Demeter Krupka and David Saunders

The spectral geometry of operators of Dirac and Laplace type

P. Gilkey

Contents

1 Introduction

The field of spectral geometry is a vibrant and active one. In these brief notes, we will sketch some of the recent developments in this area. Our choice is somewhat idiosyncratic and owing to constraints of space necessarily incomplete. It is impossible to give a complete bibliography for such a survey. We refer Carslaw and Jaeger [41] for a comprehensive discussion of problems associated with heat flow, to Gilkey [54] and to Melrose [91] for a discussion of heat equation methods related to the index theorem, to Gilkey [56] and to Kirsten [84] for a calculation of various heat trace and heat content asymptotic formulas, to Gordon [66] for a survey of isospectral manifolds, to Grubb [73] for a discussion of the pseudo-differential calculus relating to boundary problems, and to Seeley [116] for an introduction to the pseudo-differential calculus. Throughout we shall work with smooth manifolds and, if present, smooth boundaries. We have also given in each section a few additional references to relevant works. The constraints of space have of necessity forced us to omit many more important references than it was possible to include and we apologize in advance for that.

We adopt the following notational conventions. Let (M,g) be a compact Riemannian manifold of dimension m with smooth boundary ∂M. Let Greek indices μ,ν range from 1 to m and index a local system of coordinates $x=(x^1,...,x^m)$ on the interior of M. Expand the metric in the form $ds^2=g_{\mu\nu}dx^\mu\circ dx^\nu$ were $g_{\mu\nu}:=g(\partial_{x_\mu},\partial_{x_\nu})$ and where we adopt the *Einstein convention* of summing over repeated indices. We let $g^{\mu\nu}$ be the inverse matrix. The Riemannian measure is given by $dx:=g\ dx^1...dx^m$ for $g:=\sqrt{\det(g_{\mu\nu})}$.

Let ∇ be the Levi-Civita connection. We expand $\nabla_{\partial_{x_\nu}}\partial_{x_\mu}=\Gamma_{\nu\mu}{}^\sigma\partial_{x_\sigma}$ where $\Gamma_{\nu\mu}{}^\sigma$ are the *Christoffel symbols*. The *curvature operator* $\mathcal{R}$ and corresponding *curvature tensor* R are may then be given by $\mathcal{R}(X,Y):=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$ and given by $R(X,Y,Z,W):=g(\mathcal{R}(X,Y)Z,W)$.

We shall let Latin indices i,j range from 1 to m and index a local orthonormal frame $\{e_1,...,e_m\}$ for the tangent bundle of M. Let R_{ijkl} be the components of the curvature tensor relative to this base; the *Ricci tensor* ρ and the *scalar curvature* τ are then given by setting $\rho_{ij}:=R_{ikkj}$ and $\tau:=\rho_{ii}=R_{ikki}$. We shall often have an auxiliary vector bundle V and an auxiliary connection given on V. We use this connection and the Levi-Civita connection to covariantly differentiate tensors of all types and we shall let ';' denote the components of multiple covariant differentiation.

Let dy be the measure of the induced metric on the boundary ∂M. We choose a local orthonormal frame near the boundary of M so that e_m is the inward unit normal. We let indices a,b range from 1 to $m-1$ and index the induced local frame $\{e_1,...,e_{m-1}\}$ for the tangent bundle of the boundary. Let $L_{ab}:=g(\nabla_{e_a}e_b,e_m)$ denote the *second fundamental form*. We sum over indices with the implicit range indicated. Thus the geodesic curvature κ_g is given by $\kappa_g:=L_{aa}$. We shall let ':' denote multiple tangential covariant differentiation with respect to the Levi-Civita connection of the boundary; the difference between ';' and ':' being, of course, measured by the second fundamental form.

2 The geometry of operators of Laplace and Dirac type

In this section, we shall establish basic definitions, discuss operators of Laplace and of Dirac type, introduce the DeRham complex, and discuss the Bochner Laplacian and the Weitzenböch formula; [55] provides a good reference for the material of this section.

Let D be a second order partial differential operator on the space of smooth sections $C^\infty(V)$ of a vector bundle V over M. Expand $D=-\{a^{\mu\nu}\partial_{x_\mu}\partial_{x_\nu}+a^\sigma\partial_{x_\sigma}+b\}$ where the coefficients $\{a^{\mu\nu},a^\mu,b\}$ are smooth endomorphisms of V; we suppress the fiber indices. We say that D is an *operator of Laplace type* if $a^{\mu\nu}=g^{\mu\nu}\,\mathrm{id}$. A first order operator A on $C^\infty(V)$ is said to be an operator of *Dirac type* if A^2 is an operator of Laplace type. If we expand $A=\gamma^\nu\partial_{x_\nu}+\gamma_0$, then A is an operator of Dirac type if and only if the endomorphisms γ^ν satisfy the *Clifford commutation relations* $\gamma^\nu\gamma^\mu+\gamma^\mu\gamma^\nu=-2g^{\mu\nu}\,\mathrm{id}$.

Let A be an operator of Dirac type and let $\xi=\xi_\nu dx^\nu$ be a smooth 1-form on M. We let $\gamma(\xi)=\xi_\nu\gamma^\nu$ define a *Clifford module structure* on V; this is independent of the particular coordinate system chosen. We can always choose a fiber metric on V so that γ is skew-adjoint. We can then construct a unitary connection ∇ on V so that $\nabla\gamma=0$. Such a connection is called *compatible*. If ∇ is compatible, we expand $A=\gamma^\nu\nabla_{\partial_{x_\nu}}+\psi_A$; the endomorphism ψ_A is tensorial and does not depend on the particular coordinate system chosen; it does, of course, depend on the particular compatible connection chosen.

2.1 The DeRham complex

The prototypical example is given by the exterior algebra. Let $C^\infty(\Lambda^p M)$ be the space of smooth p forms. Let $d : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M)$ be *exterior differentiation* and let $\delta = d^*$ be the adjoint operator, *interior differentiation*. If ξ is a cotangent vector, let $\mathrm{ext}(\xi) : \omega \to \xi \wedge \omega$ denote exterior multiplication, and let $\mathrm{int}(\xi)$ be the dual, interior multiplication. Let $\gamma(\xi) := \mathrm{ext}(\xi) - \mathrm{int}(\xi)$ define a Clifford module on the exterior algebra $\Lambda(M)$. Since $d + \delta = \gamma(dx^\nu)\nabla_{\partial_{x_\nu}}$, $d + \delta$ is an operator of Diract type. The associated Laplacian $\Delta_M := (d + \delta)^2 = \Delta_M^0 \oplus ... \oplus \Delta_M^p \oplus ... \oplus \Delta_M^m$ decomposes as the direct sum of operators of Laplace type Δ_M^p on the space of smooth p forms $C^\infty(\Lambda^p M)$. One has $\Delta_M^0 = -g^{-1}\partial_{x_\mu} g g^{\mu\nu}\partial_{x_\nu}$.

It is possible to write the p-form valued Laplacian in an invariant form. Extend the Levi-Civita connection to act on tensors of all types. Let $\tilde\Delta_M \omega := -g^{\mu\nu}\omega_{;\mu\nu}$ define the *Bochner* or *reduced Laplacian*. Let $\mathcal{R}$ give the associated action of the curvature tensor. The *Weitzenböck* formula then permits us to express the ordinary Laplacian in terms of the Bochner Laplacian in the form $\Delta_M = \tilde\Delta_M + \frac{1}{2}\gamma(dx^\mu)\gamma(dx^\nu)\mathcal{R}_{\mu\nu}$. This formalism can be applied more generally:

Lemma 2.1 *Let D be an operator of Laplace type on a Riemannian manifold. There exists a unique connection ∇ on V and there exists a unique endomorphism E of V so that $D\phi = -\phi_{;ii} - E\phi$. If we express D locally in the form $D = -\{g^{\mu\nu}\partial_{x_\nu}\partial_{x_\mu} + a^\mu\partial_{x_\mu} + b\}$ then the connection 1-form ω of ∇ and the endomorphism E are given by*

$$\omega_\nu = \tfrac{1}{2}\left(g_{\nu\mu}a^\mu + g^{\sigma\varepsilon}\Gamma_{\sigma\varepsilon\nu}\,\mathrm{id}\right) \quad \textit{and} \quad E = b - g^{\nu\mu}\left(\partial_{x_\nu}\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma\right).$$

Let V be equipped with an auxiliary fiber metric. Then D is self-adjoint if and only if ∇ is unitary and E is self-adjoint. We note that if D is the *Spin Laplacian*, then ∇ is the spin connection on the spinor bundle and the Lichnerowicz formula [86] yields, with our sign convention, that $E = -\frac{1}{4}\tau\,\mathrm{id}$ where τ is the scalar curvature.

3 Heat trace asymptotics for closed manifolds

Throughout this section, we shall assume that D is an operator of Laplace type on a closed Riemannian manifold (M, g). We shall discuss the L^2 spectral resolution if D is self-adjoint, define the heat equation, introduce the heat trace and the heat trace asymptotics, present the leading terms in the heat trace asymptotics, and discuss the form valued Laplacian; [41, 54, 116] are good references for the material of this section and other references will be cited as needed.

We suppose that D is self-adjoint. There is then a *complete spectral resolution* of D on $L^2(V)$. This means that we can find a complete orthonormal basis $\{\phi_n\}$ for $L^2(V)$ where the ϕ_n are smooth sections to V which satisfy the equation $D\phi_n = \lambda_n\phi_n$. Let $||_k$ denote the C^k-norm.

Theorem 3.1 *Let $\phi \in L^2(V)$. Expand $\phi = \sum_{n=1}^\infty c_n\phi_n$ in the L^2 sense where one has $c_n := \int_M (\phi, \phi_n)$. If $\phi \in C^\infty(V)$, then this series converges in the C^k topology for any*

k; $\phi \in C^\infty(V)$ if and only if $\lim_{n\to\infty} n^k c_n < \infty$ for any k. The set of eigenvalues is discrete. Each eigenvalue appears with finite multiplicity and there are only a finite number of negative eigenvalues. If we enumerate the eigenvalues so that $\lambda_1 \le \lambda_2 \le \ldots$, then $\lambda_n \sim n^{2/m}$ as $n \to \infty$. There exist constants $\nu_k > 0$ and $C_k > 0$ so that one has norm estimates $||\phi_n||_k \le C_k n^{\nu_k}$ for all k, n.

This yields the familiar *Weyl asymptotic formula* [127] giving the eigenvalue growth. For example, if $D = -\partial_\theta^2$ on the circle, then the eigenvalues grow quadratically since the associated spectral resolution is given by $\{n^2, \frac{1}{\sqrt{2\pi}} e^{in\theta}\}_{n\in\mathbb{Z}}$. The L^2 expansion of Theorem 3.1 in this setting then becomes the usual Fourier series expansion for ϕ and one has the familiar result that a function on the circle is smooth if and only if its Fourier coefficients are rapidly decreasing.

Let the initial temperature distribution be given by $\phi \in L^2(V)$. Impose the classical time evolution for the subsequent temperature distribution without additional heat input:

$$(\partial_t + D)u = 0 \text{ for } t > 0 \quad \text{and} \quad \lim_{t\downarrow 0} u(t,\cdot) = \phi \text{ in } L^2 .$$

Then $u(t,\cdot) = e^{-tD}\phi$ where e^{-tD} is given by the functional calculus. This operator is infinitely smoothing; we have $u(t,x) = \int_M K(t,x,\tilde{x})\phi(\tilde{x})d\tilde{x}$ for a smooth kernel function K. If D is self-adjoint, let $\{\lambda_n, \phi_n\}$ be a spectral resolution of D. Then

$$K(t,x,\tilde{x}) := \sum_n e^{-t\lambda_n}\phi_n(x) \otimes \phi_n(\tilde{x}) : V_{\tilde{x}} \to V_x .$$

Theorem 3.1 implies this series converges uniformly in the C^k topology for $t \ge \varepsilon > 0$.

Let $F \in C^\infty(\mathrm{End}(V))$ be an auxiliary endomorphism used for localizing; F is often referred to as a *smearing endomorphism*. The localized heat trace $\mathrm{Tr}_{L^2}\left\{Fe^{-tD}\right\}$ is analytic for $t > 0$. As $t \downarrow 0$, there is a complete asymptotic expansion [117]

$$\mathrm{Tr}_{L^2}\left\{Fe^{-tD}\right\} \sim \sum_{n=0}^{\infty} a_n(F,D)t^{(n-m)/2} .$$

The coefficients $a_n(F,D)$ are the *heat trace asymptotics*; $a_n(F,D) = 0$ if n is odd. In Section 5 we will consider manifolds with boundary and the corresponding invariants are non-trivial for n both even and odd. There exist locally computable endomorphisms $e_n^M(D)(x)$ of V which are defined for all $x \in M$ so that

$$a_n(F,D) = \int_M \mathrm{Tr}\{Fe_n^M(D)\}(x)dx . \tag{3.a}$$

The invariants $e_n^M(D)$ are uniquely characterized by Equation (3.a).

We use Lemma 2.1 to express $D = D(g,\nabla,E)$ where ∇ is a uniquely defined connection on V and where E is a uniquely defined auxiliary endomorphism of V. Let Ω_{ij} be the endomorphism valued components of the curvature defined by the connection ∇.

Theorem 3.2 *Let $F \in C^\infty(\mathrm{End}(V))$ be a smearing endomorphism.*

(1) $a_0(F,D) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\{F\}dx$.

(2) $a_2(F,D) = (4\pi)^{-m/2}\frac{1}{6}\int_M \mathrm{Tr}\{F(6E + \tau\,\mathrm{id})\}dx$.

(3) $a_4(F,D)=(4\pi)^{-m/2}\frac{1}{360}\int_M \operatorname{Tr}\{F(60E_{;kk}+60\tau E+180E^2$
$+12\tau_{;kk}\operatorname{id}+5\tau^2\operatorname{id}-2|\rho|^2\operatorname{id}+2|R|^2\operatorname{id}+30\Omega_{ij}\Omega_{ij})\}dx.$

(4) $a_6(F,D)=\int_M \operatorname{Tr}\{F((\frac{18}{7!}\tau_{;iijj}+\frac{17}{7!}\tau_{;k}\tau_{;k}-\frac{2}{7!}\rho_{ij;k}\rho_{ij;k}-\frac{4}{7!}\rho_{jk;n}\rho_{jn;k}$
$+\frac{9}{7!}R_{ijkl;n}R_{ijkl;n}+\frac{28}{7!}\tau\tau_{;nn}-\frac{8}{7!}\rho_{jk}\rho_{jk;nn}+\frac{24}{7!}\rho_{jk}\rho_{jn;kn}+\frac{12}{7!}R_{ijkl}R_{ijkl;nn}$
$+\frac{35}{9\cdot 7!}\tau^3-\frac{14}{3\cdot 7!}\tau|\rho|^2+\frac{14}{3\cdot 7!}\tau|R|^2-\frac{208}{9\cdot 7!}\rho_{jk}\rho_{jn}\rho_{kn}-\frac{64}{3\cdot 7!}\rho_{ij}\rho_{kl}R_{ikjl}$
$-\frac{16}{3\cdot 7!}\rho_{jk}R_{jnli}R_{knli}-\frac{44}{9\cdot 7!}R_{ijkn}R_{ijlp}R_{knlp}-\frac{80}{9\cdot 7!}R_{ijkn}R_{ilkp}R_{jlnp})\operatorname{id}$
$+\frac{1}{45}\Omega_{ij;k}\Omega_{ij;k}+\frac{1}{180}\Omega_{ij;j}\Omega_{ik;k}+\frac{1}{60}\Omega_{ij;kk}\Omega_{ij}+\frac{1}{60}\Omega_{ij}\Omega_{ij;kk}-\frac{1}{30}\Omega_{ij}\Omega_{jk}\Omega_{ki}$
$-\frac{1}{60}R_{ijkn}\Omega_{ij}\Omega_{kn}-\frac{1}{90}\rho_{jk}\Omega_{jn}\Omega_{kn}+\frac{1}{72}\tau\Omega_{kn}\Omega_{kn}+\frac{1}{60}E_{;iijj}+\frac{1}{12}EE_{;ii}$
$+\frac{1}{12}E_{;ii}E+\frac{1}{12}E_{;i}E_{;i}+\frac{1}{6}E^3+\frac{1}{30}E\Omega_{ij}\Omega_{ij}+\frac{1}{60}\Omega_{ij}E\Omega_j+\frac{1}{30}\Omega_{ij}\Omega_{ij}E$
$+\frac{1}{36}\tau E_{;kk}+\frac{1}{90}\rho_{jk}E_{;jk}+\frac{1}{30}\tau_{;k}E_{;k}-\frac{1}{60}E_{;j}\Omega_{ij;i}+\frac{1}{60}\Omega_{ij;i}E_{;j}+\frac{1}{12}EE\tau$
$+\frac{1}{30}E\tau_{;kk}+\frac{1}{72}E\tau^2-\frac{1}{180}E|\rho|^2+\frac{1}{180}E|R|^2)\}dx.$

There are formulas available for a_8 and a_{10}; we refer to Amsterdamski, Berkin, and O'Connor[1], to Avramidi [9], and to van de Ven [124] for further details.

There is also information available about the general form of the heat trace asymptotics a_n for all values of n; we refer to Avramidi [10] and to Branson et al. [36] for further details. These formulas play an important role in the compactness results we shall discuss presently in Theorem 4.6. Let D be an operator of Laplace type on a closed Riemannian manifold M. Let $\Delta E=-E_{;kk}$. Set $\epsilon_n=(-1)^n/\{2^{n+1}\cdot 1\cdot 3\cdot\ldots\cdot(2n+1)\}$.

Theorem 3.3 *Let '+...' denote lower order terms.*

(1) *If* $n\geq 1$, *then* $a_{2n}(F,D)=\epsilon_n(4\pi)^{-m/2}\int_M \operatorname{Tr}\{F(-(8n+4)\Delta^{n-1}E$
$-2n\Delta^{n-1}\tau\operatorname{id}+\ldots\}dx.$

(2) *If* $n\geq 3$, *then* $a_{2n}(D)=\epsilon_n(4\pi)^{-m/2}\operatorname{Tr}\{(n^2-n-1)|\nabla^{n-2}\tau|^2\operatorname{id}$
$+2|\nabla^{n-2}\rho|^2\operatorname{id}+4(2n+1)(n-1)\nabla^{n-2}\tau\cdot\nabla^{n-2}E$
$+2(2n+1)\nabla^{n-2}\Omega\cdot\nabla^{n-2}\Omega+4(2n+1)(2n-1)\nabla^{n-2}E\cdot\nabla^{n-2}E+\ldots\}dx.$

We note that Polterovich [109, 110] has introduced a formalism for computing in closed form the heat trace asymptotics a_n for all n.

If one specializes these formulas for a_0, a_2, and a_4 to the case in which D is the form valued Laplacian, one has the following result of Patodi [106]. Introduce constants:

$$\begin{aligned}
&c(m,p)=\tfrac{m!}{p!(m-p)!},\\
&c_0(m,p)=c(m,p)-6c(m-2,p-1),\\
&c_1(m,p)=5c(m,p)-60c(m-2,p-1)+180c(m-4,p-2),\\
&c_2(m,p)=-2c(m,p)+180c(m-2,p-1)-720c(m-4,p-2),\\
&c_3(m,p)=2c(m,p)-30c(m-2,p-1)+180c(m-4,p-2)\,.
\end{aligned}$$

Theorem 3.4 (1) $a_0(\Delta_M^p)=(4\pi)^{-m/2}c(m,p)\operatorname{Vol}(M)$.

(2) $a_2(\Delta_M^p)=(4\pi)^{-m/2}\frac{1}{6}c_0(m,p)\int_M\tau dx$.

(3) $a_4(\Delta_M^p)=(4\pi)^{-m/2}\frac{1}{360}\int_M\{c_1(m,p)\tau^2+c_2(m,p)\rho^2+c_3(m,p)R^2\}dx$.

