{\pi_{t+j}^e} \beta^{-(s-(T_1-1))}, \quad \text{for } s \geq T_1.
\]

We assume that the “truncation” parameter \( T > T_1, T_p \).

Incorporating the assumption that expectations \( y^e, \pi^e \) return to their perceived steady-state values after \( T \) periods, we arrive at the aggregate demand temporary equilibrium equation

\[
y_t = (1 - \xi) g_t + (\beta^{-1} - 1) \sum_{s=1}^{T} (D_{t,t+s}^e)^{-1} (y_{t+s}^e - (1 - \xi) g_{t+s}^e) + \quad (29)
\]

\[
(y^e - \bar{g} (1 - \xi)) \sum_{s=T+1}^{\infty} (D_{t,t+s}^e)^{-1}, \quad \text{where}
\]

\[
\sum_{s=T+1}^{\infty} (D_{t,t+s}^e)^{-1} = (D_{t,t+T_1-1}^e)^{-1} (1 - \beta)^{-1} \beta^{T-T_1+2}.
\]

The latter equation is obtained from

\[
\sum_{s=T+1}^{\infty} (D_{t,t+s}^e)^{-1} = (D_{t,t+T_1-1}^e)^{-1} \sum_{s=T+1}^{\infty} \beta^{s-T_1+1}.
\]

The forward-looking \( R \)-rule has the advantage that \( \pi_t, y_t \) can be solved explicitly using the above equations (28) and (29). As already noted, the results for the contemporaneous rule are similar.

Next, we discuss the interpretation of the numerical magnitudes pointed out at the end of Section 4. To understand the magnitude of the expectation shocks given in Table 1, it is helpful to consider a reinterpretation of the role of \( y^e \) in the temporary equilibrium model. For the consumption function (3), assuming the representative agent case with \( \Xi_t^e \equiv 1 \), it can be seen that consumption, and hence temporary

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equilibrium output $y_t$, depend to first-order on $\{y^e_{t+s}\}_{s=1}^{\infty}$ through its present value

$$\text{PV} (\{y^e_{t+s}\}_{s=1}^{\infty}) = \sum_{s=1}^{\infty} (D^e_{t,t+s})^{-1} y^e_{t+s} \approx \sum_{s=1}^{\infty} \beta^s y^e_{t+s}. $$

We have interpreted steady state learning as agents acting as if $y^e_{t+s} = y^e_t$ for all horizons $s = 1, 2, 3 \ldots$. However this is behaviorally equivalent to assuming that agents have an expected output profile with the same present value as $\text{PV} (\{y^e_{t+s} = y^e_t\}_{s=1}^{\infty})$.

In particular suppose that agents believe $y^e_{t+s} = \hat{y} < y^*$ for $s = 1, \ldots, L$ periods, interpreted as quarters, followed by $y^e_{t+s} = y^*$ for $s > L$, i.e. a recession of $L$ periods followed by a return to targeted steady state. Then the PV of the $L$-period recession output expectation sequence equals the PV of a constant sequence $\tilde{y}^e < y^*$ when

$$\sum_{s=1}^{\infty} \beta^s \tilde{y}^e = \sum_{s=1}^{L} \beta^s \hat{y}^e + \sum_{s=L+1}^{\infty} \beta^s y^* \text{ or } \tilde{y}^e = \hat{y} + \beta^L (y^* - \hat{y}). \tag{30}$$

As an example assume $\beta = 0.99$ and consider an expectations shock $\hat{y}$ lasting two years i.e. $L = 8$. Then compute $\hat{y}$ that is equivalent to permanent shock at $\tilde{y}^e = 0.99745$ which in Table 1 is slightly above the boundary of the stochastic domain of attraction at $\pi^e = \pi^*$ for the targeted steady state. Then (30) yields $\hat{y} = 0.96663$ which corresponds to an expected recession approximately equal to over 3.34% reduction of expected GDP relative to target $y^*$ during two years, followed by a return to normal value.

