ANALYTIC CONTINUATION, THE CHERN-GAUSS-BONNET THEOREM, AND THE EULER-LAGRANGE EQUATIONS IN LOVELOCK THEORY FOR INDEFINITE SIGNATURE METRICS

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Abstract. We use analytic continuation to derive the Euler-Lagrange equations associated to the Pfaffian in indefinite signature (p, q) directly from the corresponding result in the Riemannian setting. We also use analytic continuation to derive the Chern-Gauss-Bonnet theorem for pseudo-Riemannian manifolds with boundary directly from the corresponding result in the Riemannian setting. Complex metrics on the tangent bundle play a crucial role in our analysis and we obtain a version of the Chern-Gauss-Bonnet theorem in this setting for certain complex metrics.
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1. Introduction

The use of analytic continuation to pass between the spacelike and the timelike settings in pseudo-Riemannian geometry has proven to be a very fruitful technical tool. In the study of Osserman geometry, García-Río et al. [14] examined complex "tangent vectors" (i.e. elements of $TM \otimes_{\mathbb{R}} \mathbb{C}$) to show that spacelike Osserman and timelike Osserman were equivalent concepts; subsequently other authors used a similar technique to show that timelike Ivanov–Petrova, spacelike Ivanov–Petrova, and mixed Ivanov–Petrova were equivalent concepts as were spacelike Szabó and timelike Szabó (see the discussion in [16] for example).

In this paper, we will use analytic continuation to show that results holding in the Riemannian context can often be extended to the pseudo-Riemannian context with relatively little additional fuss. For example, if (M, g, f) is a gradient Ricci solution and if $\mathcal{H}_f$ is the Hessian, then

$$(\nabla_{\nabla f}\operatorname{Ric}) + \operatorname{Ric}\circ\mathcal{H}_f = R(\nabla f, \cdot)\nabla f + \tfrac{1}{2}\nabla\nabla\tau\,.$$

This result was first proved in the Riemannian setting [26] but extends to the pseudo-Riemannian setting via analytic continuation [7].

In all cases, one first extends the relevant notions to complex "metrics" (i.e. sections g to $S^2(T^*M)\otimes_{\mathbb{R}}\mathbb{C}$ satisfying $\det(g) \neq 0$) or complex "tangent vectors", applies analytic continuation, and thereby passes from one signature to another. This general meta principle is best illustrated by example and we shall take as our example the circle of ideas related to Chern-Gauss-Bonnet theorem. In this paper we shall derive the Chern-Gauss-Bonnet theorem for manifolds with non-degenerate boundary in the pseudo-Riemannian context from the corresponding result in the Riemannian setting; this gives what we believe is an elegant and conceptual proof that makes direct use of the results of [8]. We shall also extend the formulae for the Euler-Lagrange equations from the Riemannian to the pseudo-Riemannian setting.

The 4-dimensional Chern-Gauss-Bonnet integrand E_4 is given by the invariant $\frac{1}{32\pi^2}\{\tau^2 - 4|\rho|^2 + |R|^2\}$ where τ is the scalar curvature, $|\rho|^2$ is the norm of the Ricci

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tensor, $|R|^2$ is the norm of the total curvature tensor, and the signature is Riemannian. E_4 has played a useful role in many papers recently in mathematical physics - especially in the study of Einstein-Gauss-Bonnet Gravity (EGBG) using the associated Euler-Lagrange equations for E_4. There has been an enormous amount of work in this subject and we can only cite a few representative papers. Lovelock [24] introduced generalized theories of this type. Chervon et al. [10] use the EGBG equations to search for new models of the Emergent Universe (EmU) scenario and study an EmU supported by two chiral cosmological fields for a spatially flat universe, and with three chiral fields when investigating open and closed universes. M. Soltani and S. Sayyahi [28] investigate the Hawking-Unruh effect on the quantum entanglement of bosonic field in background of a spherically symmetric black hole of Gauss-Bonnet gravity beyond the single mode approximation. K. Bamba, A. Makarenko, A. Myagky, and S. Odintsov [4] explore the bounce cosmology in $F(G)$ gravity with the Gauss-Bonnet invariant G and construct an $F(G)$ gravity theory realizing the bouncing behavior of the early universe.