Such formulas play an important role in the study of spectral geometry. There is a long history involved in computing these invariants. Weyl [127] discovered the leading term in the asymptotic expansion, a_0. Minakshisundaram and Pleijel [93, 94] examined the asymptotic expansion for the scalar Laplacian in some detail. The a_2 and a_4 terms in the asymptotic expansion were investigated by McKean and Singer [90] in the scalar case and by Patodi [105] for the form valued Laplacian. The a_6 term for the scalar Laplacian was determined by Sakai [111] and the general expression for a_2, a_4 and a_6 for arbitrary operators of Laplace type was worked out in [53]. As noted above, there are formulas for a_8 and a_{10}. The literature is a vast one and we refer to [54, 56] more details and additional references.

We now discuss the relationship between the heat trace asyptotics and the eta and zeta functions in a quite general context. Let P be a positive, self-adjoint elliptic partial differential operator on a closed Riemannian manifold M. Then e^{-tP} is an infinitely smoothing operator which is given by a smooth kernel function. Let Q be an auxiliary partial differential operator. Then $\operatorname{Tr}_{L^2}\{Qe^{-tP}\}$ is analytic for $t>0$ and as $t\downarrow 0$, there is a complete asymptotic expansion with locally computable coefficients:

$$\operatorname{Tr}_{L^2}\{Qe^{-tP}) \sim \sum_{n=0}^{\infty} a_n(P,Q)t^{(n-m-\operatorname{ord}(Q))/\operatorname{ord}(P)}$$

The generalized zeta function is given by:

$$\zeta(s,P,Q) := \operatorname{Tr}_{L^2}(QP^{-s}) \quad \text{for} \quad \Re(s) >> 0\,.$$

The Mellin transform may be used to relate the zeta function to the heat kernel. Let Γ be the classical Gamma function. We refer to Seeley [116, 117] for the proof of Assertions (1) and (2) and to [50] for the proof of Assertion (3) in the following result. Assertion (2) generalizes eigenvalue growth estimates of Weyl [127] given previously in Theorem 3.1.

Theorem 3.5 (1) *If* $\operatorname{Re}(s) >> 0$*, then* $\zeta(s,P,Q) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}\operatorname{Tr}_{L^2}(Qe^{-tP})dt$. $\Gamma(s)\zeta(s,P,Q)$ *has a meromorphic extension to the complex plane with isolated simple poles at* $s=(m+\operatorname{ord}(Q)-n)/\operatorname{ord}(P)$ *for* $n=0,1,\ldots$ *and*

$$\operatorname{Res}_{s=(m+\operatorname{ord}(Q)-n)/\operatorname{ord}(P)}\Gamma(s)\zeta(s,P,Q) = a_n(P,Q).$$

(2) *The leading heat trace coefficient* $a_0(P)$ *is non-zero. Let* $\lambda_1\le\ldots\le\lambda_n\le\ldots$ *be the eigenvalues of* P*. Then* $\lim_{n\to\infty} n\lambda_n^{-m/\operatorname{ord}(P)} = \Gamma(\frac{m}{\operatorname{ord}(P)})^{-1}a_0(P)$.

(3) *Let* $A(t)$ *and* $B(t)$ *be polynomials of degree* $a\ge 0$ *and* $b>0$ *where* B *is monic. There are constants so* $a_n(B(P),A(P)) = \Sigma_{k\le k(n)}c(k,n,m,A,B)a_k(P)$.

4 Hearing the shape of a drum

Let $\operatorname{Spec}(D)=\{\lambda_1\le\lambda_2\le\ldots\}$ denote the set of eigenvalues of a self-adjoint operator of Laplace type, repeated according to multiplicity. One is interested in what geometric and topological properties of M are reflected by the spectrum. Good references for this section are [26, 54, 66]; other references will be cited as appropriate.

One says that M and $\tilde M$ are *isospectral* if $\operatorname{Spec}(\Delta^0_M)=\operatorname{Spec}(\Delta^0_{\tilde M})$; p*-isospectral* refers to Δ^p. M. Kac [81] in his seminal article raised the question of determining the geometry, at least in part, of the underlying manifold from the spectrum of the scalar Laplace operator Δ_M^0. It is not possible in general to completely determine the geometry:

Theorem 4.1 (1) *Milnor [92]: There exist isospectral non isometric flat tori of dimension 16.*

(2) *Vigneras [125]: There exist isospectral non-isometric hyperbolic Riemann surfaces. Furthermore, if $m \geq 3$, there exist isospectral hyperbolic manifolds with different fundamental groups.*

(3) *Ikeda [79]: There exist isospectral non-isometric spherical space forms.*

(4) *Urakawa [123]: There exist regions Ω_i in flat space for $m \geq 4$ which are isospectral for the Laplacian with both Dirichlet and Neumann boundary conditions but which are not isometric.*

These examples listed above come in finite families. We say that a family of metrics g_t on M is a non-trivial family of isospectral manifolds if (M, g_t) and (M, g_s) are isospectral for every s, t, but (M, g_t) is not isometric to (M, g_s) for $s \neq t$.

Theorem 4.2 (1) *Gordon-Wilson [67]: There exists a non-trivial family of isospectral metrics on a smooth manifold M which are not conformally equivalent.*

(2) *Brooks-Gordon [37]: There exists a non-trivial family of isospectral metrics on a smooth manifold M which are conformally equivalent.*

There is a vast literature in the subject. In particular, Sunada [121] gave a general method for attacking the problem which has been exploited by many authors.

Despite this somewhat discouraging prospect, there are a number of positive results available. For example Berger [27] and Tanno [122] showed that a sphere or projective space is characterized by its spectral geometry, at least in low dimensions:

Theorem 4.3 *Let $\mathcal{M}_i$ and $\mathcal{M}_2$ be closed Riemannian manifolds of dimension $m \leq 6$ which are isospectral. If M_1 has constant sectional curvature c, so does M_2.*

Patodi [106] showed additional geometrical properties are determined by the form valued Laplacian. The following is an easy consequence of Theorem 3.4.

Theorem 4.4 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be closed Riemannian manifolds which are p-isospectral for $p = 0, 1, 2$. Then:*

(1) *If M_1 has constant scalar curvature $\tau = c$, then so does M_2.*

(2) *If M_1 is Einstein, so is M_2.*

(3) *If M_1 has constant sectional curvature c, then so does M_2.*

For manifolds with boundary, suitable boundary conditions must be imposed. Formulas that will be discussed presently in Section 5 have been used by Park [104] to show:

Theorem 4.5 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be compact Einstein Riemannian manifolds with smooth boundaries with the same constant scalar curvatures $\tau_{\mathcal{M}_1} = \tau_{\mathcal{M}_2}$. Also assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isospectral for both Neumann and Dirichlet boundary conditions. Then:*

(1) *If $\mathcal{M}_1$ has totally geodesic boundary, then so does $\mathcal{M}_2$.*

(2) *If $\mathcal{M}_1$ has minimal boundary, then so does $\mathcal{M}_2$.*

(3) *If $\mathcal{M}_1$ has totally umbillic boundary, then so does $\mathcal{M}_2$.*

(4) *If $\mathcal{M}_1$ has strongly totally umbillic boundary, then so does $\mathcal{M}_2$.*

There are also a number of compactness results. Theorem 3.3 plays a central role in the following results:

Theorem 4.6 (1) *Osgood, Phillips, and Sarnak [102]: Families of isospectral metrics on Riemann surfaces are compact modulo gauge equivalence.*

(2) *Brooks, Perry, and Yang [39] and Chang and Yang [42]: If $m = 3$, then families of isospectral metrics within a conformal class are compact modulo gauge equivalence.*

(3) *Brooks, Perry, and Petersen [38]: Isospectral negative curvature manifolds contain only a finite number of topological types.*

5 Heat trace asymptotics of manifolds with boundary

In previous sections, we have concentrated on closed Riemannian manifolds. Let D be an operator of Laplace type on a compact Riemannian manifold M with smooth boundary ∂M. Good basic references for the material of this section are [56, 73, 84]. Many authors have contributed to the material discussed here; we refer in particular to the work of [40, 76, 78, 82, 83, 90, 93, 94, 96, 120, 127].

We impose suitable boundary conditions $\mathcal{B}$ to have a well posed problem; $\mathcal{B}$ must satisfy a condition called the *strong Lopatenski-Shapiro condition*. We shall suppress technical details for the most part in the interests of simplicity. The boundary conditions we shall consider have physical underpinnings. Dirichlet boundary conditions correspond to immersing the boundary in ice water; Neumann boundary conditions correspond to an insulated boundary. Robin boundary conditions are a generalization of Neumann boundary conditions where the heat flow across the boundary is proportional to the temperature on the boundary. Transmission boundary conditions arise in the study of heat conduction problems between closely coupled membranes. Transfer boundary conditions arise in the study of branes. Both these conditions reflect the heat flow between two inhomogeneous mediums coupled along a common boundary or brane. Transmission boundary conditions correspond to having the two components pressed tightly together. By contrast, heat transfer boundary conditions correspond to a loose coupling between the two components. We refer to Carslaw and Jaeger [41] for further details.

Through out the remainder of this section, we let $F \in C^\infty(\text{End}(V))$ define a localizing or smearing endomorphism and let $\mathcal{B}$ denote a suitable boundary operator; in what follows, we shall give a number of examples. Let $D_\mathcal{B}$ be the realization of an operator D of Laplace type with respect $\mathcal{B}$; the domain of $D_\mathcal{B}$ is then the set of all functions ϕ in a suitable Schwarz space so that ϕ satisfies the appropriate boundary conditions, i.e. so that $\mathcal{B}\phi = 0$. Greiner [68, 69] and Seeley [118, 119] showed that there was a full asymptotic expansion as $t \downarrow 0$ of the form:

$$\text{Tr}_{L^2}\{Fe^{-tD_\mathcal{B}}\} \sim \sum_{n=0}^{\infty} a_n(F, D, \mathcal{B}) t^{(n-m)/2} .$$

There are locally computable endomorphisms $e_n(D)(x)$ defined on the interior and locally computable endomorphisms $e^{\partial M}_{n,k}(D,\mathcal{B})(y)$ defined on the boundary so that

$$\begin{aligned} a_n(F,D,\mathcal{B}) &= \int_M \mathrm{Tr}\{F e^M_n(D)\}(x)dx \\ &+ \sum_{k=0}^{n-1}\int_{\partial M}\mathrm{Tr}\{(\nabla^k_{e_m}F)e^{\partial M}_{n,k}(D,\mathcal{B})\}(y)dy\,. \end{aligned}$$

The invariants $e^M_n(D)$ and $e^{\partial M}_{n,k}(D,\mathcal{B})$ are uniquely characterized by this identity; the interior invariants $e^M_n(D)$ are not sensitive to the boundary condition and agree with those considered previously in Equation (3.a). The remainder of Section 5 is devoted to giving explicit combinatorial formulas for these invariants.

A function ϕ satisfies Dirichlet boundary conditions if ϕ vanishes on ∂M. Thus the Dirichlet boundary operator is defined by:

$$\mathcal{B}\phi := \phi|_{\partial M}\,. \tag{5.a}$$

Theorem 5.1 *[Dirichlet boundary conditions] Let* $F\in C^\infty(\mathrm{End}(V))$.

(1) $a_0(F,D,\mathcal{B}) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\{F\}dx$.

(2) $a_1(F,D,\mathcal{B}) = -(4\pi)^{-(m-1)/2}\frac14\int_{\partial M}\mathrm{Tr}\{F\}dy$.

(3) $a_2(F,D,\mathcal{B}) = (4\pi)^{-m/2}\frac16\int_M \mathrm{Tr}\{F(6E+\tau)\}dx + (4\pi)^{-m/2}\frac16\int_{\partial M}\mathrm{Tr}\{2FL_{aa} -3F_{;m}\}dy$.

(4) $a_3(F,D,\mathcal{B}) = -\frac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\{96FE + F(16\tau+8R_{amam}+7L_{aa}L_{bb} -10L_{ab}L_{ab}) - 30F_{;m}L_{aa} + 24F_{;mm}\}dy$.

(5) $a_4(F,D,\mathcal{B}) = (4\pi)^{-m/2}\frac{1}{360}\int_M \mathrm{Tr}\{F(60E_{;kk}+60\tau E+180E^2+30\Omega^2+12\tau_{;kk} +5\tau^2-2|\rho^2|+2|R^2|)\}dx + (4\pi)^{-m/2}\frac{1}{360}\int_{\partial M}\mathrm{Tr}\{F(-120E_{;m}+120EL_{aa} -18\tau_{;m}+20\tau L_{aa}+4R_{amam}L_{bb}-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}+24L_{aa:bb} +\frac{40}{21}L_{aa}L_{bb}L_{cc}-\frac{88}{7}L_{ab}L_{ab}L_{cc}+\frac{320}{21}L_{ab}L_{bc}L_{ac}) + F_{;m}(-180E-30\tau -\frac{180}{7}L_{aa}L_{bb}+\frac{60}{7}L_{ab}L_{ab}) + 24F_{;mm}L_{aa} - 30F_{;iim}\}dy$.

Neumann boundary conditions are defined by the operator $\mathcal{B}_N\phi := \phi_{;m}|_{\partial M}$; the associated boundary conditions define a perfectly insulated boundary with no heat flow across the boundary. It is convenient in many applications to consider slightly more general conditions called Robin boundary conditions that permit the heat flow to be proportional to the temperature. Let S be an auxiliary endomorphism of V over ∂M. The Robin boundary operator is defined by:

$$\mathcal{B}_S\phi := (\phi_{;m}+S\phi)|_{\partial M}\,. \tag{5.b}$$

Theorem 5.2 *[Robin boundary conditions] Let* $F\in C^\infty(\mathrm{End}(V))$.

(1) $a_0(F,D,\mathcal{B}_S) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\{F\}dx$.

(2) $a_1(F,D,\mathcal{B}_S)=(4\pi)^{(1-m)/2}\frac14\int_{\partial M}\operatorname{Tr}\{F\}dy$.

(3) $a_2(F,D,\mathcal{B}_S)=(4\pi)^{-m/2}\frac16\int_M\operatorname{Tr}\{F(6E+\tau)\}dx+(4\pi)^{-m/2}\frac16\int_{\partial M}\operatorname{Tr}\{F(2L_{aa}$ $+12S)+3F_{;m}\}dy$.

(4) $a_3(F,D,\mathcal{B}_S)=(4\pi)^{(1-m)/2}\frac{1}{384}\int_{\partial M}\operatorname{Tr}\{F(96E+16\tau+8R_{amam}+13L_{aa}L_{bb}$ $+2L_{ab}L_{ab}+96SL_{aa}+192S^2+F_{;m}(6L_{aa}+96S)+24F_{;mm}\}dy$.

(5) $a_4(F,D,\mathcal{B}_S)=(4\pi)^{-m/2}\frac{1}{360}\int_M\operatorname{Tr}\{F(60E_{;kk}+60\tau E+180E^2+30\Omega^2+12\tau_{;kk}$ $+5\tau^2-2|\rho|^2+2|R|^2)\}dx+(4\pi)^{-m/2}\frac{1}{360}\int_{\partial M}\operatorname{Tr}\{F(240E_{;m}+42\tau_{;m}$ $+24L_{aa:bb}+120EL_{aa}+20\tau L_{aa}+4R_{amam}L_{bb}-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}$ $+\frac{40}{3}L_{aa}L_{bb}L_{cc}+8L_{ab}L_{ab}L_{cc}+\frac{32}{3}L_{ab}L_{bc}L_{ac}+360(SE+ES)+120S\tau$ $+144SL_{aa}L_{bb}+48SL_{ab}L_{ab}+480S^2L_{aa}+480S^3+120S_{:aa})+F_{;m}(180E$ $+30\tau+12L_{aa}L_{bb}+12L_{ab}L_{ab}+72SL_{aa}+240S^2)+F_{;mm}(24L_{aa}+120S)$ $+30F_{;iim}\}dy$.

When discussing the Euler characteristic of a manifold with boundary in Section 6 subsequently, it will useful to consider absolute and relative boundary conditions. Let r be the geodesic distance to the boundary. Near the boundary, decompose a differential form $\omega\in C^\infty(\Lambda(M))$ in the form $\omega=\omega_1+dr\wedge\omega_2$ where the ω_i are tangential differential forms. We define the relative boundary operator $\mathcal{B}_r$ and the absolute boundary operator $\mathcal{B}_a$ for the operator $d+\delta$ by setting:

$$\mathcal{B}_r(\omega)=\omega_1|_{\partial M}\text{ and }\mathcal{B}_a(\omega)=\omega_2|_{\partial M}. \tag{5.c}$$

There are induced boundary conditions for the associated Laplacian $(d+\delta)^2$. They are defined by the operator $\bar{\mathcal{B}}_{r/a}\phi:=\mathcal{B}_{r/a}\phi\oplus\mathcal{B}_{r/a}(d+\delta)\phi$.

The boundary conditions defined by the operators $\bar{\mathcal{B}}_{r/a}$ provide examples of a more general boundary condition which are called *mixed boundary conditions*. We can combine Theorems 5.1 and 5.2 into a single result by using such boundary conditions. We assume given a decomposition $V|_{\partial M}=V_+\oplus V_-$. Extend the bundles $V_\pm$ to a collared neighborhood of ∂M by parallel translation along the inward unit geodesic rays. Set $\chi:=\Pi_+-\Pi_-$. Let S be an auxiliary endomorphism of V_+ over ∂M. The *mixed boundary operator* may then be defined by setting

$$\mathcal{B}_{\chi,S}\phi:=\Pi_+(\phi_{;m}+S\phi)|_{\partial M}\oplus\Pi_-\phi|_{\partial M}\,. \tag{5.d}$$

One sets $\chi=\mathrm{id}$, $\Pi_+=\mathrm{id}$, and $\Pi_-=0$ to obtain the Robin boundary operator of Equation (5.b); one sets $\chi=-\mathrm{id}$, $\Pi_+=0$, and $\Pi_-=\mathrm{id}$ to obtain the Dirichlet boundary operator of Equation (5.a). The formulas of Theorem 5.1 and Theorem 5.2 then be obtained by this specialization.

Theorem 5.3 *[Mixed boundary conditions] Let $F=f\,\mathrm{id}$ for $f\in C^\infty(M)$. Then:*

(1) $a_0(F,D,\mathcal{B}_{\chi,S})=(4\pi)^{-m/2}\int_M\operatorname{Tr}\{F\}dx$.

(2) $a_1(F,D,\mathcal{B}_{\chi,S})=(4\pi)^{-(m-1)/2}\frac14\int_{\partial M}\operatorname{Tr}\{F\chi\}dy$.