## C Calibration details

The parameter values are $\pi^* = 1.005$, $\beta = 0.99$, $\alpha = 0.7$, $\xi = 0.4$, $\bar{A} = 1.113$, $\nu = 13.5$, $\phi = 75$, $\psi = 20$, $\varepsilon = 1$, $\bar{g} = 0.2$, $B = 1.5/R^*$, $\phi_y = 8.25$. We set $\phi = 75$, $\psi = 20$ based on comparing Linex-type functions to a quadratic adjustment cost function at the most common range for $\pi$. Here are some comments about these values.

The parameter values in the main text are chosen as follows. $\alpha = 0.7$, $\beta = 0.99$ and $\varepsilon = 1$ are standard. There are various suggestions for $\xi$ and we set $\xi = 0.4$. The frequency of price change is that $1/3$ ($= 1 - \eta$) of firms change prices per quarter. This is consistent with Nakamura and Steinsson (2008) and Kehoe and Midrigan (2015).

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48 The computation assumes unchanged fiscal policy and unchanged monetary policy given by the specified Taylor-rule.
Various estimates of $\nu$ or of the markup $\nu/(\nu - 1)$ have been used with estimates of $\nu$ ranging from 21 to 3.5. Keen and Wang (2007) give the relation between these parameters and the Rotemberg quadratic adjustment cost parameter

$$\gamma = \frac{(\nu - 1) \eta}{(1 - \eta)(1 - \beta \eta)}.$$  

We choose $\nu = 13.5$, a markup of about 8%, and $\eta = 0.67$ which gives $\gamma = 75$.

For Linex adjustment cost functions the parameter estimates for $\phi$, $\psi$ vary widely. Note that $\phi \rightarrow \gamma$ as $\psi \rightarrow 0$. In most papers adjustment costs are assumed to be proportional to output or profit, whereas we use a non-proportional setup to avoid multiplicities. However as we normalize steady-state target output to $\gamma = 1$ then the parameters are comparable. Also near the steady state marginal utilities drop out to first order. We choose $\phi = 75$ and $\psi = 20$, which gives a fairly close approximation to the quadratic in the range $\pi = 1.00$ to $\pi = 1.01$, i.e. 0 to 4% annual inflation.

For technology we set $\bar{A} = 1.113$. A high steady state is $\bar{y} \approx 1.00003 \approx 1$ with $\bar{g} = 0.2$. For productivity and mark-up shock calibrations we set first-order autocorrelation parameters to $\rho_A = \rho_\nu = 0.5$ and standard deviations for the log innovations, in decimal form, to $\sigma_A = 0.0015$ and $\sigma_\nu = 0.0001$. Both the serial correlation and autocorrelation parameters are smaller than those found by Smets and Wouters (2007), but their estimates are for models under RE and with additional frictions. Adaptive learning dynamics add additional volatility relative to RE, particularly in purely forward-looking models.

## D Calibrating the Taylor rule

To calibrate the interest rate rule

$$R_t = R(\pi_{t+1}, y_{t+1}) = 1 + (R^* - 1) \left( \frac{\pi_{t+1}^e}{\pi^*} \right)^{BR^*/(R^* - 1)} \left( \frac{y_{t+1}^e}{y^*} \right)^{\phi_\nu},$$

(31)

where $y^*$ is output level at the target steady state, we relate (31) to the usual Taylor rule. Rearranging and taking logs we get

$$\log(R_t - 1) - \log(R^* - 1) = \frac{BR^*}{R^* - 1}(\log \pi_{t+1}^e - \log \pi^*) + \phi_y^e(\log y_{t+1}^e - \log y^*).$$
Multiplying by \((R^* - 1)\) and approximating log differences by percentage changes we get
\[
R_t - R^* = BR^* \left( \frac{\pi_{t+1}^e - \pi^*}{\pi^*} \right) + (R^* - 1)\phi_y \left( \frac{y_{t+1}^e - y^*}{y^*} \right).
\]

Thus \(BR^*\) is the inflation coefficient and \((R^* - 1)\phi_y\) is the output coefficient in the usual linear Taylor rule. Assuming a quarterly calibration one should have
\[
BR^* = 1.5, \quad (R^* - 1)\phi_y = \frac{0.5}{4}.
\]

At the target steady state \(R^* = \beta^{-1}\pi^*\) we get
\[
\phi_y = \frac{0.5}{4}/(0.01515) \approx 8.25
\]
when \(\beta = 0.99\) and \(\pi^* = 1.005\).