Higher dimensional examples also are important. Although a-priori the Euler-Lagrange equations defined by E_m can involve the 4^{th} derivatives of the metric in dimensions $n > m$, Berger [5] conjectured it only involved curvature; this was subsequently verified by Kuz'mina [19] and Labbi [20, 21, 22]. Following de Lima and de Santos [23], one says that a compact Riemannian n-manifold is $2k$-Einstein for $2 \leq 2k < n$ if it is a critical metric for the Einstein-Hilbert-Lovelock functional $L_{2k}(g) = \int_M E_{2k}\,\text{dvol}$ when restricted to metrics on M with unit volume. This involves, of course, examining the associated Euler-Lagrange equations for this functional. The Euler-Lagrange equations were determined very explicitly in the Riemannian setting in [17] using invariant theory. In this paper we use analytic continuation to extend these results to the pseudo-Riemannian context without significant additional effort; the original treatment in [18] involved redoing the analysis in [17] and we think this an elegant independent derivation.

1.1. **Riemannian manifolds without boundary.**

The classical 2-dimensional Gauss-Bonnet formula

$$\chi(M^2) = \frac{1}{4\pi}\int_{M^2} \tau\,\text{dvol}_g \tag{1.a}$$

has been generalized to the higher dimensional setting by Chern [8] (see related work by Allendoerfer and Weil [1]). For example, in the crucial 4-dimensional setting, one has as noted above:

$$\chi(M^4) = \frac{1}{32\pi^2}\int_{M^4}\{\tau^2 - 4|\rho|^2 + |R|^2\}\,\text{dvol}_g\ .$$

Let (M, g) be a smooth compact Riemannian manifold of dimension m without boundary; we shall consider the case when M has boundary presently in Section 1.3. If $\vec{x} = (x^1, \dots, x^m)$ is a system of local coordinates, let

$$\text{dvol}_g = \det(g_{ij})^{1/2} dx^1 \dots dx^m$$

be the Riemannian measure. Let ∇ be the Levi-Civita connection of M and let

$$R_{ijkl} := g((\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})e_k, e_l)$$

be the components of the Riemann curvature tensor relative to an arbitrary local frame field $\{e_i\}$ for the tangent bundle TM. We adopt the *Einstein* convention and sum over repeated indices. Let $m = 2\ell$ be even. Define the *Pfaffian* by setting:

$$E_m(g) := \frac{1}{(8\pi)^\ell \ell!} R_{i_1 i_2 j_2 j_1} \dots R_{i_{2\ell-1} i_{2\ell} j_{2\ell} j_{2\ell-1}}\, g(e^{i_1} \wedge \dots \wedge e^{i_{2\ell}}, e^{j_1} \wedge \dots \wedge e^{j_{2\ell}})\ . \tag{1.b}$$

We refer to Chern [8] for the proof of the following result; there is also a heat equation proof due to Patodi [25]. Note that the Euler characteristic $\chi(M)$ of any compact manifold without boundary of odd dimension vanishes so only the even dimensional case is of interest.

Theorem 1.1. *Let (M,g) be a compact Riemannian manifold without boundary of even dimension m. Then*

$$\chi(M)=\int_M E_m(g)\,\mathrm{dvol}_g\ .$$

1.2. **Pseudo-Riemannian manifolds without boundary.** Avez [3] and Chern [9] independently extended Theorem 1.1 to the indefinite setting (there is a slight mistake in Chern's paper as the the sign change $(-1)^{p/2}$ is not present in [9]). Let (M,g) be a compact pseudo-Riemannian manifold without boundary of signature (p,q). The volume element is then given by

$$|\,\mathrm{dvol}_g\,|=|\det(g_{ij})|^{1/2}dx^1\dots dx^m\ .$$

We use Equation (1.b) to define $E_m(g)$ without change. One then has:

Theorem 1.2. *Let (M,g) be a compact pseudo-Riemannian manifold of signature (p,q) without boundary of even dimension m. If p is odd, then $\chi(M)$ vanishes. If p is even, then*

$$\chi(M)=(-1)^{p/2}\int_M E_m|\,\mathrm{dvol}_g\,|.$$

We refer to work by Bonome et al. [6] and by Derdzinski and Roter [11] for a discussion of the role that Theorem 1.2 plays in pseudo-Riemannian geometry.