(3) $a_2(F,D,\mathcal{B}_{\chi,S})=(4\pi)^{-m/2}\frac16\int_M \mathrm{Tr}\{F(6E+\tau)\}dx$
$+(4\pi)^{-m/2}\frac16\int_{\partial M}\mathrm{Tr}\{2FL_{aa}+3F_{;m}\chi+12FS\}dy.$

(4) $a_3(F,D,\mathcal{B}_{\chi,S})=(4\pi)^{-(m-1)/2}\frac{1}{384}\int_{\partial M}\mathrm{Tr}\{F(96\chi E+16\chi\tau+8\chi R_{amam}$
$+[13\Pi_+-7\Pi_-]L_{aa}L_{bb}+[2\Pi_++10\Pi_-]L_{ab}L_{ab}+96SL_{aa}+192S^2$
$-12\chi_{:a}\chi_{:a})+F_{;m}([6\Pi_++30\Pi_-]L_{aa}+96S)+24\chi F_{;mm}\}dy.$

(5) $a_4(F,D,\mathcal{B}_{\chi,S})=(4\pi)^{-m/2}\frac{1}{360}\int_M\mathrm{Tr}\{F(60E_{;kk}+60\tau E+180E^2$
$+30\Omega^2+12\tau_{;kk}+5\tau^2-2|\rho|^2+2|R|^2)\}dx+(4\pi)^{-m/2}\frac{1}{360}\int_M\mathrm{Tr}\{F([240\Pi_+$
$-120\Pi_-]E_{;m}+[42\Pi_+-18\Pi_-]\tau_{;m}+120EL_{aa}+24L_{aa:bb}+20\tau L_{aa}$
$+4R_{amam}L_{bb}-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}+720ES+120S\tau+[\frac{280}{21}\Pi_+$
$+\frac{40}{21}\Pi_-]L_{aa}L_{bb}L_{cc}+[\frac{168}{21}\Pi_+-\frac{264}{21}\Pi_-]L_{ab}L_{ab}L_{cc}+[\frac{224}{21}\Pi_++\frac{320}{21}\Pi_-]$
$\times L_{ab}L_{bc}L_{ac}+144SL_{aa}L_{bb}+48SL_{ab}L_{ab}+480S^2L_{aa}+480S^3+120S_{:aa}$
$+60\chi\chi_{:a}\Omega_{am}-12\chi_{:a}\chi_{:a}L_{bb}-24\chi_{:a}\chi_{:b}L_{ab}-120\chi_{:a}\chi_{:a}S)+F_{;m}(180\chi E$
$+30\chi\tau+[\frac{84}{7}\Pi_+-\frac{180}{7}\Pi_-]L_{aa}L_{bb}+240S^2+[\frac{84}{7}\Pi_++\frac{60}{7}\Pi_-]L_{ab}L_{ab}$
$+72SL_{aa}-18\chi_{:a}\chi_{:a})+F_{;mm}(24L_{aa}+120S)+30F_{;iim}\chi\}dy.$

(6) $a_5(F,D,\mathcal{B}_{\chi,S})=(4\pi)^{-(m-1)/2}\frac{1}{5760}\int_{\partial M}\mathrm{Tr}\{F\{360\chi E_{;mm}+1440E_{;m}S$
$+720\chi E^2+240\chi E_{:aa}+240\chi\tau E+48\chi\tau_{;ii}+20\chi\tau^2-8\chi\rho_{ij}\rho_{ij}$
$+8\chi R_{ijkl}R_{ijkl}-120\chi\rho_{mm}E-20\chi\rho_{mm}\tau+480\tau S^2+12\chi\tau_{;mm}$
$+24\chi\rho_{mm:aa}+15\chi\rho_{mm;mm}+270\tau_{;m}S+120\rho_{mm}S^2+960SS_{:aa}$
$+16\chi R_{ammb}\rho_{ab}-17\chi\rho_{mm}\rho_{mm}-10\chi R_{ammb}R_{ammb}+2880ES^2$
$+1440S^4+(90\Pi_++450\Pi_-)L_{aa}E_{;m}+(\frac{111}{2}\Pi_++42\Pi_-)L_{aa}\tau_{;m}$
$+30\Pi_+L_{ab}R_{ammb;m}+240L_{aa}S_{:bb}+420L_{ab}S_{:ab}+390L_{aa:b}S_{:b}$
$+480L_{ab:a}S_{:b}+420L_{aa:bb}S+60L_{ab:ab}S+(\frac{487}{16}\Pi_++\frac{413}{16}\Pi_-)L_{aa:b}L_{cc:b}$
$+(238\Pi_+-58\Pi_-)L_{ab:a}L_{cc:b}+(\frac{49}{4}\Pi_++\frac{11}{4}\Pi_-)L_{ab:a}L_{bc:c}$
$+(\frac{535}{8}\Pi_+-\frac{355}{8}\Pi_-)L_{ab:c}L_{ab:c}+(\frac{151}{4}\Pi_++\frac{29}{4}\Pi_-)L_{ab:c}L_{ac:b}$
$+(111\Pi_+-6\Pi_-)L_{aa:bb}L_{cc}+(-15\Pi_++30\Pi_-)L_{ab:ab}L_{cc}$
$+(-\frac{15}{2}\Pi_++\frac{75}{2}\Pi_-)L_{ab:ac}L_{bc}+(\frac{945}{4}\Pi_+-\frac{285}{4}\Pi_-)L_{aa:bc}L_{bc}$
$+(114\Pi_+-54\Pi_-)L_{bc:aa}L_{bc}+1440L_{aa}SE+30L_{aa}S\rho_{mm}+240L_{aa}S\tau$
$-60L_{ab}\rho_{ab}S+180L_{ab}SR_{ammb}+(195\Pi_+-105\Pi_-)L_{aa}L_{bb}E$
$+(30\Pi_++150\Pi_-)L_{ab}L_{ab}E+(\frac{195}{6}\Pi_+-\frac{105}{6}\Pi_-)L_{aa}L_{bb}\tau$
$+(5\Pi_++25\Pi_-)L_{ab}L_{ab}\tau+(-\frac{275}{16}\Pi_++\frac{215}{16}\Pi_-)L_{aa}L_{bb}\rho_{mm}$
$+(-\frac{275}{8}\Pi_++\frac{215}{8}\Pi_-)L_{ab}L_{ab}\rho_{mm}+(-\Pi_+-14\Pi_-)L_{cc}L_{ab}\rho_{ab}$
$+(\frac{109}{4}\Pi_+-\frac{49}{4}\Pi_-)L_{cc}L_{ab}R_{ammb}+16\chi L_{ab}L_{ac}\rho_{bc}$
$+(\frac{133}{2}\Pi_++\frac{47}{2}\Pi_-)L_{ab}L_{ac}R_{bmmc}-32\chi L_{ab}L_{cd}R_{acbd}$
$+\frac{315}{2}L_{cc}L_{ab}L_{ab}S+(\frac{2041}{128}\Pi_++\frac{65}{128}\Pi_-)L_{aa}L_{bb}L_{cc}L_{dd}$

$$
\begin{aligned}
&+150L_{ab}L_{bc}L_{ac}S+(\tfrac{417}{32}\Pi_++\tfrac{141}{32}\Pi_-)L_{cc}L_{dd}L_{ab}L_{ab}\\
&+1080L_{aa}L_{bb}S^2+360L_{ab}L_{ab}S^2+(\tfrac{375}{32}\Pi_+-\tfrac{777}{32}\Pi_-)L_{ab}L_{ab}L_{cd}L_{cd}\\
&+\tfrac{885}{4}L_{aa}L_{bb}L_{cc}S+(25\Pi_+-\tfrac{17}{2}\Pi_-)L_{dd}L_{ab}L_{bc}L_{ac}+2160L_{aa}S^3\\
&+(\tfrac{231}{8}\Pi_++\tfrac{327}{8}\Pi_-)L_{ab}L_{bc}L_{cd}L_{da}-180E^2+180\chi E\chi E-120S_{:a}S_{:a}\\
&+720\chi S_{:a}S_{:a}-\tfrac{105}{4}\Omega_{ab}\Omega_{ab}+120\chi\Omega_{ab}\Omega_{ab}+\tfrac{105}{4}\chi\Omega_{ab}\chi\Omega_{ab}-45\Omega_{am}\Omega_{am}\\
&+180\chi\Omega_{am}\Omega_{am}-45\chi\Omega_{am}\chi\Omega_{am}+360(\Omega_{am}\chi S_{:a}-\Omega_{am}S_{:a}\chi)\\
&+45\chi\chi_{:a}\Omega_{am}L_{cc}-180\chi_{:a}\chi_{:b}\Omega_{ab}+90\chi\chi_{:a}\chi_{:b}\Omega_{ab}+90\chi\chi_{:a}\Omega_{am;m}\\
&+120\chi\chi_{:a}\Omega_{ab:b}+180\chi\chi_{:a}\Omega_{bm}L_{ab}+300\chi_{:a}E_{:a}-180\chi_{:a}\chi_{:a}E-90\chi\chi_{:a}\chi_{:a}E\\
&+240\chi_{:aa}E-30\chi_{:a}\chi_{:a}\tau-60\chi_{:a}\chi_{:b}\rho_{ab}+30\chi_{:a}\chi_{:b}R_{mabm}-\tfrac{675}{32}\chi_{:a}\chi_{:a}L_{bb}L_{cc}\\
&-\tfrac{75}{4}\chi_{:a}\chi_{:b}L_{ac}L_{bc}-\tfrac{195}{16}\chi_{:a}\chi_{:a}L_{cd}L_{cd}-\tfrac{675}{8}\chi_{:a}\chi_{:b}L_{ab}L_{cc}-330\chi_{:a}S_{:a}L_{cc}\\
&-300\chi_{:a}S_{:b}L_{ab}+\tfrac{15}{4}\chi_{:a}\chi_{:a}\chi_{:b}\chi_{:b}+\tfrac{15}{8}\chi_{:a}\chi_{:b}\chi_{:a}\chi_{:b}-\tfrac{15}{4}\chi_{:aa}\chi_{:bb}-\tfrac{105}{2}\chi_{:ab}\chi_{:ab}\\
&-15\chi_{:a}\chi_{:a}\chi_{:bb}-\tfrac{135}{2}\chi_{:b}\chi_{:aab}\}+F_{;m}\{(\tfrac{195}{2}\Pi_+-60\Pi_-)\tau_{;m}+240\tau S-90\rho_{mm}S\\
&+270S_{:aa}+(630\Pi_+-450\Pi_-)E_{;m}+1440ES+720S^3+(90\Pi_++450\Pi_-)\\
&\times L_{aa}E+(-\tfrac{165}{8}\Pi_+-\tfrac{255}{8}\Pi_-)L_{aa}\rho_{mm}+(15\Pi_++75\Pi_-)L_{aa}\tau+600L_{aa}S^2\\
&+(\tfrac{1215}{8}\Pi_+-\tfrac{315}{8}\Pi_-)L_{aa:bb}-\tfrac{45}{4}\chi L_{ab:ab}+(15\Pi_+-30\Pi_-)L_{ab}\rho_{ab}+(-\tfrac{165}{4}\Pi_+\\
&+\tfrac{465}{4}\Pi_-)L_{ab}R_{ammb}+\tfrac{705}{4}L_{aa}L_{bb}S-\tfrac{75}{2}L_{ab}L_{ab}S+(\tfrac{459}{32}\Pi_++\tfrac{495}{32}\Pi_-)\\
&\times L_{aa}L_{bb}L_{cc}+(\tfrac{267}{16}\Pi_+-\tfrac{1485}{16}\Pi_-)L_{cc}L_{ab}L_{ab}+(-54\Pi_++\tfrac{225}{2}\Pi_-)L_{ab}L_{bc}L_{ac}\\
&-210\chi_{:a}S_{:a}-\tfrac{165}{16}\chi_{:a}\chi_{:a}L_{cc}-\tfrac{405}{8}\chi_{:a}\chi_{:b}L_{ab}+135\chi\chi_{:a}\Omega_{am}\}+F_{;mm}\{30L_{aa}S\\
&+(\tfrac{315}{16}\Pi_+-\tfrac{1215}{16}\Pi_-)L_{aa}L_{bb}+(-\tfrac{645}{8}\Pi_++\tfrac{945}{8}\Pi_-)L_{ab}L_{ab}+60\chi\tau-90\chi\rho_{mm}\\
&+360\chi E+360S^2-30\chi_{:a}\chi_{:a}\}+F_{;mmm}\{180S+(-30\Pi_++105\Pi_-)L_{aa}\}\\
&+45\chi F_{;mmmm}\}dy.
\end{aligned}
$$

We now consider transmission and transfer boundary conditions. Let M_+ and M_- be two manifolds which are coupled along a common boundary $\Sigma := \partial M_+ = \partial M_-$. We have metrics $g_\pm$ and operators $D_\pm$ of Laplace type on $M_\pm$. We have scalar smearing functions $f_\pm$ over $M_\pm$. Transmission boundary conditions arise in the study of heat conduction problems between closely coupled membranes. We impose the compatibility conditions

$$g_+|_\Sigma = g_-|_\Sigma, \quad V_+|_\Sigma = V_-|_\Sigma = V_\Sigma, \quad f_+|_\Sigma = f_-|_\Sigma\,.$$

No matching condition is assumed on the normal derivatives of f or of g on the interface Σ. Assume given an impedance matching endomorphism U defined on the hypersurface Σ. The *transmission boundary operator* is given by:

$$
\begin{aligned}
&\mathcal{B}_U\phi := \big\{\phi_+|_\Sigma - \phi_-|_\Sigma\big\} \quad \oplus \quad \big\{\nabla_{\nu_+}\phi_+|_\Sigma + \nabla_{\nu_-}\phi_-|_\Sigma - U\phi_+|_\Sigma\big\}, \qquad (5.e)\\
&\omega_a := \nabla_a^+ - \nabla_a^-\,.
\end{aligned}
$$

Since the difference of two connections is tensorial, ω_a is a well defined endomorphism of V_Σ. The tensor ω_a is *chiral*; it changes sign if the roles of $+$ and $-$ are reversed. On the other hand, the tensor field U is *non-chiral* as it is not sensitive to the roles of $+$ and $-$.

The following result is due to Gilkey, Kirsten, and Vassilevich [62]; see also related work by Bordag and Vassilevich [31] and by Moss [95]. Define:

$$\begin{aligned}
&\mathcal{L}^{\rm even}_{ab}:=L^+_{ab}+L^-_{ab}, && \mathcal{L}^{\rm odd}_{ab}:=L^+_{ab}-L^-_{ab},\\
&\mathcal{F}^{\rm even}_{;\nu}:=f_{;\nu^+}+f_{;\nu^-}, && \mathcal{F}^{\rm odd}_{;\nu}:=f_{;\nu^+}-f_{;\nu^-},\\
&\mathcal{F}^{\rm even}_{;\nu\nu}:=f_{;\nu^+\nu^+}+f_{;\nu^-\nu^-}, && \mathcal{F}^{\rm odd}_{;\nu\nu}:=f_{;\nu^+\nu^+}-f_{;\nu^-\nu^-},\\
&\mathcal{E}^{\rm even}:=E^++E^-, && \mathcal{E}^{\rm odd}:=E^+-E^-,\\
&\mathcal{E}^{\rm even}_{;\nu}:=E^+_{;\nu^+}+E^-_{;\nu^-}, && \mathcal{E}^{\rm odd}_{;\nu}:=E^+_{;\nu^+}-E^-_{;\nu^-},\\
&\mathcal{R}^{\rm even}_{ijkl}:=R^+_{ijkl}+R^-_{ijkl}, && \mathcal{R}^{\rm odd}_{ijkl}:=R^+_{ijkl}-R^-_{ijkl}\\
&\Omega^{\rm even}_{ij}:=\Omega^+_{ij}+\Omega^-_{ij}, && \Omega^{\rm odd}_{ij}:=\Omega^+_{ij}-\Omega^-_{ij}\,.
\end{aligned}$$

Theorem 5.4 *[Transmission boundary conditions]*

(1) $a_0(f,D,\mathcal{B}_U)=(4\pi)^{-m/2}\int_M f\,\mathrm{Tr}(\mathrm{id})dx.$

(2) $a_1(f,D,\mathcal{B}_U)=0.$

(3) $a_2(f,D,\mathcal{B}_U)=(4\pi)^{-m/2}\frac16\int_M f\,\mathrm{Tr}\{\tau\,\mathrm{id}+6E\}dx$
$+(4\pi)^{-m/2}\frac16\int_\Sigma 2f\,\mathrm{Tr}\{\mathcal{L}^{\rm even}_{aa}\,\mathrm{id}-6U\}dy.$

(4) $a_3(f,D,\mathcal{B}_U)=(4\pi)^{(1-m)/2}\frac{1}{384}\int_\Sigma \mathrm{Tr}\{f[\frac32\mathcal{L}^{\rm even}_{aa}\mathcal{L}^{\rm even}_{bb}+3\mathcal{L}^{\rm even}_{ab}\mathcal{L}^{\rm even}_{ab}]\,\mathrm{id}$
$+9\mathcal{L}^{\rm even}_{aa}\mathcal{F}^{\rm even}_{;\nu}\,\mathrm{id}+48fU^2+24f\omega_a\omega_a-24f\mathcal{L}^{\rm even}_{aa}U-24\mathcal{F}^{\rm even}_{;\nu}U\}dy.$

(5) $a_4(f,D,\mathcal{B}_U)=(4\pi)^{-m/2}\frac{1}{360}\int_M f\,\mathrm{Tr}\{60E_{;kk}+60R_{ijji}E+180E^2$
$+30\Omega_{ij}\Omega_{ij}+[12\tau_{;kk}+5\tau^2-2|\rho|^2+2|R|^2]\,\mathrm{id}\}dx$
$+(4\pi)^{-m/2}\frac{1}{360}\int_\Sigma \mathrm{Tr}\{[-5\mathcal{R}^{\rm odd}_{ijji}\mathcal{F}^{\rm odd}_{;\nu}+2\mathcal{R}^{\rm odd}_{a\nu a\nu}\mathcal{F}^{\rm odd}_{;\nu}$
$-5\mathcal{L}^{\rm odd}_{aa}\mathcal{L}^{\rm even}_{bb}\mathcal{F}^{\rm odd}_{;\nu}-\mathcal{L}^{\rm odd}_{ab}\mathcal{L}^{\rm even}_{ab}\mathcal{F}^{\rm odd}_{;\nu}+\frac{18}{7}\mathcal{L}^{\rm even}_{ab}\mathcal{L}^{\rm even}_{ab}\mathcal{F}^{\rm even}_{;\nu}$
$-\frac{12}{7}\mathcal{L}^{\rm even}_{aa}\mathcal{L}^{\rm even}_{bb}\mathcal{F}^{\rm even}_{;\nu}+12\mathcal{L}^{\rm even}_{aa}\mathcal{F}^{\rm even}_{;\nu\nu}]\,\mathrm{id}+f[-\mathcal{L}^{\rm odd}_{ab}\mathcal{L}^{\rm odd}_{ab}\mathcal{L}^{\rm even}_{cc}$
$-\mathcal{L}^{\rm even}_{ab}\mathcal{L}^{\rm odd}_{ab}\mathcal{L}^{\rm odd}_{cc}+2\mathcal{L}^{\rm odd}_{ab}\mathcal{L}^{\rm odd}_{bc}\mathcal{L}^{\rm even}_{ac}+2\mathcal{R}^{\rm odd}_{abcb}\mathcal{L}^{\rm odd}_{ac}+12\mathcal{R}^{\rm even}_{ijji;\nu}$
$+\frac{40}{21}\mathcal{L}^{\rm even}_{aa}\mathcal{L}^{\rm even}_{bb}\mathcal{L}^{\rm even}_{cc}-\frac47\mathcal{L}^{\rm even}_{ab}\mathcal{L}^{\rm even}_{ab}\mathcal{L}^{\rm even}_{cc}+\frac{68}{21}\mathcal{L}^{\rm even}_{ab}\mathcal{L}^{\rm even}_{bc}\mathcal{L}^{\rm even}_{ac}$
$+24\mathcal{L}^{\rm even}_{aa:bb}+10\mathcal{R}^{\rm even}_{ijji}\mathcal{L}^{\rm even}_{aa}+2\mathcal{R}^{\rm even}_{a\nu a\nu}\mathcal{L}^{\rm even}_{aa}-6\mathcal{R}^{\rm even}_{a\nu b\nu}\mathcal{L}^{\rm even}_{ab}$
$+2\mathcal{R}^{\rm even}_{abcb}\mathcal{L}^{\rm even}_{ac}]\,\mathrm{id}+18\omega_a^2\mathcal{F}^{\rm even}_{;\nu}-30\mathcal{E}^{\rm odd}\mathcal{F}^{\rm odd}_{;\nu}+15U\mathcal{L}^{\rm odd}_{aa}\mathcal{F}^{\rm odd}_{;\nu}$
$-30U\mathcal{F}^{\rm even}_{;\nu\nu}-9U\mathcal{L}^{\rm even}_{aa}\mathcal{F}^{\rm even}_{;\nu}+30U^2\mathcal{F}^{\rm even}_{;\nu}+f[12\omega_a^2\mathcal{L}^{\rm even}_{bb}$
$+24\omega_a\omega_b\mathcal{L}^{\rm even}_{ab}+60\mathcal{E}^{\rm even}_{;\nu}-60\omega_a\Omega^{odd}_{a\nu}+60\mathcal{E}^{\rm even}\mathcal{L}^{\rm even}_{aa}-60U^3$
$-30U\mathcal{R}^{\rm even}_{ijji}-180U\mathcal{E}^{\rm even}-60U_{:aa}-18U\mathcal{L}^{\rm even}_{aa}\mathcal{L}^{\rm even}_{bb}$
$-6U\mathcal{L}^{\rm even}_{ab}\mathcal{L}^{\rm even}_{ab}+60U^2\mathcal{L}^{\rm even}_{aa}-60U\omega_a^2]\}dy.$