### E Proof of Proposition 1

We start by computing the partial derivatives of the right-hand sides of differential equations (12)-(13):
\[
\frac{\partial F_y}{\partial \pi^e} = D_y G_1 \frac{\partial G_2}{\partial y^e} - 1, \quad \frac{\partial F_y}{\partial y^e} = D_y G_1 + D_y G_1 \frac{\partial G_2}{\partial y^e}.
\]

and
\[
\frac{\partial F_y}{\partial \pi^e} = D_y G_2, \quad \frac{\partial F_y}{\partial y^e} = D_y G_2 - 1.
\]

The E-stability differential equations in vector form are
\[
\left( \begin{array}{c} \frac{\partial \pi^*}{\partial \pi^e} \\ \frac{\partial \pi^*}{\partial y^e} \end{array} \right) = \left( \begin{array}{c} F_y(\pi^e, y^e) \\ F_y(\pi^e, y^e) \end{array} \right),
\]

where \(F_y(.,.)\) and \(F_y(.,.)\) are given in (12) and (13). We get the Jacobian
\[
DFI = \left( \begin{array}{cc} D_y G_1 D_{\pi^e} G_2 - 1 & D_y G_1 + D_y G_1 D_{\pi^e} G_2 \\ D_y G_2 & D_y G_2 - 1 \end{array} \right).
\]

**Proof of Proposition 1:** (a) (i) Consider the case \(\phi_y = 0\). Calculating the
derivatives of the Jacobian at the target steady state we get

\[(Q^{-1})' = (\Phi'' \pi + \Phi')^{-1} = (\phi \pi^*)^{-1} > 0\] so

\[D_y G_1 = (Q^{-1})' \left( \frac{\nu(1+\varepsilon)}{\alpha^2} (y^*/A)^{(1+\varepsilon)/\alpha-1} + \frac{(\nu-1)(1-\xi)g}{(y^* - (1-\xi)g)^2} \right) > 0.\]

\[D_{\pi^*} G_2 = (\beta^{-1} - 1)(y^* - g(1 - \xi)) \left( \frac{R(\pi_t^*, y_t^*) - \pi_t^* D_\pi R(\pi_t^*, y_t^*)}{(R(\pi_t^*, y_t^*) - \pi_t^*)^2} \right) < 0.\]

As

\[D_\pi R(\pi, y) = \frac{BR^*}{\pi^*} \left( \frac{\pi}{\pi^*} \right)^{(R^*(B-1)+1)/(R^*-1)} \left( \frac{y}{y^*} \right) \phi_y\]

we have \(D_\pi R(\pi, y) = \frac{BR^*}{\pi^*}\) and so \(R(\pi^*, y^*) - \pi^* D_\pi R(\pi^*, y^*) = \beta^{-1} \pi^*(1 - B) < 0\) at the target steady state. Also

\[D_{y^*} G_2 = 1\]

if \(\phi_y = 0\). So in this case we get

\[DFI = \begin{pmatrix} - & + & 0 \\ - & 0 & 0 \end{pmatrix}\]

which has negative trace and positive determinant and is thus a stable matrix. The result follows by continuity of eigenvalues.