Rather than invoking the Chern-Weil homomorphism as was done by Chern [9] or by going back to the original proof of Chern as was done by Avez [3], we shall derive Theorem 1.2 directly from Theorem 1.1 using analytic continuation. In Section 3, we shall consider complex "metrics" on the tangent bundle and show the associated Euler-Lagrange equations for the Pfaffian E_m vanish. This will enable us to extend Theorem 1.1 to a restricted class of complex metrics – see Theorem 3.2. We will then derive Theorem 1.2 from Theorem 3.2. We shall not deal with arbitrary complex metrics since it may not be possible to extract an appropriate square root but shall content ourselves with considering complex metrics which can be connected by a smooth family to Riemannian metrics in order to avoid problems with holonomy.

1.3. **Riemannian manifolds with boundary.** If M is a 2-dimensional manifold with smooth boundary, then Equation (1.a) must be adjusted to include a boundary contribution. Let κ_g be the geodesic curvature. We then have

$$\chi(M^2)=\frac{1}{4\pi}\int_{M^2}\tau dA+\frac{1}{2\pi}\int_{\partial M^2}\kappa_g ds\,.$$

Chern's original paper [8] also gives a formula for the Euler characteristic in the context of Riemannian manifolds of dimension m with smooth boundary. Near the boundary ∂M, let X be an inward pointing normal vector field and let $\{e_2,\dots,e_m\}$ be an arbitrary local frame field for the tangent bundle of the boundary. This gives a local frame field $\{X,e_2,\dots,e_m\}$ for TM. The components of the second fundamental form are then given by

$$L_{ab}:=g(X,X)^{-1/2}g(\nabla_{e_a}e_b,X)\,. \tag{1.c}$$

The *transgression* of the Pfaffian is defined by summing over indices which range from 2 to m and by summing over relevant expressions in the second fundamental form:

$$TE_m(g) := \sum_\mu \left\{ \frac{R_{a_1 a_2 b_2 b_1} \dots R_{a_{2\mu-1} a_{2\mu} b_{2\mu} b_{2\mu-1}} L_{a_{2\mu+1} b_{2\mu+1}} \dots L_{a_{m-1} b_{m-1}}}{(8\pi)^\mu \mu! \operatorname{Vol}(S^{m-1-2\mu})(m-1-2\mu)!} \right. \\ \left. \times g(e^{a_1} \wedge \dots \wedge e^{a_{m-1}}, e^{b_1} \wedge \dots \wedge e^{b_{m-1}}) \right\}. \tag{1.d}$$

Note that if m is odd, then $\chi(M) = \frac{1}{2}\chi(\partial M)$ so we may apply Theorem 1.1 to compute $\chi(\partial M)$ and thereby express $\chi(M)$ in terms of curvature. We therefore assume m is even.

Theorem 1.3. *Let (M, g) be a compact smooth manifold Riemannian manifold of even dimension m with smooth boundary ∂M. Then:*

$$\chi(M) = \int_M E_m(g) \operatorname{dvol}_g + \int_{\partial M} TE_m(g) \operatorname{dvol}_{g|_{\partial M}}.$$

Alty [2] generalized this result to the case of pseudo-Riemannian manifolds with boundary under the assumption that the normal vector was either spacelike, timelike, or null on each boundary component by combining the analysis of Avez [3] with the original discussion of Chern [8]. We shall not deal with the null case and in the interests of simplicity shall simply assume the normal vector to be either timelike or spacelike or, equivalently, that the restriction of the metric to the boundary is non-degenerate. We use Equation (1.c) and Equation (1.d) to define TE_m where we replace $g(X, X)^{1/2}$ by $|g(X, X)|^{1/2}$ just like we replaced $\det(g)^{1/2}$ by $|\det(g)|^{1/2}$ previously when defining the Pfaffian. We then have:

Theorem 1.4. *Let (M, g) be a compact smooth pseudo-Riemannian manifold of even dimension m and signature (p, q) which has smooth boundary ∂M. Assume $g|_{\partial M}$ is non-degenerate. If p is odd, then $\chi(M) = 0$. Otherwise*

$$\chi(M) = (-1)^{p/2} \left\{ \int_M E_m(g) |\operatorname{dvol}_g| + \int_{\partial M} TE_m(g) |\operatorname{dvol}_{g|_{\partial M}}| \right\}.$$

We refer to the articles by Dappiaggi, Hack, and Pinamonti [12]; Saa [27]; and Dunn, Harriott, and Williams [13] for a discussion of the role Theorem 1.4 plays in various applications. Section 4 will be devoted to a derivation of this result from Theorem 1.3 using analytic continuation.