We now examine transfer boundary conditions. As previously, we take structures $(M,g,V,D)=((M_+,g_+,V_+,D_+),(M_-,g_-,V_-,D_-))$. We now assume the compatibility conditions

$$\partial M_+=\partial M_-=\Sigma\quad\text{and}\quad g_+|_\Sigma=g_-|_\Sigma\,.$$

We no longer assume an identification of $V_+|_\Sigma$ with $V_-|_\Sigma$. Let $F_\pm$ be smooth smearing endomorphisms of $V_\pm$; there is no assumed relation between F_+ and F_-. Let $\mathrm{Tr}_\pm$ denote the fiber trace on $V_\pm$. We suppose given auxiliary impedance matching endomorphisms $\mathfrak{S} := \{S_{\pm\pm}\}$ from $V_\pm$ to $V_\pm$. The *transfer boundary operator* is defined by setting:

$$\mathcal{B}_{\mathfrak{S}}\phi := \left\{ \begin{pmatrix} \nabla^+_{\nu_+} + S_{++} & S_{+-} \\ S_{-+} & \nabla^-_{\nu_-} + S_{--} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \right\} \bigg|_\Sigma . \tag{5.f}$$

We set $S_{+-} = S_{-+} = 0$ to introduce the associated decoupled Robin boundary conditions

$$\begin{aligned} &\mathcal{B}_{R(S_{++})}\phi_+ := (\nabla^+_{\nu_+} + S_{++})\phi_+|_\Sigma, \quad \text{and} \\ &\mathcal{B}_{R(S_{--})}\phi_- := (\nabla^-_{\nu_-} + S_{--})\phi_-|_\Sigma . \end{aligned}$$

Define the correction term $a_n(F, D, S)(y)$ by means of the identity

$$\begin{aligned} a_n(F, D, \mathcal{B}_{\mathfrak{S}}) &= \int_M a_n(F, D)(x)dx + \int_\Sigma a_n(F_+, D_+, \mathcal{B}_{R(S_{++})})dy \\ &+ \int_\Sigma a_n(F_-, D_-, \mathcal{B}_{R(S_{--})})dy + \int_\Sigma a_n(F, D, S)(y)dy . \end{aligned}$$

As the interior invariants $a_n(F, D)$ are discussed in Theorem 3.4 and as the Robin invariants $a_n(F, D, \mathcal{B}_{R(S_{++})})$ and $a_n(F, D, \mathcal{B}_{R(S_{--})})$ are discussed in Theorem 5.2, we must only determine the invariant $a_n(F, D, S)$ which measures the new interactions that arise from S_{+-} and S_{-+}. We refer to [63] for the proof of the following result:

Theorem 5.5 *[Transfer boundary conditions]*

(1) $a_n(F, D, \mathcal{B}_{\mathfrak{S}})(y) = 0$ *for* $n \leq 2$.

(2) $a_3(F, D, \mathcal{B}_{\mathfrak{S}})(y) = (4\pi)^{(1-m)/2}\frac{1}{2}\big\{ \mathrm{Tr}_+(F_+S_{+-}S_{-+}) + \mathrm{Tr}_-(F_-S_{-+}S_{+-})\big\}$.

(3) $$\begin{aligned} a_4(F, D, \mathcal{B}_{\mathfrak{S}})(y) = (4\pi)^{-m/2}\tfrac{1}{360}\big\{ &\mathrm{Tr}_+\{480(F_+S_{++} + S_{++}F_+)S_{+-}S_{-+} \\ &+480F_+S_{+-}S_{--}S_{-+} \\ &+(288F_+L^+_{aa} + 192F_+L^-_{aa} + 240F_{+;\nu_+})S_{+-}S_{-+}\} \\ &+ \mathrm{Tr}_-\{480(F_-S_{--} + S_{--}F_-)S_{-+}S_{+-} + 480F_-S_{-+}S_{++}S_{+-} \\ &+(288F_-L^-_{aa} + 192F_-L^+_{aa} + 240F_{-;\nu_-})S_{-+}S_{+-}\}\big\}. \end{aligned}$$

We now take up *spectral asymmetry*. We refer to [33, 34] for the material of this section. Let M be a compact Riemannian manifold. Let A be an operator of Dirac type and let $D = A^2$ be the associated operator of Laplace type. Instead of studying $\mathrm{Tr}_{L^2}(e^{-tD})$, we study $\mathrm{Tr}_{L^2}(Ae^{-tD})$; this provides a measure of the spectral asymmetry of A.

Let ∇ be a compatible connection; this means that $\nabla\gamma = 0$ and that if there is a fiber metric on V that ∇ is unitary. Expand $A = \gamma^\nu\nabla_{\partial_{x_\nu}} + \psi_A$. If ∂M is non-empty, we shall use local boundary conditions; we postpone until a subsequent section the question of spectral boundary conditions. Let $\{e_1, ..., e_m\}$ be a local orthonormal frame for the tangent bundle near ∂M which is normalized so e_m is the inward unit geodesic normal vector field. Suppose there exists an endomorphism χ of $V|_{\partial M}$ so that χ is self-adjoint and so that

$$\chi^2 = 1, \quad \chi\gamma_m + \gamma_m\chi = 0, \quad \text{and} \quad \chi\gamma_a = \gamma_a\chi \quad \text{for} \quad 1 \leq a \leq m-1 .$$

Such a χ always exists if M is orientable and if m is even as, for example, one could take $\chi=\varepsilon\gamma_1...\gamma_{m-1}$ where ε is a suitable 4^{th} root of unity. There are topological obstructions to the existence of χ if m is odd; if ∂M is empty, χ plays no role. Let $\Pi_\chi^\pm:=\frac12(\text{id}\pm\chi)$ be orthonormal projection on the ±1 eigenspaces of χ. We let $\mathcal{B}\phi:=\Pi_\chi^-\phi|_{\partial M}$. The associated boundary condition for $D:=A^2$ is defined by the operator $\mathcal{B}_1\phi:=\mathcal{B}\phi\oplus\mathcal{B}A\phi$ and is equivalent to a mixed boundary operator $\mathcal{B}_{\chi,S}$ where

$$S=\tfrac12\Pi_+(\gamma_m\psi_A-\psi_A\gamma_m-L_{aa}\chi)\Pi_+\,.$$

As $t\downarrow0$, there is an asymptotic expansion

$$\text{Tr}_{L^2}(FAe^{-tA_{\mathcal{B}}^2})\sim\sum_{n=0}^\infty a_n^\eta(F,A,\mathcal{B})t^{(n-m-1)/2}\,.$$

These invariants measure the spectral asymmetry of A; $a_n^\eta(F,A,\mathcal{B})=-a_n^\eta(F,-A,\mathcal{B})$.

Theorem 5.6 *Let* $W_{ij}:=\Omega_{ij}-\frac14R_{ijkl}\gamma_k\gamma_l$ *where* Ω *is the curvature of* ∇. *Let* $F=f\,\text{id}$ *for* $f\in C^\infty(M)$.

(1) $a_0^\eta(f,A,\mathcal{B})=0$.

(2) $a_1^\eta(f,A,\mathcal{B})=-(4\pi)^{-m/2}(m-1)\int_M f\,\text{Tr}\{\psi_A\}dx$.

(3) $a_2^\eta(f,A,\mathcal{B})=\frac14(4\pi)^{-(m-1)/2}\int_{\partial M}(2-m)f\,\text{Tr}\{\psi_A\chi\}dy$.

(4) $a_3^\eta(f,A,\mathcal{B})=-\frac1{12}(4\pi)^{-m/2}\int_M f\big\{\,\text{Tr}\{2(m-1)\nabla_{e_i}\psi_A+3(4-m)\psi_A\gamma_i\psi_A$
$+3\gamma_j\psi_A\gamma_j\gamma_i\psi_A\}_{;i}+(m-3)\,\text{Tr}\{-\tau\psi_A-6\gamma_i\gamma_jW_{ij}\psi_A+6\gamma_i\psi_A\nabla_{e_i}\psi_A$
$+(m-4)\psi_A^3-3\psi_A^2\gamma_j\psi_A\gamma_j\}\big\}dx-\frac1{12}(4\pi)^{-m/2}\int_{\partial M}\text{Tr}\{6(m-2)f_{;m}\chi\psi_A$
$+f[(6m-18)\chi\nabla_{e_m}\psi_A+2(m-1)\nabla_{e_m}\psi_A+6\chi\gamma_m\gamma_a\nabla_{e_a}\psi_A$
$+6(2-m)\chi\psi_AL_{aa}+2(3-m)\psi_AL_{aa}+6(3-m)\chi\gamma_m\psi_A^2+3\gamma_m\psi_A\gamma_a\psi_A\gamma_a$
$+3(3-m)\chi\gamma_m\psi_A\chi\psi_A+6\chi\gamma_aW_{am}]\}dy$.

6 Heat trace asymptotics and index theory

We refer to [54] for a more exhaustive treatment; the classical results may be found in [2, 3, 4, 7, 8]. In this section, we only present a brief introduction to the subject as it relates to heat trace asymptotics. Let $P:C^\infty(V_1)\to C^\infty(V_2)$ be a first order partial differential operator on a closed Riemannian manifold M. We assume V_1 and V_2 are equipped with fiber metrics. We say that the triple $\mathcal{C}:=(P,V_1,V_2)$ is an *elliptic complex of Dirac type* if the associated second order operators $D_1:=P^*P$ and $D_2:=PP^*$ are of Laplace type. One may then define $\text{Index}(\mathcal{C}):=\dim\ker(D_1)-\dim\ker(D_2)$

Bott noted that $\text{Tr}_{L^2}\{e^{-tD_1}\}-Tr_{L^2}\{e^{-tD_2}\}=\text{Index}(\mathcal{C})$ was independent of the parameter t. He then used the asymptotic expansion of the heat equation to obtain a local formula for the index in terms of heat trace asymptotics. Following the notation of Equation (3.a), one may define the heat trace asymptotics of P by setting:

$$a_n^M(P)(x):=\big\{\,\text{Tr}\{e_n^M(D_1)\}-\text{Tr}\{e_n^M(D_2)\}\big\}(x)\,.$$

One then has a local formula for the index:

Theorem 6.1 *Let $\mathcal{C}$ be an elliptic complex of Dirac type over a closed Riemannian manifold M. Then:*

$$\int_M a_n^M(P)(x)dx = \begin{cases} \operatorname{Index}(\mathcal{C}) & \textit{if} \quad n = m, \\ 0 & \textit{if} \quad n \neq m\,. \end{cases}$$

The critical term $a_m^M(P)(x)dx$ is often referred to as the *index density*. The other terms are in divergence form since they integrate to zero. They need not, however, vanish identically.

The existence of a local formula for the index implies the index is constant under deformations. It also yields, less trivially, that the index is multiplicative under finite coverings and additive with respect to connected sums. In the next section, we shall see that the index of the DeRham complex is the Euler-Poincare characteristic $\chi(M)$ of the manifold. Thus if $F \to M_1 \to M_2$ is a finite covering, then $\chi(M_1) = |F| \cdot \chi(M_2)$. Similarly, if $M = M_1 \# M_2$ is a connected sum, then $\chi(M) + \chi(S^m) = \chi(M_1) + \chi(M_2)$. Analogous formulas hold for the Hirzebruch signature of a manifold.

We define DeRham complex as follows. Let $d : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M)$ be exterior differentiation and let $\delta : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p-1} M)$ be the dual, interior multiplication. We may then define a 2-term elliptic complex of Dirac type:

$$\begin{aligned} &(d+\delta) : C^\infty(\Lambda^e M) \to C^\infty(\Lambda^o M) \quad \text{where} \\ &\Lambda^e(M) := \oplus_n \Lambda^{2n}(M) \quad \text{and} \quad \Lambda^o(M) := \oplus_n \Lambda^{2n+1}(M)\,. \end{aligned} \tag{6.a}$$

Let R_{ijkl} be the curvature tensor. Let $m = 2\bar{m}$ be even. Let $\{e_1, ..., e_m\}$ be a local orthonormal frame for the tangent bundle. We sum over repeated indices to define the *Pfaffian*

$$\mathcal{PF}_m : \;=\; \frac{g(e^{i_1} \wedge ... \wedge e^{i_m}, e^{j_1} \wedge ... \wedge e^{j_m})}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!} R_{i_1 i_2 j_1 j_2} ... R_{i_{m-1} i_m j_{m-1} j_m}\,.$$

Set $\mathcal{PF}_m = 0$ if m is odd. The following result of Patodi [105] recovers the classical *Gauss-Bonnet theorem* of Chern [43]:

Theorem 6.2 *Let M be a closed even dimensional Riemannian manifold. Then*

(1) $a_n^M(d+\delta)(x) = 0$ *for* $n < m$.

(2) $a_m^M(d+\delta)(x) = \mathcal{PF}_m(x)$.

(3) $\chi(M) = \int_M \mathcal{PF}_m(x)dx$.

One can discuss Gauss-Bonnet theorem for manifolds with boundary similarly. On the boundary, normalize the orthonormal frame so e_m is the inward unit normal and let indices a, b range from 1 to $m-1$ and index the induced frame for the tangent bundle of the boundary. Let L_{ab} be the components of the second fundamental form. Define the *transgression* of the Pfaffian by setting:

$$\begin{aligned} T\mathcal{PF}_m : \;=\; & \sum_k \frac{g(e^{a_1} \wedge ... \wedge e^{a_{m-1}}, e^{b_1} \wedge ... \wedge e^{b_{m-1}})}{\pi^k 8^k k! (m-1-2k)! \operatorname{vol}(S^{m-1-2k})} \\ \times \;\; & R_{a_1 a_2 b_1 b_2} ... R_{a_{2k-1} a_{2k} b_{2k-1} b_{2k}} L_{a_{2k+1} b_{2k+1}} ... L_{a_{m-1} b_{m-1}}\,. \end{aligned}$$

If we impose absolute boundary conditions as discussed in Equation (5.c) to define the elliptic complex, we recover the Chern-Gauss-Bonnet theorem for manifolds with boundary [44]. Let Δ_M^e and Δ_M^o denote the Laplacians on the space of smooth differential forms of even and odd degrees, respectively. Let

$$a_n^{\partial M}(d+\delta)(y) = \left\{ \operatorname{Tr}\{e_n^{\partial M}(\Delta_M^e, \mathcal{B}_a)\} - \operatorname{Tr}\{e_n^{\partial M}(\Delta_M^0, \mathcal{B}_a)\}\right\}(y)\,.$$

Theorem 6.1 extends to this setting to become:

$$\int_M a_n^M(d+\delta)(x)dx + \int_{\partial M} a_n^{\partial M}(d+\delta)(y)dy = \begin{cases} 0 & \text{if} \quad n \neq m, \\ \chi(M) & \text{if} \quad n = m\,. \end{cases}$$

Theorem 6.2 then extends to this setting to yield:

Theorem 6.3 (1) $a_n^{\partial \mathrm{M}}(d+\delta)(y) = 0$ *for* $n < m$.

(2) $a_m^{\partial \mathrm{M}}(d+\delta)(y) = T\mathcal{P}\mathcal{F}_m$.

(3) $\chi(M) = \int_M \mathcal{P}\mathcal{F}_m(x)dx + \int_{\partial M} T\mathcal{P}\mathcal{F}_m(y)dy$.

The local index invariants $a_{m+2}^M(d+\delta)(x)$ are in divergence form but do not vanish identically. Set

$$\begin{aligned} \Phi_m \quad &= \quad \frac{\bar{m}}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!}\{R_{i_1 i_2 j_1 k;k} R_{i_3 i_4 j_3 j_4} ... R_{i_{m-1} i_m j_{m-1} j_m}\}_{;j_2} \\ &\times \quad g(e^{i_1}\wedge ... \wedge e^{i_m}, e^{j_1} \wedge ... \wedge e^{j_m})\,. \end{aligned}$$

Theorem 6.4 *If M is even, then $a_{m+2}^M(d+\delta) = \frac{1}{12}\mathcal{P}\mathcal{F}_{m;kk} + \frac{1}{6}\Phi_m$.*

Spectral boundary conditions plan an important role in index theory. We suppose given an elliptic complex of Dirac type $P : C^\infty(V_1) \to C^\infty(V_2)$. Let γ be the leading symbol of P. Then

$$\begin{pmatrix} 0 & \gamma^* \\ \gamma & 0 \end{pmatrix}$$

defines a *unitary Clifford module structure* on $V_1 \oplus V_2$. We may choose a unitary connection ∇ on $V_1 \oplus V_2$ which is compatible with respect to this Clifford module structure and which respects the splitting and induces connections ∇_1 and ∇_2 on the bundles V_1 and V_2, respectively. Decompose $P = \gamma_i \nabla_{e_i} + \psi$. Near the boundary, the structures depend on the normal variable. We set the normal variable x_m to zero to define a tangential operator of Dirac type

$$B(y) := \gamma_m(y,0)^{-1}\left(\gamma_a(y,0)\nabla_{e_a} + \psi(y,0)\right) \text{ on } C^\infty(V_1|_{\partial M})\,.$$

Let B^* be the adjoint of B on $L^2(V_1|_{\partial M})$;

$$B^* = \gamma_m(y,0)^{-1}\gamma_a(y,0)\nabla_{e_a} + \psi_B^*$$

where $\psi_B := \gamma_m(y,0)^{-1}\psi(y,0)$. Let Θ be an auxiliary self-adjoint endomorphism of V_1. We set

$$\begin{aligned} A &:= \tfrac{1}{2}(B + B^*) + \Theta \quad \text{on} \quad C^\infty(V_1|_{\partial M}), \\ A^\# &:= -\gamma^m A(\gamma^m)^{-1} \quad \text{on} \quad C^\infty(V_2|_{\partial M})\,. \end{aligned}$$

The leading symbol of A is then given by $\gamma_a^T := \gamma_m^{-1}\gamma_a$ which is a unitary Clifford module structure on $V_1|_{\partial M}$. Thus A is a self-adjoint operator of Dirac type on $C^\infty(V_1|_{\partial M})$; similarly $A^\#$ is a self-adjoint operator of Dirac type on $C^\infty(V_2|_{\partial M})$.

Let Π_A^+ (resp. $\Pi_{A^\#}^+$) be spectral projection on the eigenspaces of A (resp. $A^\#$) corresponding to the positive (resp. non-negative) eigenvalues; there is always a bit of technical fuss concerning the harmonic eigenspaces that we will ignore as it does not affect the heat trace asymptotic coefficients that we shall be considering. Introduce the associated spectral boundary operators by

$$\begin{aligned}
\mathcal{B}_1\phi_1 &:= \Pi_A^+(\phi_1|_{\partial M}) && \text{for} \quad \phi_1 \in C^\infty(V_1),\\
\mathcal{B}_2\phi_2 &:= \Pi_{A^\#}^+(\phi_2|_{\partial M}) && \text{for} \quad \phi_2 \in C^\infty(V_2),\\
\mathcal{B}_\Theta\phi_1 &:= \mathcal{B}_1\phi_1 \oplus \mathcal{B}_2(P\phi_1) && \text{for} \quad \phi_1 \in C^\infty(V_1)\,.
\end{aligned}$$

If $P_{\mathcal{B}_1}$, $P_{\mathcal{B}_2}^*$, and $D_{1,\mathcal{B}}$ are the realizations of P, of P^*, and of D_1 with respect to the boundary conditions $\mathcal{B}_1$,$\mathcal{B}_2$, and $\mathcal{B}_\Theta$, respectively, then

$$(P_{\mathcal{B}_1})^* = P_{\mathcal{B}_2}^* \qquad \text{and} \qquad D_{1,\mathcal{B}_\Theta} = P_{\mathcal{B}_1}^* P_{\mathcal{B}_1}\,.$$

We will discuss these boundary conditions in further detail in Section 10.