If \(\phi_y\) is not zero, we have

\[D_{y^*} G_2 = 1 + (\beta^{-1} - 1) \left[ (1 - g(1 - \xi)) \left( \frac{\pi}{D_y R(\pi, y) - \pi} \right) \right].\]

For the interest rate rule we get

\[D_y R(\pi, y) = (R^* - 1) \left( \frac{\pi}{\pi^*} \right)^{(R^*/B)/(R^*-1)} \frac{\phi_y}{y^*} \left( \frac{y}{y^*} \right) \phi_y^{-1},\]

so

\[D_y R(\pi^*, y^*) = (R^* - 1) \phi_y / y^*\]
at the target steady state and thus

\[ D_{y^*}G_2 = 1 + (\beta^{-1} - 1) \left[ (1 - g(1 - \xi)) \frac{\pi^*}{(R^* - 1)\phi_y/y^* - \pi^*} \right]. \]

Now \( R^* - 1 \gtrsim 0 \) is small, something like 0.02, while \( \phi_y \approx 8 \) and \( y^* \approx 1 \), whereas \( \pi^* \gtrsim 1 \). Then \( D_{y^*}G_2 < 1 \) and the targeted steady state is E-stable.

(ii) Doing calculations similar to above, set first \( \phi_y = 0 \) and we get

\[
(Q^{-1})' = \left( \frac{\phi(1 - (1 - \psi \pi_L) \exp(\pi_L - \pi^*))}{\psi} \right)^{-1} > 0 \text{ normally so}
\]

\[ D_{\pi^*}G_1 = (Q^{-1})' \left( \frac{\nu(1 + \varepsilon)}{\alpha^2} (y_L/A)^{1+\varepsilon-\alpha/\alpha} + \frac{(\nu - 1)(1 - \xi)g}{(y_L - (1 - \xi)g)^2} \right) > 0. \]

\[ D_{\pi^*}G_2 = (\beta^{-1} - 1)(y_L - g(1 - \xi)) \left( \frac{R(\pi_L, y_L) - \pi_L D_{\pi^*}R(\pi_L, y_L)}{(R(\pi_L, y_L) - \pi_L)^2} \right) < 0, \]

as

\[
1 + (R^* - 1) \left( \frac{\pi_L}{\pi^*} \right)^{BR^*/(R^*-1)} \left( \frac{y_L}{y^*} \right)^{\phi_y} < B \beta^{-1} \pi^* \left( \frac{\pi_L}{\pi^*} \right)^{BR^*/(R^*-1)} \left( \frac{y_L}{y^*} \right)^{\phi_y} \text{ normally.}
\]

Also

\[
D_{y^*}G_1 = (Q^{-1})' \frac{\beta}{1 - \beta} \left[ \frac{\nu(1 + \varepsilon)}{\alpha^2} (y_L/A)^{1+\varepsilon-\alpha/\alpha} + \frac{(\nu - 1)(1 - \xi)g}{(y_L - (1 - \xi)g)^2} \right] > 0
\]

\[ D_{y^*}G_2 = 1 - \frac{\phi_y(y_L/y^*)(\beta^{-1} \pi_L - 1)}{(\beta^{-1} - 1)\pi_L}. \]

Normally, \( D_{y^*}G_1 + D_{y^*}G_1D_{y^*}G_2 > 0 \) and considering the case \( \phi_y = 0 \), we get

\[ DFI = \left( \begin{array}{ccc} ? & + & + \\ + & 0 & + \end{array} \right). \]

It is seen that the determinant of \( DFI \) is negative, so the \((\pi_L, y_L)\) is not E-stable.

The case \( \phi_y > 0 \) but not too large also leads to instability depending on the parameter values. This is true in numerical analyses where \( \phi_y = 8.25 \).

(iii) At the stagnation steady state \( y_t = G_2(\pi_t^e, y_t^e) \) has to be locally constant, so
\[ D_{\pi^c}G_2 = D_y G_2 = 0. \] Then the Jacobian matrix becomes
\[ DFI = \begin{pmatrix} -1 & D_{\pi^c}G_1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & + \\ 0 & -1 \end{pmatrix} \]
as
\[ D_y G_1 = (Q^{-1})' \frac{\beta}{1 - \beta} \left[ \frac{\nu(1+\varepsilon)}{\alpha^2} (g)^{1+\varepsilon-\alpha}/\alpha + \frac{(\nu - 1)(1 - \xi)g}{(\xi g)^2} \right] > 0. \]

Given the sign pattern for DFI, local stability is evident.