1.4. **Euler-Lagrange Equations.** We use the formula of Theorem 1.1 to define the Pfaffian and to consider $\int_M E_{2\ell}(g) \operatorname{dvol}_g$. Let g_ϵ be a smooth 1-parameter family of Riemannian metrics. Let $h := \partial_\epsilon g_\epsilon|_{\epsilon=\epsilon_0}$. We differentiate with respect to the parameter ϵ and integrate by parts to define the Euler-Lagrange equations:

$$\partial_\epsilon \left\{ \int_M E_{2\ell}(g_\epsilon) \operatorname{dvol}_{g_\epsilon} \right\} \Big|_{\epsilon=\epsilon_0} = \int_M \langle \mathcal{E}_{2\ell}(g_{\epsilon_0}), h \rangle \operatorname{dvol}_{g_{\epsilon_0}} \tag{1.e}$$

for $\mathcal{E}_{2\ell}(\cdot) \in C^\infty(S^2(TM))$ where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $S^2(TM)$ and $S^2(T^*M)$. Although a-priori $\mathcal{E}_{2\ell}$ can involve the 4^{th} derivatives of the metric in dimensions $m > 2\ell$, Berger [5] conjectured it only involved curvature; this was subsequently verified by Kuz'mina [19] and Labbi [20, 21, 22]. An explicit formula was derived for this invariant in [17]:

Theorem 1.5. *Let (M, g) be a Riemannian manifold of dimension $m > 2\ell$. Then:*

$$\mathcal{E}_{2\ell} := \frac{1}{(8\pi)^\ell \ell!} R_{i_1 i_2 j_2 j_1} \dots R_{i_{2\ell-1} i_{2\ell} j_{2\ell} j_{2\ell-1}} e_{i_{2\ell+1}} \circ e_{j_{2\ell+1}} g(e^{i_1} \wedge \dots \wedge e^{i_{2\ell+1}}, e^{j_1} \wedge \dots \wedge e^{j_{2\ell+1}}).$$

In Section 2 we will use analytic continuation to extend this to the pseudo-Riemannian setting and establish a result originally established in [18] using a direct computation:

Theorem 1.6. *Let (M,g) be a pseudo-Riemannian manifold of dimension $m>2\ell$. Then:*

$$\mathcal{E}_{2\ell} := \frac{1}{(8\pi)^\ell \ell!} R_{i_1 i_2 j_2 j_1} \dots R_{i_{2\ell-1} i_{2\ell} j_{2\ell} j_{2\ell-1}} e_{i_{2\ell+1}} \circ e_{j_{2\ell+1}} g(e^{i_1}\wedge \dots \wedge e^{i_{2\ell+1}}, e^{j_1}\wedge\dots\wedge e^{j_{2\ell+1}}) .$$

2. The proof of Theorem 1.6

Let $g\in C^\infty(S^2(T^*M)\otimes\mathbb{C})$ be a complex "metric". We assume $\det(g)\neq 0$ as a non-degeneracy condition. The Levi-Civita connection and curvature tensor may then be defined. To maintain analyticity, we set $\operatorname{dvol} := \det(g_{ij})^{1/2} dx^1 \dots dx^m$; we do not take the absolute value. There is a subtlety here since, of course, there are 2 branches of the square root function. We shall ignore this for the moment in the interests of simplifying the argument and return to this point in a moment. The invariant $\mathcal{E}_{2\ell}(g)$ is then defined by Equation (1.e). Here $\mathcal{E}_{2\ell}(\cdot)\in C^\infty(S^2(TM)\otimes\mathbb{C})$ defines the Euler-Lagrange equations for the Lovelock functional of the Pfaffian $E_{2\ell}$.