The local index density for the twisted signature and for the twisted spin complex has been identified using methods of invariance theory; see, for example, the discussion in Atiyah, Bott, and Patodi [5]. This identification of the local index density has been used to give a heat equation proof of the Atiyah-Singer index theorem in complete generality and has led to the proof of the index theorem for manifolds with boundary of Atiyah, Patodi, and Singer [6]. Unlike the DeRham complex, a salient feature of these complexes is the necessity to introduce spectral boundary conditions for the twisted signature and twisted spin complexes – there is a topological obstruction which prevents using local boundary conditions. The eta invariant plays an essential role in this development. We also refer to N. Berline, N. Getzler, and M. Vergne [28], to Bismut [30], and to Melrose [91] for other treatments of the local index theorem.

The Dolbeault complex is a bit different. Patodi [106] showed the heat trace invariants agreed with the classical Riemann-Roch invariant for a Kaehler manifold; it should be noted that this is not the case for an arbitrary Hermitian manifold. The Lefschetz fixed point formulas can also be established using heat equation methods.

7 Heat content asymptotics

We refer to [41, 56] for further details concerning the material of this section; we note that the asymptotic series for the heat content function is established by van den Berg et al [24] in a very general setting. Let D be an operator of Laplace type on a smooth vector bundle V over a smooth Riemannian manifold. Let $\langle\cdot,\cdot\rangle$ denote the natural pairing between V and the dual bundle $\tilde V$. Let $\rho \in C^\infty(\tilde V)$ be the specific heat and let $\phi \in C^\infty(V)$ be the initial heat temperature distribution of the manifold. Impose suitable boundary conditions $\mathcal{B}$; we shall denote the dual boundary conditions for the dual operator $\tilde D$ on $C^\infty(\tilde V)$ by $\tilde{\mathcal{B}}$. Let ∇ be the connection determined by D and E the associated endomorphism. Then the dual connection $\tilde\nabla$ and the dual endomorphism $\tilde E$ are the connection and the endomorphism determined by $\tilde D$.

The total heat energy content of the manifold is given by:

$$\beta(\phi,\rho,D,\mathcal{B})(t)=\beta(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})(t):=\int_M\langle\rho,e^{-tD}\phi\rangle dx\,.$$

As $t\downarrow 0$, there is a complete asymptotic expansion of the form

$$\beta(\phi,\rho,D,\mathcal{B})(t)\sim\sum_{n=0}^{\infty}\beta_n(\phi,\rho,D,\mathcal{B})t^{n/2}\,.$$

There are local interior invariants β_n^M and boundary invariants $\beta_n^{\partial M}$ so that

$$\beta_n(\phi,\rho,D,\mathcal{B})=\int_M\beta_n^M(\phi,\rho,D)(x)dx+\int_{\partial M}\beta_n^{\partial M}(\phi,\rho,D,\mathcal{B})(y)dy\,.$$

These invariants are not uniquely characterized by this formula; divergence terms in the interior can be compensated by corresponding boundary terms.

We now study the heat content asymptotics of the disk D^m in $\mathbb{R}^m$ and the hemisphere H^m in S^m. We let D be the scalar Laplacian, $\phi=\rho=1$, and impose Dirichlet boundary conditions to define $\beta_n(M):=\beta_n(1,1,\Delta_M^0,\mathcal{B}_D)$. One has [16, 17] that:

Theorem 7.1 *Let D^m be the unit disk in $\mathbb{R}^m$. Then:*

(1) $\beta_0(D^m)=\frac{\pi^{m/2}}{\Gamma(\frac{(2+m)}{2})}$.

(2) $\beta_1(D^m)=-4\frac{\pi^{(m-1)/2}}{\Gamma(\frac{m}{2})}$.

(3) $\beta_2(D^m)=\frac{\pi^{m/2}}{\Gamma(\frac{m}{2})}(m-1)$.

(4) $\beta_3(D^m)=-\frac{\pi^{(m-1)/2}}{3\Gamma(\frac{m}{2})}(m-1)(m-3)$.

(5) $\beta_4(D^m)=-\frac{\pi^{m/2}}{8\Gamma(\frac{m}{2})}(m-1)(m-3)$.

(6) $\beta_5(D^m)=\frac{\pi^{(m-1)/2}}{120\Gamma(\frac{m}{2})}(m-1)(m-3)(m+3)(m-7)$.

(7) $\beta_6(D^m)=\frac{\pi^{m/2}}{96\Gamma(\frac{m}{2})}(m-1)(m-3)(m^2-4m-9)$.

(8) $\beta_7(D^m)=-\frac{\pi^{(m-1)/2}}{3360\Gamma(\frac{m}{2})}(m-1)(m-3)(m^4-8m^3-90m^2+424m+633)$.

Theorem 7.2 *Let H^m be the upper hemisphere of the unit sphere S^m. Then*

(1) $\beta_{2k}(H^m)=0$ *for any m if $k>0$.*

(2) $\beta_{2k+1}(H^3)=\frac{8\pi^{1/2}}{k!(2k-1)(2k+1)}$.

(3) $\beta_{2k+1}(H^5)=\frac{\pi^{3/2}2^{2k+3}(2-k)}{3k!(2k-1)(2k+1)}$.

(4) $\beta_{2k+1}(H^7)=\frac{\pi^{5/2}}{30}\{\frac{(67-54k)9^k}{k!(2k-1)(2k+1)}+\sum_{\ell=0}^k\frac{3\cdot 2^{3\ell}}{\ell!(k-\ell)!(2k-2\ell+1)}\}$.

We now study the heat content asymptotics with Dirichlet boundary conditions. Let $\mathcal{B}_D$ be the Dirichlet boundary operator of Equation (5.a). We refer to [16, 19] for the proof of:

Theorem 7.3 *[Dirichlet boundary conditions]*

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_D)=\int_M\langle\phi,\rho\rangle dx$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_D)=-\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\phi,\rho\rangle dy$.

(3) $\beta_2(\phi,\rho,D,\mathcal{B}_D)=-\int_M\langle D\phi,\rho\rangle dx+\int_{\partial M}\{\langle\frac12 L_{aa}\phi,\rho\rangle-\langle\phi,\rho_{;m}\rangle\}dy$.

(4) $\beta_3(\phi,\rho,D,\mathcal{B}_D)=-\frac{2}{\sqrt{\pi}}\int_{\partial M}\{\frac23\langle\phi_{;mm},\rho\rangle+\frac23\langle\phi,\rho_{;mm}\rangle-\langle\phi_{:a},\rho_{:a}\rangle+\langle E\phi,\rho\rangle$
$-\frac23 L_{aa}\langle\phi,\rho\rangle_{;m}+\langle(\frac{1}{12}L_{aa}L_{bb}-\frac16 L_{ab}L_{ab}-\frac16 R_{amma})\phi,\rho\rangle\}dy$.

(5) $\beta_4(\phi,\rho,D,\mathcal{B}_D)=\frac12\int_M\langle D\phi,\tilde D\rho\rangle dx+\int_{\partial M}\{\frac12\langle(D\phi)_{;m},\rho\rangle+\frac12\langle\phi,(\tilde D\rho)_{;m}\rangle$
$-\frac14\langle L_{aa}D\phi,\rho\rangle-\frac14\langle L_{aa}\phi,\tilde D\rho\rangle+\langle(\frac18 E_{;m}-\frac1{16}L_{ab}L_{ab}L_{cc}+\frac18 L_{ab}L_{ac}L_{bc}$
$-\frac1{16}R_{ambm}L_{ab}+\frac1{16}R_{abcb}L_{ac}+\frac1{32}\tau_{;m}+\frac1{16}L_{ab:ab})\phi,\rho\rangle$
$-\frac14 L_{ab}\langle\phi_{:a},\rho_{:b}\rangle-\frac18\langle\Omega_{am}\phi_{:a},\rho\rangle+\frac18\langle\Omega_{am}\phi,\rho_{:a}\rangle\}dy$.

We may compute $\beta_n(M)$ for $n\le 4$ by setting $\phi=\rho=1$ and $E=\Omega=0$ in Theorem 7.3. One has a formula [18] for $\beta_5(M)$; $\beta_5(\phi,\rho,D,\mathcal{B}_D)$ is not known in full generality.

Theorem 7.4 $\beta_5(M)=-\frac{1}{240\sqrt{\pi}}\int_{\partial M}\{8\rho_{mm;mm}-8L_{aa}\rho_{mm;m}+16L_{ab}R_{ammb;m}$
$-4\rho_{mm}^2+16R_{ammb}R_{ammb}-4L_{aa}L_{bb}\rho_{mm}-8L_{ab}L_{ab}\rho_{mm}+64L_{ab}L_{ac}R_{mbcm}$
$-16L_{aa}L_{bc}R_{mbcm}-8L_{ab}L_{ac}R_{bddc}-8L_{ab}L_{cd}R_{acbd}+4R_{abcm}R_{abcm}$
$+8R_{abbm}R_{accm}-16L_{aa:b}R_{bccm}-8L_{ab:c}L_{ab:c}+L_{aa}L_{bb}L_{cc}L_{dd}$
$-4L_{aa}L_{bb}L_{cd}L_{cd}+4L_{ab}L_{ab}L_{cd}L_{cd}-24L_{aa}L_{bc}L_{cd}L_{db}+48L_{ab}L_{bc}L_{cd}L_{da}\}dy$.

The invariants $\beta_0(M)$, $\beta_1(M)$, and $\beta_2(M)$ were computed by van den Berg and Davies [20] and by van den Berg and Le Gall [21] for domains in $\mathbb{R}^m$. The invariants $\beta_0(M)$, $\beta_1(M)$, and $\beta_2(M)$ were computed by van den Berg [14] for the upper hemisphere of the unit sphere. The general case where D is an arbitrary operator of Laplace type and where ϕ and ρ are arbitrary was studied in [16, 19]. Savo [112, 113, 114, 115] has given a closed formula for all the heat content asymptotics $\beta_k(M)$ that is combinatorially quite different in nature from the formulas we have presented here. There is also important related work of McAvity [87, 88], of McDonald and Meyers [89], and of Phillips and Jansons [108].

We now study heat content asymptotics for Robin boundary conditions. Let $\mathcal{B}_S$ be the Robin boundary operator of Equation (5.b); the dual boundary condition is then given by $\tilde{\mathcal{B}}_S\rho=\mathcal{B}_{\tilde S}\rho=(\rho_{;m}+\tilde S\rho)|_{\partial M}$ where, of course, we use the dual connection on $\tilde V$ to define $\rho_{;m}$. The following result is proved in [19, 45]:

Theorem 7.5 *[Robin boundary conditions]*

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_S)=\int_M\langle\phi,\rho\rangle dx$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_S)=0$.

(3) $\beta_2(\phi,\rho,D,\mathcal{B}_S)=-\int_M\langle D\phi,\rho\rangle dx+\int_{\partial M}\langle\mathcal{B}_S\phi,\rho\rangle dy$.

(4) $\beta_3(\phi,\rho,D,\mathcal{B}_S)=\frac{2}{3}\cdot\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\mathcal{B}_S\phi,\mathcal{B}_{\tilde S}\rho\rangle dy$.

(5) $\beta_4(\phi,\rho,D,\mathcal{B}_S)=\frac{1}{2}\int_M\langle D\phi,\tilde D\rho\rangle dx+\int_{\partial M}\{-\frac{1}{2}\langle\mathcal{B}_S\phi,\tilde D\rho\rangle-\frac{1}{2}\langle D\phi,\mathcal{B}_{\tilde S}\rho\rangle$
$+\langle(\frac{1}{2}S+\frac{1}{4}L_{aa})\mathcal{B}_S\phi,\mathcal{B}_{\tilde S}\rho\rangle\}dy$.

(6) $\beta_5(\phi,\rho,D,\mathcal{B}_S)=\frac{2}{\sqrt{\pi}}\int_{\partial M}\{-\frac{4}{15}(\langle\mathcal{B}_SD\phi,\mathcal{B}_{\tilde S}\rho\rangle+\langle\mathcal{B}_S\phi,\mathcal{B}_{\tilde S}\tilde D\rho\rangle)$
$-\frac{2}{15}\langle(\mathcal{B}_S\phi)_{:a},(\mathcal{B}_{\tilde S}\rho)_{:a}\rangle+\langle(\frac{2}{15}E+\frac{4}{15}S^2+\frac{4}{15}SL_{aa}+\frac{1}{30}L_{aa}L_{bb}$
$+\frac{1}{15}L_{ab}L_{ab}-\frac{1}{15}R_{amam})\mathcal{B}_S\phi,\mathcal{B}_{\tilde S}\rho\rangle\}dy$.

(7) $\beta_6(\phi,\rho,D,\mathcal{B}_S)=-\frac{1}{6}\int_M\langle D^2\phi,\tilde D\rho\rangle dx+\int_{\partial M}\{\frac{1}{6}\langle\mathcal{B}_SD\phi,\tilde D\rho\rangle+\frac{1}{6}\langle D^2\phi,\tilde{\mathcal{B}}_S\rho\rangle$
$+\frac{1}{6}\langle\mathcal{B}_S\phi,\tilde D^2\rho\rangle-\frac{1}{6}\langle S\mathcal{B}_SD\phi,\tilde{\mathcal{B}}_S\rho\rangle-\frac{1}{6}\langle S\mathcal{B}_S\phi,\tilde{\mathcal{B}}_S\tilde D\rho\rangle-\frac{1}{12}\langle L_{aa}\mathcal{B}_SD\phi,\tilde{\mathcal{B}}_S\rho\rangle$
$-\frac{1}{12}\langle L_{aa}\mathcal{B}_S\phi,\tilde{\mathcal{B}}_S\tilde D\rho\rangle+\langle(\frac{1}{24}E_{;m}+\frac{1}{12}EL_{aa}+\frac{1}{48}L_{ab}L_{ab}L_{cc}+\frac{1}{24}L_{ab}L_{ac}L_{bc}$
$-\frac{1}{48}R_{ambm}L_{ab}+\frac{1}{48}R_{abcb}L_{ac}-\frac{1}{24}R_{amam}L_{bb}+\frac{1}{96}\tau_{;m}+\frac{1}{48}L_{ab:ab}+\frac{1}{12}SL_{aa}L_{bb}$
$+\frac{1}{12}SL_{ab}L_{ab}-\frac{1}{12}SR_{amam}+\frac{1}{12}(SE+ES)+\frac{1}{4}S^2L_{aa}+\frac{1}{6}S^3$
$+\frac{1}{6}S_{:aa})\mathcal{B}_S\phi,\tilde{\mathcal{B}}_S\rho\rangle-\frac{1}{12}L_{aa}\langle(\mathcal{B}_S\phi)_{:b},(\tilde{\mathcal{B}}_S\rho)_{:b}\rangle-\frac{1}{12}L_{ab}\langle(\mathcal{B}_S\phi)_{:a},(\tilde{\mathcal{B}}_S\rho)_{:b}\rangle$
$-\frac{1}{6}\langle S(\mathcal{B}_S\phi)_{:a},(\tilde{\mathcal{B}}_S\rho)_{:a}\rangle-\frac{1}{24}\langle\Omega_{am}(\mathcal{B}_S\phi)_{:a},\tilde{\mathcal{B}}_S\rho\rangle$
$+\frac{1}{24}\langle\Omega_{am}\mathcal{B}_S\phi,(\tilde{\mathcal{B}}_S\rho)_{:a}\rangle\}dy$.

We now turn our attention to mixed boundary conditions. We use Equation (5.d) to defined the mixed boundary operator $\mathcal{B}_{\chi,S}$. The dual boundary operator on $\tilde V$ is given by $\tilde{\mathcal{B}}_{\chi,S}\rho:=\tilde\Pi_+(\rho_{;m}+\tilde S\rho)|_{\partial M}\oplus\tilde\Pi_-\rho|_{\partial M}$. Extend χ to a collared neighborhood of M to be parallel along the inward normal geodesic rays. Then $\chi_{;m}=0$. Let $\phi_\pm:=\Pi_\pm\phi$ and $\rho_\pm:=\Pi_\pm\rho$. Since $\chi_{;m}=0$, $\phi_{\pm;m}=\Pi_\pm(\phi_{;m})$ and $\rho_{\pm;m}=\tilde\Pi_\pm(\phi_{;m})$. As $\chi_{:a}$ need not vanish in general, we need not have equality between $\phi_{\pm:a}$ and $\Pi_\pm(\phi_{:a})$ or between $\rho_{\pm:a}$ and $\tilde\Pi_\pm(\rho_{:a})$. We refer to [45] for the proof of:

Theorem 7.6 *[Mixed boundary conditions]*

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_{\chi,S})=\int_M\langle\phi,\rho\rangle dx$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_{\chi,S})=-\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\phi_-,\rho_-\rangle dy$.

(3) $\beta_2(\phi,\rho,D,\mathcal{B}_{\chi,S})=-\int_M\langle D\phi,\rho\rangle dx+\int_{\partial M}\{\langle\phi_{+;m}+S\phi_+,\rho_+\rangle$
$+\langle\frac{1}{2}L_{aa}\phi_-,\rho_-\rangle-\langle\phi_-,\rho_{-;m}\rangle\}dy$.

(4) $\beta_3(\phi,\rho,D,\mathcal{B}_{\chi,S})=\frac{2}{\sqrt{\pi}}\int_{\partial M}\{-\frac{2}{3}\langle\phi_{-;mm},\rho_-\rangle-\frac{2}{3}\langle\phi_-,\rho_{-;mm}\rangle+\frac{2}{3}L_{aa}\langle\phi_-,\rho_-\rangle_{;m}$
$+\langle(-\frac{1}{12}L_{aa}L_{bb}+\frac{1}{6}L_{ab}L_{ab}+\frac{1}{6}R_{amma})\phi_-,\rho_-\rangle+\frac{2}{3}\langle\phi_{+;m}+S\phi_+,\rho_{+;m}+\tilde S\rho_+\rangle$
$-\langle E\phi_-,\rho_-\rangle+\langle\phi_{-:a},\rho_{-:a}\rangle+\frac{2}{3}\langle\phi_{+:a},\rho_{-:a}\rangle+\frac{2}{3}\langle\phi_{-:a},\rho_{+:a}\rangle$
$-\frac{2}{3}\langle E\phi_-,\rho_+\rangle-\frac{2}{3}\langle E\phi_+,\rho_-\rangle\}dy$.

We adopt the notation of Equation (5.e) to define the transmission boundary operator $\mathcal{B}_U$ and the tensor ω.

Theorem 7.7 *[Transmission boundary conditions]*

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_U)=\int_{M_+}\langle\phi_+,\rho_+\rangle dx_+ +\int_{M_-}\langle\phi_-,\rho_-\rangle dx_-$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_U)=-\frac{1}{\sqrt{\pi}}\int_\Sigma\langle\phi_+-\phi_-,\rho_+-\rho_-\rangle dy$.