(b) In non-stochastic models, with constant-gain, steady-state learning is locally stable for sufficiently small gains if and only if E-stability holds. We provide a sketch of the proof. Consider a linear(-ized) model \( y_t = Ty^e_t \), assuming zero intercept without loss of generality, and the constant-gain rule \( a_t = a_{t-1} + \gamma(y_t - a_{t-1}) \), where \( 0 < \gamma < 1 \). The PLM is \( y^e_t = a_{t-1} \). Then \( a_t = a_{t-1} + \gamma(Ta_{t-1} - a_{t-1}) \) and the system is convergent if matrix \( \gamma T + (1 - \gamma)I \) has eigenvalues \( t_i \) inside the unit circle; equivalently \( T + \gamma^{-1}(1 - \gamma)I \) has eigenvalues inside the circle with radius \( \gamma^{-1} \). Eigenvalues of the latter matrix are equal to \( t_i + \gamma^{-1}(1 - \gamma) \), so the \( t_i \) values must lie inside unit circle with origin at \((1 - \gamma^{-1}, 0)\) and radius \( \gamma^{-1} \). Letting \( \gamma \to 0 \) yields the E-stability condition that real parts of \( t_i \) must be less than 1. \( \blacksquare \)

### F Figure 1 and Figure 7 Details

Figure 1 data details:

Figure 1, left panel: The interest-rate rule curve takes the form \( I = A \ast \exp(B \Pi) \), where \( \Pi \) denotes net inflation and \( I \) denotes the net interest rate. Japan switched the policy target in 2013 to monetary base.

Figure 1, right panel: Macrobond data base which in turn utilizes standard data sources. GDP data is volume data with 2010 as reference year and in local currency. GDP data is annualized. This was specifically done for the Euro area by multiplying quarterly data by 4. Population data is total population and it is interpolated for quarters.

Figure 7 simulation details:

In the simulations used for Figure 7, the pricing friction \( \Phi \) is modified so that \( Q(\pi) \equiv \Phi'(\pi)\pi \) is replaced by \( QQ(\pi) \). Over \( \pi_{Lc1} < 1 < \pi_{Rc1} \) \( Q \) is unchanged, i.e. \( QQ(\pi) = Q(\pi) \). For \( \pi_{Lc2} < \pi_{Lc1} \) and for \( \pi_{Rc1} < \pi_{Rc2} \) the function \( QQ(\pi) \) modifies \( Q(\pi) \) using logit-type asymptotes that give elastic effective bounds on inflation and deflation. Because \( \pi_{Lc1} < \pi_L < \pi^* < \pi_{Rc1} \), the targeted and unintended steady states \((\pi^*, y^*)\) and \((\pi_L, y_L)\) are unchanged, as are their local dynamics under adaptive
learning. The stagnation steady state remains locally stable under learning but has a deflation rate corresponding to $\pi_{L<2}$.

The Figure 7 scatterplots combine simulated data from three stochastic simulations, each of 80 periods length. These correspond to three expectations starting points: (i) $(\pi^e, y^e)$ close to $(\pi_L, y_L)$ but with $\pi^e < \pi_L$ and $y^e < y_L$; (ii) initial $(\pi^e, y^e)$ close to $(\pi^*, y^*)$, but with initial $y^e$ slightly below $y^*$, (iii) initial $(\pi^e, y^e)$ close to $(\pi^*, y^*)$, but with initial $\pi^e$ slightly below $\pi^*$. In Figure 7 simulations we set $T_1 = 8$, which also acts to moderate the response of output and inflation to expectations.

For simulations (ii) and (iii) normal policy is followed. For simulation (i) stimulative monetary and fiscal policies are followed after delays. Specifically, in simulation (i) normal policies are initially followed for 12 quarters. During this period inflation and output, and their expectations, gradually fall, with the negative output gap reaching 3%. After this delay monetary policy drops the net interest rate to an “effective lower bound” of 0.8% per year and, using forward guidance, holds it there for 14 quarters. (If a credit friction were included a similar outcome would arise with a policy net interest rate at zero). This policy is not enough to begin a sustained recovery, and in period 26 policy adds a large fiscal stimulus (increasing $g$ from 0.20 to 0.34 for 18 quarters). These measures together increase output and inflation substantially, eventually returning the economy to the targeted steady state.