We regard $\mathcal{E}_{2\ell}$, the curvature tensor R, the covariant derivative of the curvature ∇R, and so forth as polynomials in the derivatives of the metric tensor with coefficients which are analytic in the g_{ij} variables. The identity of Theorem 1.5 is then an identity between two analytic expressions in the variables $\{g_{ij}, g_{ij/k}, g_{ij/kl}, \dots\}$ where $\det(g_{ij})\neq 0$. Since the zeros of $\det(g_{ij})$ have codimension 2 in the linear space of symmetric 2-tensors, the condition $\det(g_{ij})\neq 0$ does not disconnect the parameter space. Consequently since the identity holds where $\det(g_{ij})\neq 0$, g_{ij} is real, and the signature is positive definite, the identity holds in general so Theorem 1.6 follows immediately from Theorem 1.5. □

3. The proof of Theorem 1.2

The following is a useful technical observation which we shall need subsequently. Although it is well known, we present the proof to keep our treatment self-contained.

Lemma 3.1. *Let (M,g_1) be a pseudo-Riemannian manifold of signature (p,q). There exist smooth complementary subbundles $V_\pm$ of TM so that $TM = V_-\oplus V_+$, so that V_+ is perpendicular to V_-, so that restriction of g_1 to V_+ is positive definite, and so that the restriction of g_1 to V_- is negative definite.*

Proof. Let g_r be an auxiliary Riemannian metric. Express $g_1(\xi_1,\xi_2)=g_r(T\xi_1,\xi_2)$ where T is an invertible linear map of the tangent bundle which is self-adjoint with respect to g_r. The bundle V_+ (resp. V_-) can then be taken to be the span of the eigenvectors of T corresponding to positive (resp. negative) eigenvalues of T. □

Let $m=2\ell$. The Chern-Gauss-Bonnet theorem shows that $\mathcal{E}_m$ vanishes on real positive definite metrics and hence using the argument of Section 2, it vanishes on complex metrics as well. The following result is now immediate from Theorem 1.1:

Theorem 3.2. *Let M be a compact manifold of dimension $m=2\ell$ without boundary. Let g_ϵ be a smooth 1-parameter family of complex metrics which contains a Riemannian metric for some ϵ. Then we can define a branch of $\det(g_\epsilon)^{1/2}$ along this family so that $\int_M E_m(g_\epsilon)\operatorname{dvol}_{g_\epsilon}$ is independent of ϵ and consequently*

$$\chi(M) = \int_M E_m(g_\epsilon)\operatorname{dvol}_{g_\epsilon} \text{ for any } \epsilon .$$

If M is not simply connected, there may be difficulties in defining $\det(g)^{1/2}$ consistently over the manifold. And furthermore, if g is not in the same homotopy class as a Riemannian metric, there may be difficulties evaluating the integral. However Theorem 3.2 as stated is sufficient to establish Theorem 1.2. We argue as follows. Let (M,g_1) be a compact pseudo-Riemannian manifold without boundary of even dimension m and signature (p,q). Apply Lemma 3.1 to decompose the tangent bundle $TM = V_-\oplus V_+$. Let $g_\pm := \pm g_1|_{V_\pm}$ so $g_1 = -g_-\oplus g_+$. Let $g_0 := g_-\oplus g_+$ be a Riemannian metric on M. We follow a circular arc from 1 to -1 in the complex plane given by $e^{\epsilon\pi\sqrt{-1}}$ to define the following variation connecting g_0 to g_1:

$$g_\epsilon := e^{\epsilon\pi\sqrt{-1}}g_-\oplus g_+\,. \tag{3.a}$$

We note $\det(g_\epsilon)=\det(g_0)e^{p\epsilon\pi\sqrt{-1}}$ so the family is admissible and we have:

$$\det(g_\epsilon)^{1/2}=e^{p\epsilon\pi\sqrt{-1}/2}\det(g_0)^{1/2}\,.$$

If p is odd, then $\det(g_1)^{1/2}$ will be purely imaginary and thus $\chi(M)$ will be purely imaginary. Since $\chi(M)$ is real, we conclude $\chi(M)=0$ in this case. On the other hand, if p is even, then $\det(g_1)^{1/2}=(-1)^{p/2}\det(g_0)$ and thus Theorem 1.2 follows from Theorem 3.2. □