(3) $$\begin{aligned}&\beta_2(\phi,\rho,D,\mathcal{B}_U)=-\int_{M_+}\langle D_+\phi_+,\rho_+\rangle dx_+ -\int_{M_-}\langle D_-\phi_-,\rho_-\rangle dx_-\\
&+\int_\Sigma\big\{\tfrac18(L^+_{aa}+L^-_{aa})(\langle\phi_+,\rho_+\rangle+\langle\phi_-,\rho_-\rangle)\\
&-\tfrac18(L^+_{aa}+L^-_{aa})(\langle\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)+\tfrac12(\langle\phi_{+;\nu},\rho_+\rangle+\langle\phi_{-;\nu}\rho_-\rangle+\langle\phi_{+;\nu},\rho_-\rangle\\
&+\langle\phi_{-;\nu},\rho_+\rangle)-\tfrac12(\langle\phi_+,\rho_{+;\nu}\rangle+\langle\phi_-,\rho_{-;\nu}\rangle)+\tfrac12(\langle\phi_+,\rho_{-;\nu}\rangle+\langle\phi_-,\rho_{+;\nu}\rangle)\\
&-\tfrac14(\langle U\phi_+,\rho_+\rangle+\langle U\phi_-,\rho_-\rangle+\langle U\phi_+,\rho_-\rangle+\langle U\phi_-,\rho_+\rangle)\big\}dy.\end{aligned}$$

(4) $$\begin{aligned}&\beta_3(\phi,\rho,D,\mathcal{B}_U)=\tfrac{1}{6\sqrt{\pi}}\int_\Sigma\{4(\langle D_+\phi_+,\rho_+\rangle+\langle\phi_+,\tilde D_+\rho_+\rangle+\langle D_-\phi_-,\rho_-\rangle\\
&+\langle\phi_-,\tilde D_-\rho_-\rangle)-4(\langle D_+\phi_+,\rho_-\rangle+\langle\phi_+,\tilde D_-\rho_-\rangle+\langle D_-\phi_-,\rho_+\rangle+\langle\phi_-,\tilde D_+\rho_+\rangle)\\
&-(\langle\omega_a\phi_{+;a},\rho_+\rangle-\langle\omega_a\phi_{-;a},\rho_-\rangle-\langle\omega_a\phi_+,\rho_{+;a}\rangle+\langle\omega_a\phi_-,\rho_{-;a}\rangle)\\
&-(\langle\omega_a\phi_{+;a},\rho_-\rangle-\langle\omega_a\phi_{-;a},\rho_+\rangle+\langle\omega_a\phi_+,\rho_{-;a}\rangle-\langle\omega_a\phi_-,\rho_{+;a}\rangle)\\
&+4(\langle\phi_{+;\nu},\rho_{+;\nu}\rangle+\langle\phi_{-;\nu},\rho_{-;\nu}\rangle+\langle\phi_{+;\nu},\rho_{-;\nu}\rangle+\langle\phi_{-;\nu},\rho_{+;\nu}\rangle)-2(\langle\phi_{+;a},\rho_{+;a}\rangle\\
&+\langle\phi_{-;a},\rho_{-;a}\rangle)+2(\langle\phi_{+;a},\rho_{-;a}\rangle+\langle\phi_{-;a},\rho_{+;a}\rangle)-2(\langle U\phi_{+;\nu},\rho_+\rangle\\
&+\langle U\phi_+,\rho_{+;\nu}\rangle+\langle U\phi_{-;\nu},\rho_-\rangle+\langle U\phi_-,\rho_{-;\nu}\rangle)-2(\langle U\phi_{-;\nu},\rho_+\rangle+\langle U\phi_-,\rho_{+;\nu}\rangle\\
&+\langle U\phi_{+;\nu},\rho_-\rangle+\langle U\phi_+,\rho_{-;\nu}\rangle)+(L^-_{aa}-L^+_{aa})(\nu_+\langle\phi_+,\rho_+\rangle-\nu_-\langle\phi_-,\rho_-\rangle)\\
&+L^+_{aa}(\langle\phi_{+;\nu},\rho_-\rangle+\langle\phi_-,\rho_{+;\nu}\rangle)+L^-_{aa}(\langle\phi_{-;\nu},\rho_+\rangle+\langle\phi_+,\rho_{-;\nu}\rangle)\\
&-(L^-_{aa}(\langle\phi_{+;\nu},\rho_-\rangle+\langle\phi_-,\rho_{+;\nu}\rangle)+L^+_{aa}(\langle\phi_{-;\nu},\rho_+\rangle+\langle\phi_+,\rho_{-;\nu}\rangle))\\
&+\langle\omega_a\omega_a\phi_+,\rho_+\rangle+\langle\omega_a\omega_a\phi_-,\rho_-\rangle-\tfrac12L^+_{aa}L^-_{bb}(\langle\phi_+,\rho_+\rangle+\langle\phi_-,\rho_-\rangle)\\
&+\tfrac12L^+_{aa}L^-_{bb}(\langle\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)+\tfrac12(L^+_{ab}L^+_{ab}\langle\phi_+,\rho_+\rangle+L^-_{ab}L^-_{ab}\langle\phi_-,\rho_-\rangle)\\
&+\tfrac12(L^-_{ab}L^-_{ab}\langle\phi_+,\rho_+\rangle+L^+_{ab}L^+_{ab}\langle\phi_-,\rho_-\rangle)-\tfrac12(L^+_{ab}L^+_{ab}+L^-_{ab}L^-_{ab})(\langle\phi_+,\rho_-\rangle\\
&+\langle\phi_-,\rho_+\rangle)+L^+_{aa}\langle U\phi_+,\rho_+\rangle+L^-_{aa}\langle U\phi_-,\rho_-\rangle-L^-_{aa}\langle U\phi_+,\rho_+\rangle\\
&-L^+_{aa}\langle U\phi_-,\rho_-\rangle+\langle U^2\phi_+,\rho_+\rangle+\langle U^2\phi_-,\rho_-\rangle+\langle U^2\phi_+,\rho_-\rangle+\langle U^2\phi_-,\rho_+\rangle\\
&+\langle E_+\phi_+,\rho_+\rangle+\langle E_-\phi_-,\rho_-\rangle+\langle E_-\phi_+,\rho_+\rangle+\langle E_+\phi_-,\rho_-\rangle\\
&-\langle(E_++E_-)\phi_+,\rho_-\rangle-\langle(E_++E_-)\phi_-,\rho_+\rangle+\tfrac12(R^+_{amma}+R^-_{amma})(\langle\phi_+,\rho_+\rangle\\
&+\langle\phi_-,\rho_-\rangle)-\tfrac12(R^+_{amma}+R^-_{amma})(\langle\phi_+,\rho_-\rangle+\langle\phi_-,\rho_+\rangle)\}dy.\end{aligned}$$

We continue our studies by examining the heat content asymptotics for transfer boundary conditions Adopt the Equation (5.f) to define the transfer boundary operator $\mathcal{B}_{\mathfrak{S}}$. Let $\tilde{\mathcal{B}_{\mathfrak{S}}}$ be the dual boundary operator

$$\tilde{\mathcal{B}_{\mathfrak{S}}}\rho:=\left\{\left(\begin{array}{cc}\tilde\nabla^+_{\nu_+}+\tilde S_{++} & \tilde S_{-+}\\ \tilde S_{+-} & \tilde\nabla^-_{\nu_-}+\tilde S_{--}\end{array}\right)\left(\begin{array}{c}\rho_+\\ \rho_-\end{array}\right)\right\}\Big|_\Sigma.$$

We refer to [57] for the proof of the following result:

Theorem 7.8 *[Transfer boundary conditions]*

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_{\mathfrak{S}})=\int_{M_+}\langle\phi_+,\rho_+\rangle dx_+ +\int_{M_-}\langle\phi_-,\rho_-\rangle dx_-$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_{\mathfrak{S}})=0$.

(3) $\beta_2(\phi,\rho,D,\mathcal{B}_{\mathfrak{S}})=-\int_{M_+}\langle D_+\phi_+,\rho_+\rangle dx_+ -\int_{M_-}\langle D_-\phi_-,\rho_-\rangle dx_-$
$+\int_\Sigma\langle\mathcal{B}_{\mathfrak{S}}\phi,\rho\rangle dy$.

(4) $\beta_3(\phi,\rho,D,\mathcal{B}_{\mathfrak{S}})=\frac{4}{3\sqrt{\pi}}\int_\Sigma\langle\mathcal{B}_{\mathfrak{S}}\phi,\tilde{\mathcal{B}_{\mathfrak{S}}}\rho\rangle)dy$.

Oblique boundary conditions are of particular interest. Let D be an operator of Laplace type on a bundle V over M. Let $\mathcal{B}_T$ be a tangential first order partial differential operator on $V|_{\partial M}$ and let $\tilde{B}_T$ be the dual operator on $\tilde{V}|_{\partial M}$. The associated *oblique boundary conditions* on V and dual boundary conditions on $\tilde{V}$ are defined by:

$$\mathcal{B}_{\mathcal{O}}\phi:=(\phi_{;m}+\mathcal{B}_T\phi)|_{\partial M}\quad\text{and}\quad\tilde{\mathcal{B}}_{\mathcal{O}}\rho:=(\rho_{;m}+\tilde{\mathcal{B}}_T\rho)|_{\partial M}\,.$$

Note that we recover Robin boundary conditions by taking $\mathcal{B}_T$ to be a 0^{th} order operator. We refer to [59] for the proof of the following result:

Theorem 7.9 *[Oblique boundary conditions]*

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_{\mathcal{O}})=\int_M\langle\phi,\rho\rangle dx$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_{\mathcal{O}})=0$.

(3) $\beta_2(\phi,\rho,D,\mathcal{B}_{\mathcal{O}})=-\int_M\langle D\phi,\rho\rangle dx+\int_{\partial M}\langle\mathcal{B}_{\mathcal{O}}\phi,\rho\rangle dy$.

(4) $\beta_3(\phi,\rho,D,\mathcal{B}_{\mathcal{O}})=\frac{4}{3\sqrt{\pi}}\int_{\partial M}\langle\mathcal{B}_{\mathcal{O}}\phi,\tilde{\mathcal{B}}_{\mathcal{O}}\rho\rangle dy$.

(5) $\beta_4(\phi,\rho,D,\mathcal{B}_{\mathcal{O}})=\frac12\int_M\langle D\phi,\tilde{D}\rho\rangle dx+\int_{\partial M}\{-\frac12\langle\mathcal{B}_{\mathcal{O}}\phi,\tilde{D}\rho\rangle$
$-\frac12\langle D\phi,\tilde{\mathcal{B}}\rho\rangle+\langle(\frac12\mathcal{B}_T+\frac14 L_{aa})\mathcal{B}_{\mathcal{O}}\phi,\tilde{\mathcal{B}}_{\mathcal{O}}\rho\rangle\}dy$.

We refer to [24] for further details concerning *Zaremba boundary conditions*. We assume given a decomposition $\partial M=C_R\cup C_D$ as the union of two closed submanifolds with common smooth boundary $C_R\cap C_D=\Sigma$. Let $\phi_{;m}$ denote the covariant derivative of ϕ with respect to the inward unit normal on ∂M. Let S be an auxiliary endomorphism of $V|_{C_R}$. We take Robin boundary conditions on C_R and Dirichlet boundary conditions on C_D arising from the boundary operator:

$$\mathcal{B}_Z\phi:=(\phi_{;m}+S\phi)|_{\{C_R-\Sigma\}}\oplus\phi|_{C_D}\,.$$

We refer to related work of Avramidi [11], of Dowker [46, 47], and of Jakobson et al. [80] concerning the heat trace asymptotics.

There is some additional technical fuss concerned with choosing a boundary condition on the interface $C_D\cap C_R$ that we will suppress in the interests of brevity. Instead, we shall simply give a classical formulation of the problem. Suppose $D=\Delta$ is the Laplacian and that $S=0$. Let $W^{1,2}(M)$ be the closure of $C^\infty(M)$ with respect to the Sobolev norm

$$||\phi||_1^2=\int_M\{|\nabla\phi|^2+|\phi|^2\}dx.$$

Let $W^{1,2}_{0,C_D}(M)$ be the closure of the set $\{\phi \in W^{1,2}(M) : \text{supp}(\phi) \cap C_D = \emptyset\}$. Let

$$N(M, C_D, \lambda) = \sup(\dim E_\lambda) \quad \text{for} \quad \lambda > 0$$

where the supremum is taken over all subspaces $E_\lambda \subset W^{1,2}_{0,C_D}(M)$ such that

$$||\nabla\phi||_{L^2(M)} < \lambda||\phi||_{L^2(M)}, \quad \forall\phi \in E_\lambda\,.$$

This is the spectral counting function for the Zaremba problem described above.

On Σ, we choose an orthonormal frame so e_m is the inward unit normal of ∂M in M and so that e_{m-1} is the inward unit normal of Σ in C_D.

Theorem 7.10 *[Zaremba boundary conditions] There exist universal constants c_1 and c_2 so that:*

(1) $\beta_0(\phi, \rho, D, \mathcal{B}) = \int_M \langle\phi, \rho\rangle dx$.

(2) $\beta_1(\phi, \rho, D, \mathcal{B}) = -\frac{2}{\sqrt{\pi}} \int_{C_D} \langle\phi, \rho\rangle dy$.

(3) $\beta_2(\phi, \rho, D, \mathcal{B}) = -\int_M \langle D\phi, \rho\rangle dx + \int_{C_R}\{\langle\phi_{;m} + S\phi, \rho\rangle\}dy$
$+\int_{C_D}\{\frac{1}{2}L_{aa}\langle\phi, \rho\rangle - \langle\phi, \rho_{;m}\rangle\}dy - \frac{1}{2}\int_\Sigma \langle\phi, \rho\rangle dz$.

(4) $\beta_3(\phi, \rho, D, \mathcal{B}) = \frac{4}{3\sqrt{\pi}} \int_{C_R} \langle\phi_{;m} + S\phi, \rho_{;m} + \tilde{S}\rho\rangle dy - \frac{2}{\sqrt{\pi}} \int_{C_D}\{\frac{2}{3}\langle\phi_{;mm}, \rho\rangle$
$+\frac{2}{3}\langle\phi, \rho_{;mm}\rangle - \langle\phi_{:a}, \rho_{:a}\rangle + \langle E\phi, \rho\rangle - \frac{2}{3}L_{aa}\langle\phi, \rho\rangle_{;m} + \langle(\frac{1}{12}L_{aa}L_{bb}$
$-\frac{1}{6}L_{ab}L_{ab} + \frac{1}{6}R_{amam})\phi, \rho\rangle\}dy + \int_\Sigma\{\langle(c_1 L_{m-1,m-1} + (\frac{1}{2}c_2 + \frac{2}{3\sqrt{\pi}})L_{uu}$
$+\frac{1}{2\sqrt{\pi}}\tilde{L}_{uu} + c_2 S)\phi, \rho\rangle + \frac{1}{2\sqrt{\pi}}\langle\phi, \rho\rangle_{;m-1} - \frac{2}{3\sqrt{\pi}}\langle\phi, \rho\rangle_{;m}\}dz$.

We conclude this section with a brief description of the non-smooth setting. We refer to van den Berg and Srisatkunarajah [25] for a discussion of the heat content asymptotics of polygonal regions in the plane. The fractal setting also an important one and we refer to van den Berg [15], to Fleckinger et al. [51], to Griffith and Lapidus [70], to Lapidus and Pang [85], and to Neuberger et al. [100] for a discussion of some asymptotic results for heat problems on the von Koch snowflake.

8 Heat content with source terms

We follow the discussion in [18, 22, 23, 56] throughout this section. Let D be an operator of Laplace type. Assume $\partial M = C_D \cup C_R$ decomposes as a disjoint union of two closed, possibly empty, disjoint subsets; in contrast to the Zaremba problem, we emphasize that $C_D \cap C_R$ is empty. Let $\mathcal{B}$ be the Dirichlet boundary operator on C_D and the Robin boundary operator on C_R. Let ϕ be the initial temperature of the manifold, let $\rho = \rho(x;t)$ be a variable specific heat, let $p = p(x;t)$ be an auxiliary smooth internal heat source and let $\psi = \psi(y;t)$ be the temperature of the boundary. We assume, for the sake of simplicity, that the underlying geometry is fixed. Let $u(x;t) = u_{\phi,p,\psi}(x;t)$ be the subsequent temperature distribution which is defined by the relations:

$$(\partial_t + D)u(x;t) = p(x;t) \quad \text{for} \quad t > 0,$$

$$\mathcal{B}u(y;t) = \psi(y;t) \quad \text{for} \quad t>0, y \in \partial M,$$
$$\lim_{t\downarrow 0} u(\cdot;t) = \phi(\cdot) \text{ in } L^2 .$$

The associated heat content function has a complete asymptotic series as $t \downarrow 0$:

$$\begin{aligned}\beta(\phi,\rho,D,\mathcal{B},p,\psi)(t) : &= \int_M \langle u_{\phi,p,\psi}(x;t),\rho(x;t)\rangle dx \\ &\sim \sum_{n=0}^{\infty} \beta_n(\phi,\rho,D,\mathcal{B},p,\psi) t^{n/2} .\end{aligned}$$

Assertions (1)-(4) in the following result are valid for quite general boundary conditions. Assertion (5) refers to the particular problem under consideration. This result when combined with the results of Theorems 7.3 and 7.4 permits evaluation of this invariant for $n \le 4$. Assertion (1) reduces to the case ρ is static and Assertion (2) decouples the invariants as a sum of 3 different invariants. Assertion (3) evaluates the invariant which is independent of $\{p,\psi\}$, Assertion (4) evaluates invariant which depends on p, and Assertion (5) evaluates the invariant which depends on ψ.

Theorem 8.1 (1) *Expand the specific heat $\rho(x;t) \sim \sum_{k\ge 0} t^k \rho_k(x)$ in a Taylor series. Then $\beta_n(\phi,\rho,D,\mathcal{B},p,\psi) = \sum_{2k\le n} \beta_{n-2k}(\phi,\rho_k,D,\mathcal{B},p,\psi)$.*

(2) *If the specific heat ρ is static, then $\beta_n(\phi,\rho,D,\mathcal{B},p,\psi) = \beta_n(\phi,\rho,D,\mathcal{B},0,0)$ $+\beta_n(0,\rho,D,\mathcal{B},p,0) + \beta_n(0,\rho,D,\mathcal{B},0,\psi)$.*

(3) *If the specific heat ρ is static, then $\beta_n(\phi,\rho,D,\mathcal{B},0,0) = \beta_n(\phi,\rho,D,\mathcal{B})$.*

(4) *Let $c_{ij} := \int_0^1 (1-s)^i s^{j/2} ds$. Expand $p(x;t) \sim \sum_{k\ge 0} t^k p_k(x)$ in a Taylor series. If the specific heat is static, then:*

a) $\beta_0(0,\rho,D,\mathcal{B},p,0) = 0$.

b) If $n>0$, then $\beta_n(0,\rho,D,\mathcal{B},p,0) = \sum_{2i+j+2=n} c_{ij}\beta_j(p_i,\rho,D,\mathcal{B})$.

(5) *Expand the boundary source term $\psi(x,t) \sim \sum_{k\ge 0} t^k \psi_k(x)$ in a Taylor series. Assume the specific heat ρ is static. Then:*

a) $\beta_0(0,\rho,D,\mathcal{B},0,\psi) = 0$.

b) $\beta_1(0,\rho,D,\mathcal{B},0,\psi) = \frac{2}{\sqrt{\pi}}\int_{C_D} \langle \psi_0,\rho\rangle dy$.

c) $\beta_2(0,\rho,D,\mathcal{B},0,\psi) = -\int_{C_D}\{\langle \frac12 L_{aa}\psi_0,\rho\rangle - \langle \psi_0,\rho_{;m}\rangle\}dy - \int_{C_R}\langle\psi_0,\rho\rangle dy$.

d) $\beta_3(0,\rho,D,\mathcal{B},0,\psi) = \frac{2}{\sqrt{\pi}}\int_{C_D}\{\frac23\langle\psi_0,\rho_{;mm}\rangle + \frac13\langle\psi_0,\rho_{:aa}\rangle + \langle\frac13 E\psi,\rho\rangle$
$-\frac23 L_{aa}\langle\psi_0,\rho_{;m}\rangle + \langle(\frac{1}{12}L_{aa}L_{bb} - \frac16 L_{ab}L_{ab} - \frac16 R_{amma})\psi_0,\rho\rangle\}dy$
$-\frac{4}{3\sqrt{\pi}}\int_{C_R}\langle\psi_0,\tilde{\mathcal{B}}\rho\rangle dy + \frac{4}{3\sqrt{\pi}}\int_{C_D}\langle\psi_1,\rho\rangle dy$.

e) $\beta_4(0,\rho,D,\mathcal{B},0,\psi) = -\int_{C_D}\{\frac12\langle\psi_0,(\tilde{D}\rho)_{;m}\rangle - \frac14\langle L_{aa}\psi_0,\tilde{D}\rho\rangle + \langle(\frac18 E_{;m}$
$-\frac{1}{16}L_{ab}L_{ab}L_{cc} + \frac18 L_{ab}L_{ac}L_{bc} - \frac{1}{16}R_{ambm}L_{ab} + \frac{1}{16}R_{abcb}L_{ac}$
$+\frac{1}{32}\tau_{;m} + \frac{1}{16}L_{ab:ab})\psi_0,\rho\rangle - \frac14 L_{ab}\langle\psi_{0:a},\rho_{:b}\rangle - \frac18\langle\Omega_{am}\psi_{0:a},\rho\rangle$
$+\frac18\langle\Omega_{am}\psi_0,\rho_{:a}\rangle + \frac14 L_{aa}\langle\psi_1,\rho\rangle - \frac12\langle\psi_1,\rho_{;m}\rangle\}dy$
$-\int_{C_R}\{-\frac12\langle\psi_0,\tilde{D}\rho\rangle + \langle(\frac12 S + \frac14 L_{aa})\psi_0,\tilde{\mathcal{B}}\rho\rangle + \frac12\langle\psi_1,\rho\rangle\}dy$.