G Further Numerical Results, Section 5

A systematic analysis of the case $y_0^c = 0.997 \times y^*$ of Section 5 is now conducted. The magnitude and length of fiscal policy are varied and the estimated probability of the economy going back to target steady state is computed. The expectation shock is $y_0^c = 0.997 \times y^*$, $\pi_0^c = \pi^*$ and Table A.1 gives the estimated probabilities of convergence to the targeted steady state (vs. eventual convergence to the stagnation steady state) for alternative values of the length $T_p$ and the magnitude $\bar{g}'$ of fiscal policy.
Table A.1: Percentage of simulations in which fiscal policy successfully results in convergence to the targeted steady state starting from pessimistic output expectations $y_0^e = 0.997 \times y^*$. Based on 100 replications with length 500.

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Table A.1 shows that the sequence of serially correlated random productivity and mark-up shocks can matter: for a fiscal policy that is usually successful, a particularly unfavorable sequence of shocks can adversely affect expectations enough to prevent the policy from working. However, for a substantial range of policies, with $\bar{g}'$ between 0.25 and 0.35 with $T_p$ between 2 and 4 quarters, a fiscal stimulus is successful approximately 100% of the time.

In these cases the cumulative fiscal spending multipliers would of course be huge, reflecting the fact that a temporary fiscal stimulus prevents the economy from descending into stagnation and pushes it back toward convergence to the targeted steady state.

It can also be seen that in many cases a fiscal stimulus that is too long can be counterproductive. For example, for $g = 0.30$ the effectiveness of the stimulus decreases greatly if $T_p$ is increased to $T_p = 7$ quarters or longer. This is a reflection of the negative effect on consumption of the tax burden associated with higher government spending, which we assume is correctly foreseen by households. In particular, the impact on aggregate output is largest in the first period when a fiscal policy of a given magnitude $\Delta g$ for $T_p$ periods is initiated. In this case the present value of the tax burden is simply $\Delta g$ and the direct impact of this on consumption is $-(1 - (1 - \xi)\beta)\Delta g$, which is small compared to the increase in aggregate demand for output from government spending $\Delta g$. For larger $T_p$ the present value of the
tax burden is larger; consequently the reduction in consumption in the initial period is greater, leading to a smaller initial increase in aggregate output and inflation. Against this, of course, a larger $T_p$ means that the increase in demand continues for a longer period of time.

H Convergence probabilities for $y_0^c = 0.985$

Each table gives the probabilities for the specified value of $T_m$ and ranges of values for $\ddot{g}$ and $T_p$ based on 100 replications.

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$g'_0 = 0.985, \text{gain} = 0.01, T_m = 7$

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$g'_0 = 0.985, \text{gain} = 0.01, T_m = 7$

### I Convergence probabilities for $g'_0 = 0.98$

Each table gives the probabilities for the specified value of $T_m$ and ranges of values for $g'$ and $T_p$ with 100 replications.
\[ y_0^* = 0.98, \text{ gain}=0.01, T_m = 1 \]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
T_p \backslash \tilde{g}' & 0.3 & 0.35 & 0.4 & 0.45 & 0.5 & 0.55 & 0.6 & 0.65 & 0.7 & 0.75 & 0.8 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 29 & 19 & 2 \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 31 & 4 & 0 & 0 \\
\hline
\end{array}
\]

\[ y_0^* = 0.98, \text{ gain}=0.01, T_m = 2 \]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
T_p \backslash \tilde{g}' & 0.3 & 0.35 & 0.4 & 0.45 & 0.5 & 0.55 & 0.6 & 0.65 & 0.7 & 0.75 & 0.8 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{array}
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\[ y_0^* = 0.98, \text{ gain}=0.01, T_m = 3 \]
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$y_0^c = 0.98$, gain = 0.01, $T_m = 6$

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$y_0^c = 0.98$, gain $= 0.01$, $T_m = 7$

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$y_0^c = 0.98$, gain $= 0.01$, $T_m = 8$

### J Additional results with higher values for $\bar{g}'$

The next three tables show the additional rows and columns (with at least one non-zero value) to the corresponding table above with results continued to be based on 100 replications. Note: column for $\bar{g}' = 0.8$ is included to see continuity to the earlier table.