It is worth considering a few examples just to check the sign. Suppose (M,g_0) is a Riemann surface. Let $g_1=-g_0$ have signature $(2,0)$. Then the Levi-Civita connection of g_1 and the Levi-Civita connection of g_0 agree so

$$R_{ijk}{}^\ell(g_1)=R_{ijk}{}^\ell(g_0)\text{ and }\tau(g_1)=g_1^{jk}R_{ijk}{}^i(g_1)=-g_0^{jk}R_{ijk}{}^i(g_0)=-\tau(g_0)\,.$$

As $|\operatorname{dvol}|(g_0)=|\operatorname{dvol}|(g_1)$, one must change the sign in the Gauss-Bonnet theorem:

$$\chi(M)=-\frac{1}{4\pi}\int_M\tau(g_1)|\operatorname{dvol}|(g_1)\,.$$

In dimension 4, if $(M,g)=(M_1,h_1)\times(M_2,h_2)$ is the product of two Riemann surfaces, then the Gauss-Bonnet theorem decouples and we have $\chi(M)=\chi(M_1)\chi(M_2)$ and $E_4(g)=E_2(h_1)E_2(h_2)$. Thus we will not need to change the sign in signature $(4,0)$ or $(0,4)$ but we will need to change the sign in signature $(2,2)$. The fact that the Euler characteristic vanishes if p and q are both odd is not, of course, new but follows from standard characteristic class theory.

4. The proof of Theorem 1.4

If (M,g) is Riemannian, Theorem 1.3 follows from Chern [9]; a heat equation proof appears in [15]. So the trick is to extend Theorem 1.3 to the pseudo-Riemannian setting directly rather than, as was done by Alty [2], redo the analysis of Chern in the pseudo-Riemannian context by examining the index of vector fields with isolated singularities. Again, we will use analytic continuation. But there is an important difference. Let (M,g_1) be a pseudo-Riemannian manifold. We suppose $g_1|_{\partial M}$ is non-degenerate. Choose a non-zero vector field X which is normal to the boundary and inward pointing. We can identify a neighborhood of the boundary ∂M in M with $[0,\epsilon)\times\partial M$ and choose local coordinates $(x^1,\dots,x^m)$ so that $X=\partial_{x_1}$ and so that $\partial M=\{x:x^1=0\}$. We then have

$$\det(g_1|_{\partial M})g_1(X,X)|_{\partial M}=\det(g_1)|_{\partial M}\,. \tag{4.a}$$

We use Lemma 3.1 to choose smooth complementary subbundles $V_\pm$ of TM so that $TM=V_-\oplus V_+$, so that V_+ is perpendicular to V_-, so that restriction of g_1 to V_+ is positive definite, and so that the restriction of g_1 to V_- is negative definite. We may further normalize the splitting to assume that if the normal vector X is spacelike, then $X\in C^\infty(V^+|_{\partial M})$ while if the normal vector X is timelike, then $X\in C^\infty(V_-|_{\partial M})$. Thus the splitting $TM=V_-\oplus V_+$ induces a corresponding splitting $T(\partial M)=W_-\oplus W_+$ where $W_\pm=T(\partial M)\cap V_\pm$.

We now consider the smooth 1-parameter of complex variations g_ϵ given above in Equation (3.a). The unit normal is then given by $X\cdot g_\epsilon(X,X)^{-1/2}$. In the expressions for TE_m, there are an odd number of terms which contain the second fundamental form L and hence $g_\epsilon(X,X)^{-1/2}$ appears. By Equation (4.a):

$$\left\{g_\epsilon(X,X)\det(g_\epsilon|_{\partial M})\right\}^{1/2}=\left\{\det(g_\epsilon)^{1/2}\right\}\Big|_{\partial M}.$$

Thus once again, we must take the square root of $(-1)^p$ in the analytic continuation. Apart from this, the remainder of the argument is the same as that used to prove Theorem 1.2. We note that an appropriate analogue of Theorem 3.2 holds. □

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