9 Time dependent phenomena

We refer to [56] for proofs of the assertions in this section and also for a more complete historical discussion. Let $\mathfrak{D} = \{D_t\}$ be a time-dependent family of operators of Laplace type. We expand $\mathfrak{D}$ in a Taylor series expansion

$$D_t u := Du + \sum_{r=1}^{\infty} t^r \left\{ \mathcal{G}_{r,ij} u_{;ij} + \mathcal{F}_{r,i} u_{;i} + \mathcal{E}_r u \right\}.$$

We use the initial operator $D := D_0$ to define a reference metric g_0. Choose local frames $\{e_i\}$ for the tangent bundle of M and local frames $\{e_a\}$ for the tangent bundle of the boundary which are orthonormal with respect to the initial metric g_0. Use g_0 to define the measures dx on M and dy on ∂M. The metric g_0 defines the curvature tensor R and the second fundamental form L. We also use D to define a background connection ∇_0 that we use to multiply covariantly differentiate tensors of all types and to define the endomorphism E.

As in Section 8, we again assume $\partial M = C_D \cup C_R$ decomposes as a disjoint union of two closed, possibly empty, disjoint subsets. We consider a 1 parameter family $\mathfrak{B} = \{\mathcal{B}_t\}$ of boundary operators which we expand formally in a Taylor series

$$\mathcal{B}_t \phi := \phi \Big|_{C_D} \oplus \left\{ \phi_{;m} + S\phi + \sum_{r>0} t^r (\Gamma_{r,a} \phi_{;a} + S_r \phi) \right\} \Big|_{C_R}.$$

The reason for including a dependence on time in the boundary condition comes, for example, by considering the dynamical Casimir effect. Slowly moving boundaries give rise to such boundary conditions. We let u be the solution of the time-dependent heat equation

$$(\partial_t + D_t) u = 0, \quad \mathcal{B}_t u = 0, \quad \lim_{t \downarrow 0} u(\cdot;t) = \phi(\cdot) \text{ in } L^2.$$

There is a smooth kernel function so that $u(x;t) = \int_M K(t,x,\bar{x},\mathfrak{D},\mathfrak{B}) \phi(\bar{x}) d\bar{x}$. The analogue of the heat trace expansion in this setting and of the heat content asymptotic expansion are given, respectively, by

$$\int_M f(x) \operatorname{Tr}_{V_x} \left\{ K(t,x,x,\mathfrak{D},\mathfrak{B}) \right\} dx \sim \sum_{n=0}^{\infty} a_n(f,\mathfrak{D},\mathfrak{B}) t^{(n-m)/2},$$

$$\int_M \langle K(t,x,\bar{x},\mathfrak{D},\mathfrak{B}) \phi(x), \rho(\bar{x}) \rangle dx d\bar{x} \sim \sum_{n=0}^{\infty} \beta_n(\phi,\rho,\mathfrak{D},\mathfrak{B}) t^{n/2}.$$

By assumption, the operators $\mathcal{G}_{r,ij}$ are scalar. The following theorem describes the additional terms in the heat trace asymptotics which arise from the structures described by $\mathcal{G}_{r,ij}$, $\mathcal{F}_{r,i}$, $\mathcal{E}_r$, $\Gamma_{r,a}$, and S_r given above.

Theorem 9.1 *[Varying geometries]*

(1) $a_0(F,\mathfrak{D},\mathfrak{B}) = a_0(F,D,\mathcal{B})$.

(2) $a_1(F,\mathfrak{D},\mathfrak{B}) = a_1(F,D,\mathcal{B})$.

(3) $a_2(F,\mathfrak{D},\mathfrak{B}) = a_2(F,D,\mathcal{B}) + (4\pi)^{-m/2} \frac{1}{6} \int_M \operatorname{Tr}\{\frac{3}{2} F \mathcal{G}_{1,ii}\} dx$.

(4) $a_3(F,\mathfrak{D},\mathfrak{B}) = a_3(F,D,\mathcal{B}) + (4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_D}\mathrm{Tr}\{-24F\mathcal{G}_{1,aa}\}dy$
$+(4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_R}\mathrm{Tr}\{24F\mathcal{G}_{1,aa}\}dy.$

(5) $a_4(F,\mathfrak{D},\mathfrak{B}) = a_4(F,D,\mathcal{B}) + (4\pi)^{-m/2}\frac{1}{360}\int_M \mathrm{Tr}\{F(\frac{45}{4}\mathcal{G}_{1,ii}\mathcal{G}_{1,jj} + \frac{45}{2}\mathcal{G}_{1,ij}\mathcal{G}_{1,ij}$
$+60\mathcal{G}_{2,ii} - 180\mathcal{E}_1 + 15\mathcal{G}_{1,ii}\tau - 30\mathcal{G}_{1,ij}\rho_{ij} + 90\mathcal{G}_{1,ii}E + 60\mathcal{F}_{1,i;i} + 15\mathcal{G}_{1,ii;jj}$
$-30\mathcal{G}_{1,ij;ij})\}dx + (4\pi)^{-m/2}\frac{1}{360}\int_{C_D}\mathrm{Tr}\{f(30\mathcal{G}_{1,aa}L_{bb} - 60\mathcal{G}_{1,mm}L_{bb}$
$+30\mathcal{G}_{1,ab}L_{ab} + 30\mathcal{G}_{1,mm;m} - 30\mathcal{G}_{1,aa;m} - 30\mathcal{F}_{1,m}) + F_{;m}(-45\mathcal{G}_{1,aa}$
$+45\mathcal{G}_{1,mm})\}dy + (4\pi)^{-m/2}\frac{1}{360}\int_{C_R}\mathrm{Tr}\{F(30\mathcal{G}_{1,aa}L_{bb} + 120\mathcal{G}_{1,mm}L_{bb}$
$-150\mathcal{G}_{1,ab}L_{ab} - 60\mathcal{G}_{1,mm;m} + 60\mathcal{G}_{1,aa;m} + 150\mathcal{F}_{1,m} + 180S\mathcal{G}_{1,aa} - 180S\mathcal{G}_{1,mm}$
$+360S_1) + F_{;m}(45\mathcal{G}_{1,aa} - 45\mathcal{G}_{1,mm})\}dy.$

Next we study the heat content asymptotics for variable geometries. We have the following formulas for Dirichlet and for Robin boundary conditions. Let $\mathcal{B} := \mathfrak{B}_0$.

Theorem 9.2 *[Dirichlet boundary conditions]*

(1) $\beta_n(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_n(\phi,\rho,D_0,\mathcal{B})$ *for* $n = 0,1,2$.

(2) $\beta_3(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_3(\phi,\rho,D_0,\mathcal{B}) + \frac{1}{2\sqrt{\pi}}\int_{C_D}\langle\mathcal{G}_{1,mm}\phi,\rho\rangle dy.$

(3) $\beta_4(\phi,\rho,\mathfrak{D},\mathcal{B}) = \beta_4(\phi,\rho,D_0,\mathcal{B}) - \frac{1}{2}\int_M\langle\mathcal{G}_{1,ij}\phi_{;ij} + \mathcal{F}_{1,i}\phi_{;i} + \mathcal{E}_1\phi,\rho\rangle dx$
$+\int_{C_D}\{\frac{7}{16}\langle\mathcal{G}_{1,mm;m}\phi,\rho\rangle - \frac{9}{16}L_{aa}\langle\mathcal{G}_{1,mm}\phi,\rho\rangle - \frac{5}{16}\langle\mathcal{F}_{1,m}\phi,\rho\rangle$
$+\frac{5}{16}L_{ab}\langle\mathcal{G}_{1,ab}\phi,\rho\rangle - \frac{5}{8}\langle\mathcal{G}_{1,am}\phi_{:a},\rho\rangle + \frac{1}{2}\langle\mathcal{G}_{1,mm}\phi,\rho_{;m}\rangle\}dy$
$+\int_{C_R}\{-\frac{1}{2}\langle\mathcal{G}_{1,mm}\mathcal{B}_0\phi,\rho\rangle + \frac{1}{2}\langle(S_1 + \Gamma_a\nabla_{e_a})\phi,\rho\rangle\}dy.$

10 Spectral boundary conditions

We adopt the notation used to discuss spectral boundary conditions in Section 6. Let $P : C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic complex of Dirac type. Let $D = P^*P$ and let $\mathcal{B}_\Theta$ be the spectral boundary conditions defined by the auxiliary self-adjoint endomorphism Θ of V_1. Let ∇ be a compatible connection. Expand $P = \gamma_i\nabla_{e_i} + \psi$.

We begin by studying the heat trace asymptotics with spectral boundary conditions. There is an asymptotic series

$$\mathrm{Tr}_{L^2}(fe^{-tD_{\mathcal{B}_\Theta}}) \sim \sum_{k=0}^{m-1} a_k(f, D_{\mathcal{B}_\Theta}, \mathcal{B}_\Theta)t^{(k-m)/2} + O(t^{-1/8}) \,.$$

Continuing further introduces non-local terms; we refer to Atiyah et al. [6], to Grubb [71, 72], and to Grubb and Seeley [74, 75] for further details. Define $\gamma_a^T := \gamma_m^{-1}\gamma_a$, $\hat\psi := \gamma_m^{-1}\psi$, and $\beta(m) := \Gamma(\frac{m}{2})\Gamma(\frac{1}{2})^{-1}\Gamma(\frac{m+1}{2})^{-1}$. We refer to [48] for the proof of the following result:

Theorem 10.1 *[Spectral boundary conditions] Let* $f \in C^\infty(M)$. *Then:*

(1) $a_0(f,D,\mathcal{B}_\Theta) = (4\pi)^{-m/2}\int_M \mathrm{Tr}(f\,\mathrm{id})dx.$

(2) *If* $m \geq 2$*, then* $a_1(f, D, \mathcal{B}_\Theta) = \frac{1}{4}[\beta(m) - 1](4\pi)^{-(m-1)/2} \int_{\partial M} \operatorname{Tr}(f \operatorname{id}) dy$.

(3) *If* $m \geq 3$*, then* $a_2(f, D, \mathcal{B}_\Theta) = (4\pi)^{-m/2} \int_M \frac{1}{6} \operatorname{Tr}\{f(\tau \operatorname{id} + 6E)\} dx$
$+(4\pi)^{-m/2} \int_{\partial M} \operatorname{Tr}\{\frac{1}{2}[\hat{\psi} + \hat{\psi}^*]f + \frac{1}{3}[1 - \frac{3}{4}\pi\beta(m)]L_{aa} f \operatorname{id}$
$-\frac{m-1}{2(m-2)}[1 - \frac{1}{2}\pi\beta(m)] f_{;m} \operatorname{id}\} dy$.

(4) *If* $m \geq 4$*, then* $a_3(f, D, \mathcal{B}_\Theta) = (4\pi)^{-(m-1)/2} \int_{\partial M} \operatorname{Tr}\{\frac{1}{32}(1 - \frac{\beta(m)}{m-2}) f(\hat{\psi}\hat{\psi} + \hat{\psi}^* \hat{\psi}^*)$
$+\frac{1}{16}(5 - 2m + \frac{7-8m+2m^2}{m-2}\beta(m)) f \hat{\psi}\hat{\psi}^* - \frac{1}{48}(\frac{m-1}{m-2}\beta(m) - 1) f \tau \operatorname{id}$
$+\frac{1}{32(m-1)}(2m - 3 - \frac{2m^2-6m+5}{m-2}\beta(m)) f(\gamma_a^T \hat{\psi} \gamma_a^T \hat{\psi} + \gamma_a^T \hat{\psi}^* \gamma_a^T \hat{\psi}^*)$
$+\frac{1}{16(m-1)}(1 + \frac{3-2m}{m-2}\beta(m)) f \gamma_a^T \hat{\psi} \gamma_a^T \hat{\psi}^* + \frac{1}{48}(1 - \frac{4m-10}{m-2}\beta(m)) f \rho_{mm} \operatorname{id}$
$+\frac{1}{48(m+1)}(\frac{17+5m}{4} + \frac{23-2m-4m^2}{m-2}\beta(m)) f L_{ab} L_{ab} \operatorname{id}$
$+\frac{1}{48(m^2-1)}(-\frac{17+7m^2}{8} + \frac{4m^3-11m^2+5m-1}{m-2}\beta(m)) f L_{aa} L_{bb} \operatorname{id}$
$+\frac{1}{8(m-2)}\beta(m) f(\Theta\Theta + \frac{1}{m-1}\gamma_a^T \Theta \gamma_a^T \Theta)\} + \frac{m-1}{16(m-3)}(2\beta(m) - 1) f_{;mm} \operatorname{id}$
$+\frac{1}{8(m-3)}(\frac{5m-7}{8} - \frac{5m-9}{3}\beta(m)) L_{aa} f_{;m} \operatorname{id}\} dy$.

We now study heat content asymptotics with spectral boundary conditions. To simplify the discussion, we suppose P is formally self-adjoint. We refer to [60, 61] for the proof of:

Theorem 10.2 (1) $\beta_0(\phi, \rho, D, \mathcal{B}_\Theta) = \int_M \langle \phi, \rho \rangle dx$.

(2) $\beta_1(\phi, \rho, D, \mathcal{B}_\Theta) = -\frac{2}{\sqrt{\pi}} \int_{\partial M} \langle \Pi_A^+ \phi, \Pi_{A^\#}^+ \rho \rangle dy$.

(3) $\beta_2(\phi, \rho, D, \mathcal{B}_\Theta) = -\int_M \langle D\phi, \rho \rangle dx + \int_{\partial M} \{-\langle \gamma_m \Pi_A^+ P\phi, \rho \rangle - \langle \gamma_m \Pi_A^+ \phi, \tilde{P}\rho \rangle$
$+\frac{1}{2}\langle (L_{aa} + A + \tilde{A}^\# - \gamma_m \psi_P + \psi_P \gamma_m - \psi_A - \tilde{\psi}_A^\#) \Pi_A^+ \phi, \Pi_{A^\#}^+ \rho \rangle\} dy$.

11 Operators which are not of Laplace type

We follow Avramidi and Branson [12], Branson et al. [32], Fulling [52], Gusynin [77], and Ørsted and Pierzchalski [101] to discuss the heat trace asymptotics of *non-minimal operators*. Let M be a compact Riemannian manifold with smooth boundary and let $\mathcal{B}$ define either absolute or relative boundary conditions. Let $E \in C^\infty(\operatorname{End}(\Lambda^p M))$ be an auxiliary endomorphism and let A and B be positive constants. Let

$$D_E^p := Ad\delta + B\delta d - E \quad \text{on} \quad C^\infty(\Lambda^p(M)),$$
$$c_{m,p}(A, B) := B^{-m} + (B^{-m} - A^{-m}) \sum_{k<p} (-1)^{k+p} \binom{m}{p}^{-1} \binom{m}{k}.$$

Theorem 11.1 (1) *If* $E = 0$*, then* $a_n(1, D^p, \mathcal{B}) = B^{(n-m)/2} a_n(1, \Delta_M^p, \mathcal{B})$
$+(B^{(n-m)/2} - A^{(n-m)/2}) \sum_{k<p} (-1)^{k+p} a_n(1, \Delta_M^p, \mathcal{B})$.

(2) *For general* E *one has:*

a) $a_0(1, D_E^p, \mathcal{B}) = a_0(1, D^p, \mathcal{B})$.

b) $a_1(1, D_E^p, \mathcal{B}) = a_1(1, D^p, \mathcal{B})$.

c) $a_2(1, D_E^p, \mathcal{B}) = a_2(1, D^p, \mathcal{B}) + (4\pi)^{-m/2} c_{m,p}(A, B) \int_M \operatorname{Tr}(E) dx$.

We follow the discussion in [56] to study the heat content asymptotics of the non-minimal operator $D := Ad\delta + B\delta d - E$ on $C^\infty(\Lambda^1(M))$. Let ϕ and ρ be smooth 1 forms; expand $\phi = \phi_i e_i$ and $\rho = \rho_i e_i$ where e_m is the inward geodesic normal.

Theorem 11.2 (1) *Let $\mathcal{B}$ define absolute boundary conditions. Then:*

a) $\beta_0(\phi, \rho, D, \mathcal{B}) = \int_M (\phi, \rho) dx$.

b) $\beta_1(\phi, \rho, D, \mathcal{B}) = -\frac{2}{\sqrt{\pi}} \sqrt{A} \int_{\partial M} \phi_m \rho_m dy$.

c)
$$\begin{aligned} \beta_2(\phi, \rho, D, \mathcal{B}) = &- \int_M \{A(\delta\phi, \delta\rho) + B(d\phi, d\rho) - E(\phi, \rho)\} dx \\ &+ \int_{\partial M} A\{-\phi_m \rho_{a:a} - \phi_{a:a}\rho_m - \phi_{m;m}\rho_m - \phi_m \rho_{m;m} \\ &+ \tfrac{3}{2} L_{aa} \phi_m \rho_m\} dy. \end{aligned}$$

(2) *Let $\mathcal{B}$ define relative boundary conditions. Then*

a) $\beta_0(\phi, \rho, D, \mathcal{B}) = \int_M (\phi, \rho) dx$.

b) $\beta_1(\phi, \rho, D, \mathcal{B}) = -\frac{2}{\sqrt{\pi}} \sqrt{B} \int_{\partial M} \phi_a \rho_a dy$.

c)
$$\begin{aligned} \beta_2(\phi, \rho, D, \mathcal{B}) = &- \int_M \{A(\delta\phi, \delta\rho) + B(d\phi, d\rho) - E(\phi, \rho)\} dx \\ &+ \int_{\partial M} B\{-\phi_{a:a}\rho_m - \phi_m \rho_{a:a} - \phi_{a;m}\rho_a - \phi_a \rho_{a;m} \\ &\quad + L_{ab}\phi_b \rho_a + \tfrac{1}{2} L_{aa} \phi_b \rho_b\} dy. \end{aligned}$$

We now turn our attention to fourth order operators. Let M be a closed Riemannian manifold. Let ∇ be a connection on a vector bundle V over a closed Riemannian manifold M. Set

$$\Gamma(\tfrac{m-n}{2})^{-1}\Gamma(\tfrac{m-n}{4}) := \lim_{s \to n} \{\Gamma(\tfrac{m-s}{2})^{-1}\Gamma(\tfrac{m-s}{4})\}\,.$$

Theorem 11.3 *Let $Pu = u_{;iijj} + p_{2,ij} u_{;ij} + p_{1,i} u_{;i} + p_0$ on a closed Riemannian manifold where $p_{2,ij} = p_{2,ji}$ and where $\{p_{2,ij}, p_{1,i}, p_0\}$ are endomorphism valued. . Then:*

(1) $a_0(1, P) = \frac{1}{2}(4\pi)^{-m/2}\Gamma(\frac{m}{2})^{-1}\Gamma(\frac{m}{4}) \int_M \operatorname{Tr}(\mathrm{id}) dx$.