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<th>$T_p \backslash \bar{g}'$</th>
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<th>0.9</th>
<th>0.95</th>
<th>1.0</th>
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<th>1.1</th>
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$y_0^c = 0.98$, gain $= 0.01$, $T_m = 5$
In this model it is possible to compute an approximation of the area of the domain of attraction (DOA). The figure below which is like Figure 3 shows the DOA in the basic NK model as the area which is inside the global stable manifold (blue curve). The global stable manifold (GSM) can be numerically computed by solving two boundary value problems for the E-stability differential equation as the unstable steady state is a saddle point. The unstable steady state is the end point for the two curves that form the GSM. The numerical solutions of these curves can be obtained by solving the end-point problems where the trajectory approaches the unstable steady state from South-East or North-West direction as $\tau \to \infty$.

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$T_p \backslash g'$ & 0.8 & 0.85 & 0.9 & 0.95 & 1.0 & 1.05 & 1.1 & 1.15 & 1.2 \\
\hline
3 & 0 & 1 & 24 & 39 & 44 & 47 & 25 & 17 & 10 \\
4 & 45 & 28 & 4 & 3 & 0 & 0 & 0 & 0 & 0 \\
5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}

$y_0^c = 0.98$, gain = 0.01, $T_m = 6$

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$T_p \backslash g'$ & 0.8 & 0.85 & 0.9 & 0.95 & 1.0 & 1.05 & 1.1 & 1.15 & 1.2 \\
\hline
3 & 17 & 35 & 42 & 36 & 20 & 8 & 2 & 0 & 1 \\
4 & 14 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}

$y_0^c = 0.98$, gain = 0.01, $T_m = 7$

\section{K Numerical Computation of Size of the Domain of Attraction}

In this model it is possible to compute an approximation of the area of the domain of attraction (DOA). The figure below which is like Figure 3 shows the DOA in the basic NK model as the area which is inside the global stable manifold (blue curve). The global stable manifold (GSM) can be numerically computed by solving two boundary value problems for the E-stability differential equation as the unstable steady state is a saddle point. The unstable steady state is the end point for the two curves that form the GSM. The numerical solutions of these curves can be obtained by solving the end-point problems where the trajectory approaches the unstable steady state from South-East or North-West direction as $\tau \to \infty$. 
Figure A.1: Numerical computation of the area of domain of attraction.

To compute the area of the DOA, the GSM is divided into three segments which are determined using the intersections of the vertical straight lines with GSM as follows. Segment I is the "top" curve between the intersections of GSM with AA and CC. Segment II is the "bottom" curve between the intersections of GSM with BB and CC. Segment III is the "bottom" curve between the intersections of GSM with AA and BB.

Using standard formula for determining the area below a curve in parametric form, one computes the line integral of each curve

$$\int_{\tau_1}^{\tau_2} y^\mu(\tau) F(\pi^\mu(\tau), y^\mu(\tau)) d\tau$$

when bounds for the independent variable $\tau$ are implicitly obtained from the relevant intersection points. Denote these integrals by $Num(i)$, where $i = I, II$ and $III$. It is seen from the Figure that an approximation for the area of the domain of attraction
is then given by
\[ \text{Num}(I) - \text{Num}(II) - \text{Num}(III). \]

When applying the formula, the numerical integration is made difficult by the fact that each curve is given implicitly by the solution to a differential equation and the solution is obtained from solving end-point problems for the E-stability differential equation. Some of the values for \( \tau_1 \) and \( \tau_2 \) in the integral must be obtained by trial and error method.

**References, Appendices.**