(2) $a_2(1, P) = \frac{1}{2}(4\pi)^{-m/2}\Gamma(\frac{m-2}{2})^{-1}\Gamma(\frac{m-2}{4})\frac{1}{6} \int_M \operatorname{Tr}\{\tau \,\mathrm{id} + \frac{3}{m} p_{2,ii}\} dx$.

(3)
$$\begin{aligned} a_4(1, P) = &\tfrac{1}{2}(4\pi)^{-m/2}\Gamma(\tfrac{m}{2})^{-1}\Gamma(\tfrac{m}{4})\tfrac{1}{360} \int_M \operatorname{Tr}\{\tfrac{90}{m+2} p_{2,ij} p_{2,ij} + \tfrac{45}{m+2} p_{2,ii} p_{2,jj} \\ &+ (m-2)(5\tau^2 \,\mathrm{id} - 2|\rho|^2 \,\mathrm{id} + 2|R|^2 \,\mathrm{id} + 30\Omega_{ij}\Omega_{ij}) + 30\tau p_{2,ii} - 60\rho_{ij} p_{2,ij} \\ &- 360 p_0\} dx. \end{aligned}$$

12 The spectral geometry of Riemannian submersions

We refer to [64] for further details concerning the material of this section; additionally see Bergery and Bourguignon[13], Besson and Bordoni [29], Goldberg and Ishihara [65] and Watson [126]. Let $\pi : Z \to Y$ be a smooth map where Z and Y are connected closed Riemannian manifolds. We say that π is a submersion if π is surjective and if $\pi_* : T_z Z \to T_{\pi z} Y$ is surjective for every $z \in Z$.

Submersions are fiber bundles. Let $\mathcal{F} := \pi^{-1}(y_0)$ be the fiber over some point $y_0 \in Y$. If $\mathcal{O}$ is a contractable open subset of Y, then $\pi^{-1}(\mathcal{O})$ is homeomorphic to $\mathcal{O} \times F$ and under this homeomorphism, π is projection on the first factor. The vertical distribution $\mathcal{V} := \ker(\pi_*)$ is a smooth subbundle of TZ. The horizontal distribution is defined by $\mathcal{H} := \mathcal{V}^\perp$. One says that π is a Riemannian submersion if $\pi_* : \mathcal{H}_z \to T_{\pi z} Y$ is an isometry for every point z in Z.

The fundamental tensors may be introduced as follows. Let $\pi : Z \to Y$ be a Riemannian submersion. We use indices a, b, c to index local orthonormal frames $\{f_a\}$, $\{f^a\}$, $\{F_a\}$, and $\{F^a\}$ for $\mathcal{H}$, $\mathcal{H}^*$, TY, and T^*, respectively. We use indices i, j, k to index local orthonormal frames $\{e_i\}$ and $\{e^i\}$ for $\mathcal{V}$ and $\mathcal{V}^*$, respectively. There are two fundamental tensors which arise naturally in this setting. The unnormalized mean curvature vector θ and the integrability tensor ω are defined by:

$$\begin{aligned}&\theta := -g_Z([e_i, f_a], e_i) f^a = {}^Z\Gamma_{iia} f^a \in C^\infty(\mathcal{H}),\\ &\omega := \omega_{abi} = \tfrac{1}{2} g_Z(e_i, [f_a, f_b]) = \tfrac{1}{2}({}^Z\Gamma_{abi} - {}^Z\Gamma_{bai})\,.\end{aligned}$$

Lemma 12.1 *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) *The following assertions are equivalent:*
a) The fibers of π are minimal. b) π is a harmonic map. c) $\theta = 0$.

(2) *The following assertions are equivalent:*
a) The distribution $\mathcal{H}$ is integrable. b) $\omega = 0$.

(3) *Let $\Theta := \pi_* \theta$ be the integration of θ along the fiber, and let $V(y)$ be the volume of the fiber. Then $\Theta = -d_Y \ln(V)$. Thus in particular, if $\theta = 0$, then the fibers have constant volume.*

By naturality $\pi^* d_Y = d_Z \pi^*$. The intertwining formulas for the coderivatives and for the Laplacians are more complicated. Let $\mathcal{E} := \omega_{abi}\operatorname{ext}_Z(e^i)\operatorname{int}_Z(f^a)\operatorname{int}_Z(f^b)$ and let $\Xi := \operatorname{int}_Z(\theta) + \mathcal{E}$.

Lemma 12.2 *Let $\pi : Z \to Y$ be a Riemannian submersion. Then $\delta_Z \pi^* - \pi^* \delta_Y = \Xi \pi^*$ and $\Delta_Z^p \pi^* - \pi^* \Delta_Y^p = \{\Xi d_Z + d_Z \Xi\} \pi^*$.*

One is interested in relating the spectrum on the base to the spectrum on the total space. The situation is particularly simple if $p = 0$:

Theorem 12.3 *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) *If $\Phi \in E(\lambda, \Delta_Y^0)$ is nontrivial and if $\pi^*\Phi \in E(\mu, \Delta_Z^0)$, then $\lambda = \mu$.*

(2) *The following conditions are equivalent:*
a) $\Delta_Z^0 \pi^* = \pi^* \Delta_Y^0$. b) For all λ, $\pi^* E(\lambda, \Delta_Y^0) \subset E(\lambda, \Delta_Z^0)$. c) $\theta = 0$.

Muto [97, 98, 99] has given examples of Riemannian principal S^1 bundles where eigenvalues can change. The study of homogeneous space also provides examples. This leads to the result:

Theorem 12.4 (1) *Let Y be a homogeneous manifold with $H^2(Y;\mathbb{R}) \neq 0$. There exists a complex line bundle L over Y with associated circle fibration $\pi_S : S(L) \to Y$, and there exists a unitary connection ${}^L\nabla$ on L so that the curvature $\mathcal{F}$ of ${}^L\nabla$ is harmonic and has constant norm $\epsilon \neq 0$ and so that $\pi_S^*\mathcal{F} \in E(\epsilon, \Delta_S^2)$.*

(2) *Let $0 \leq \lambda \leq \mu$ and let $p \geq 2$ be given. There exists a principal circle bundle $\pi : P \to Y$ over some manifold Y, and there exists $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ so that $\pi^*\Phi \in E(\mu, \Delta_Z^p)$.*

The case $p = 1$ is unsettled; it is not known if eigenvalues can change if $p = 1$. On the other hand, one can show that eigenvalue can never decrease.

Theorem 12.5 *Let $\pi : Z \to Y$ be a Riemannian submersion of closed smooth manifolds. Let $1 \leq p \leq \dim(Y)$. If $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and if $\pi^*\Phi \in E(\mu, \Delta_Z^p)$, then $\lambda \leq \mu$. The following conditions are equivalent:*
a) We have $\Delta_Z^p\pi^* = \pi^*\Delta_Y^p$.
b) For all λ, we have $\pi^*E(\lambda, \Delta_Y^p) \subset E(\lambda, \Delta_Z^p)$.
c) For all λ, there exists $\mu = \mu(\lambda)$ so $\pi^*E(\lambda, \Delta_Y^p) \subset E(\mu, \Delta_Z^p)$.
d) We have $\theta = 0$ and $\omega = 0$.

Results of Park [103] show this if Neumann boundary conditions are imposed on a manifolds with boundary, then eigenvalues can decrease.

There are results related to finite Fourier series. We have $L^2(\Lambda^p M) = \oplus_\lambda E(\lambda, \Delta_M^p)$. Thus if ϕ is a smooth p-form, we may decompose $\phi = \sum_\lambda \phi_\lambda$ for $\phi_\lambda \in E(\lambda, \Delta_M^p)$. Let $\nu(\phi)$ be the number of λ so that $\phi_\lambda \neq 0$. We say that ϕ has *finite Fourier series* if $\nu(\phi) < \infty$. For example, if $M = S^1$, then ϕ has finite Fourier series if and only if ϕ is a trignometric polynomial. The first assertion in the following result is an immediate consequence of the Peter-Weyl theorem; the second result follows from [49].

Theorem 12.6 (1) *Let $\pi : G \to G/H$ be a homogeneous space where G/H is equipped with a G invariant metric and where G is equipped with a left invariant metric. If ϕ is a smooth p-form on G/H with finite Fourier series, then $\pi^*\phi$ has finite Fourier series on G.*

(2) *Let $1 \leq p$, $0 < \lambda$, and $2 \leq \mu_0$ be given. There exists $\pi : G \to G/H$ and there exists $\phi \in E(\lambda, \Delta_{G/H}^p)$ so that $\mu_G(\pi^*\phi) = \nu_0$.*

In general, there is no relation between the heat trace asymptotics on the base, fiber, and total space of a Riemannian submersion. McKean and Singer [90] have determined the heat equation asymptotics for the sphere S^n. Let

$$Z(M,t) := \frac{(4\pi t)^{m/2}}{\text{Vol}(M)} \operatorname{Tr}_{L^2} e^{-t\Delta_M^0} \sim \sum_{n\geq 0} \frac{(4\pi t)^{m/2}}{\text{Vol}(M)} a_n(\Delta_M^0) t^{n/2}$$

be the normalized heat trace; with this normalization, $Z(M,t)$ is regular at the origin and has leading coefficient 1. Their results (see page 63 of McKean and Singer [90]) show that

$$Z(S^1, t) = 1 + O(t^k) \text{ for any k}$$

$$Z(S^2,t) = \frac{e^{t/4}}{\sqrt{\pi t}} \int_0^1 \frac{e^{-x/t}}{\sin\sqrt{x}} dx = 1 + \frac{t}{3} + \frac{t^2}{15} + ...$$
$$Z(S^1 \times S^2, t) = Z(S^2,t) Z(S^1,t) = 1 + \frac{t}{3} + \frac{t^2}{15} + ...$$
$$Z(S^3,t) = e^t = 1 + t + \frac{1}{2}t^2 + ...$$

The two fibrations $\pi : S^1 \times S^2 \to S^2$ and $\pi : S^3 \to S^2$ have base S^2 and minimal fibers S^1. However, the heat trace asymptotics are entirely different.

On the other hand, the following result shows that the heat content asymptotics on Z *are* determined by the heat content asymptotics of the base and by the volume of the fiber if $\theta = 0$; Lemma 12.1 shows the volume V of the fiber is independent of the point in question in this setting.

Theorem 12.7 *Let $\pi : Z \to Y$ be a Riemannian submersion of compact manifolds with smooth boundary. Let $\rho_Z := \pi^*\rho_Y$ and let $\phi_Z := \pi^*\phi_Y$. If $\theta = 0$ and if $\mathcal{B} = \mathcal{B}_D$ or $\mathcal{B} = \mathcal{B}_N$, then $\beta_n(\rho_Z, \phi_Z, \Delta_Z^0, \mathcal{B}) = \beta_n(\rho_Y, \phi_Y, \Delta_Y^0, \mathcal{B}) \cdot V$.*

Acknowledgments

Research of P. Gilkey was partially supported by the Max Planck Institute in the Mathematical Sciences (Leipzig, Germany) and by Project MTM2006-01432 (Spain). It is a pleasure to acknowledge helpful conversations with E. Puffini and L. Loro concerning these matters.

Bibliography

[1] P. Amsterdamski, A. Berkin and D. O'Connor: b_8 Hamidew coefficient for a scalar field *Classical Quantum Gravity* **6** (1989) 1981–1991

[2] M. F. Atiyah and R. Bott: The index problem for manifolds with boundary *In: Differential Analysis - papers presented at the Bombay Coll. (1964)* (Oxford Univ. Press, 1965) 337–351

[3] M. F. Atiyah and R. Bott: A Lefschetz fixed point formula for elliptic differential operators *Bull Amer. Math. Soc.* **72** (1966) 245–256

[4] M. F. Atiyah and R. Bott: A Lefschetz fixed point formula for elliptic complexes (I,II) *Ann. of Math.* **86** (1967) 374–407; *Ann. of Math.* **88** (1968) 451–491

[5] M. F. Atiyah, R. Bott and V. K. Patodi: On the heat equation and the index theorem *Invent. Math.* **19** (1973) 279–330; *Errata* **28** (1975) 277–280

[6] M. F. Atiyah, V. K. Patodi and I. M. Singer: Spectral asymmetry and Riemannian geometry I, II, III *Math. Proc. Cambridge Philos. Soc.* **77** (1975) 43–69; *Math. Proc. Cambridge Philos. Soc.* **78** (1975) 405–432; *Math. Proc. Cambridge Philos. Soc.* **79** (1976) 71–99

[7] M. F. Atiyah and G. B. Segal: The index of elliptic operators III *Ann. of Math.* **87** (1968) 531–545

[8] M. F. Atiyah and I. M. Singer: The index of elliptic operators I, II, IV, V *Ann. of Math.* **87** (1968) 484–530; *Ann. of Math.* **87** (1968) 546–604; *Ann. of Math.* **93** (1971) 119–138; *Ann. of Math.* **93** (1971) 139–149

[9] I. G. Avramidi: The covariant technique for the calculation of the heat kernel asymptotic expansion *Phys. Lett. B* **238** (1990) 92–97

[10] I. G. Avramidi: Covariant methods of studying the nonlocal structure of an effective action *Sov. J. Nucl. Phys.* **49** (1989) 735–739

[11] I. G. Avramidi: Heat kernel asymptotics of Zaremba boundary value problem *Mathematical Physics, Analysis and Geometry* **7** (2004) 9-46

[12] I. G. Avramidi and T. Branson: Heat kernel asymptotics of operators with non-Laplace principal part *Rev. Math. Phys.* **13** (2001) 847–890

[13] L. Bérard Bergery and J. P. Bourguignon: Laplacians and Riemannian submersions with totally geodesic fibers *Illinois J. Math.* **26** (1982) 181–200

[14] M. van den Berg: Heat equation on a hemisphere *Proc. Edinb. Math. Soc.* **118A** (1991) 5–12

[15] M. van den Berg: Heat equation on the arithmetic von Koch snowflake *Probab. Theory Related Fields* **118** (2000) 17–36

[16] M. van den Berg and P. Gilkey: Heat content asymptotics of a Riemannian manifold with boundary *J. Funct. Anal.* **120** (1994) 48–71

[17] M. van den Berg and P. Gilkey: Heat invariants for odd dimensional hemispheres *Proc. Edinb. Math. Soc.* **126A** (1996) 187–193

[18] M. van den Berg and P. Gilkey: The heat equation with inhomogeneous Dirichlet boundary conditions *Comm. Anal. Geom.* **7** (1999) 279–294

[19] M. van den Berg, S. Desjardins and P. Gilkey: Functorality and heat content asymptotics for operators of Laplace type *Topol. Methods Nonlinear Anal.* **2** (1993) 147–162

[20] M. van den Berg and E. B. Davies: Heat flow out of regions in $\mathbb{R}^m$ *Math. Z.* **202** (1989) 463–482

[21] M. van den Berg and J-F Le Gall: Mean curvature and the heat equation *Math. Z.* **215** (1994) 437–464

[22] M. van den Berg and P. Gilkey: The heat content asymptotics of a time dependent process *Proc. Edinb. Math. Soc.* **130A** (2000) 307–312

[23] M. van den Berg and P. Gilkey: Heat content asymptotics with inhomogeneous Neumann and Dirichlet boundary conditions *Potential Anal.* **14** (2001) 269–274

[24] M. van den Berg, P. Gilkey, K. Kirsten and V. A. Kozlov: Heat Content Asymptotics for Riemannian manifolds with Zaremba boundary conditions *to appear Potential Anal.* (2005) 1–24; `arXiv:math-ph/0506076`

[25] M. van den Berg and S. Srisatkunarajah: Heat flow and Brownian motion for a region in $\mathbb{R}^2$ with a polygonal boundary *Probab. Th. Rel. Fields* **86** (1990) 41–52

[26] P. Bérard: Variétés Riemanniennes isospectrales non isométriques *In: Séminaire Bourbaki* (41, 1989)

[27] M. Berger: Sur le spectre des variétés Riemanniennes *Rev. Roum. Math. Pure Appl.* **13** (1969) 915–931

[28] N. Berline, N. Getzler and M. Vergne: *Heat kernels and Dirac operators* (Grundlehren Band 298, Springer-Verlag, Berlin, 1991)

[29] G. Besson and M. Bordoni: On the spectrum of Riemannian submersions with totally geodesic fibers *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl.* **1** (1990) 335–340

[30] J.-M. Bismut: Index theory and the heat equation *In: Proc. ICM Berkeley 1986* (Vol. I, 1987) 491–504

[31] M. Bordag and D. V. Vassilevich: Heat kernel expansion for semitransparent boundaries *J. Phys. A* **32** (1999) 8247–8259

[32] T. Branson, S. Fulling and P. Gilkey: Heat equation asymptotics of 'nonminimal' operators on differential forms *J. Math. Phys.* **32** (1991) 2089–2091
[33] T. Branson and P. Gilkey: Residues of the eta function for an operator of Dirac Type *J. Funct. Anal.* **108** (1992) 47–87
[34] T. Branson and P. Gilkey: Residues of the eta function for an operator of Dirac type with local boundary conditions *Differential Geom. Appl.* **2** (1992) 249–267
[35] T. Branson, P. Gilkey, K. Kirsten and D. Vassilevich: Heat kernel asymptotics with mixed boundary conditions *Nuclear Phys. B* **563** (1999) 603–626
[36] T. Branson, P. Gilkey and B. Ørsted: Leading terms in the heat invariants *Proc. Amer. Math. Soc.* **109** (1990) 437–450
[37] R. Brooks and C. Gordon: Isospectral families of conformally equivalent Riemannian metrics *Bull. Amer. Math. Soc.* **23** (1990) 433–436
[38] R. Brooks, P. Perry and P. Petersen: Finiteness of diffeomorphism types of isospectral manifolds *In: Proc. Symp. Pure Math.* (54, Part 3 1993) 89–94
[39] R. Brooks, P. Perry and P. Yang: Isospectral sets of conformally equivalent metrics *Duke Math. J.* **58** (1989) 131–150
[40] T. Carleman: Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes *In: Åttonde Skand Matematikerkongressen i Stockholm, 1934* (, 1935) 34–44
[41] H. S. Carslaw and J. C. Jaeger: *Conduction of Heat in Solids* (Oxford Science Publications, Oxford Univ. Press, 1946)
[42] A. Chang and P. Yang: Isospectral conformal metrics on 3 manifolds *J. Amer. Math. Soc.* **3** (1990) 117–145
[43] S. Chern: A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds *Ann. of Math.* **45** (1944) 741–752
[44] S. Chern: On the curvature integra in a Riemannian manifold *Ann. of Math.* **46** (1945) 674–684
[45] S. Desjardins and P. Gilkey: Heat content asymptotics for operators of Laplace type with Neumann boundary conditions *Math. Z.* **215** (1994) 251–268
[46] J. S. Dowker: The hybrid spectral problem and Robin boundary conditions *J. Phys. A* **38** (2005) 4735–4754
[47] J. S. Dowker: The $N \cup D$ problem *Preprint* `arXiv:hep-th/0007129` (2000)
[48] J. S. Dowker, P. Gilkey and K. Kirsten: Heat asymptotics with spectral boundary conditions *Contemp. Math.* **242** (1999) 107–124
[49] C. Dunn, P. Gilkey and J. H. Park: The spectral geometry of the canonical Riemannian submersion of a compact Lie Group *Preprint* `arXiv:math.AP/0610788` (2006)
[50] H. Fegan and P. Gilkey: Invariants of the heat equation *Pacific Journal of Mathematics* **117** (1985) 233–254
[51] J. Fleckinger, M. Levitin and D. Vassiliev: Heat content of the triadic von Koch snowflake *Internat. J. Appl. Sci. Comput.* **2** (1995) 289–305
[52] S. Fulling: Kernel asymptotics of exotic second-order operators *In: Proceedings of the Third International Colloquium on Differential Equations* (Plovdiv, 1992) 63–76
[53] P. Gilkey: The spectral geometry of a Riemannian manifold *J. Differential Geom.* **10** (1975) 601–618
[54] P. Gilkey: *Invariance Theory, the Heat Equation, and the Atiyah-Singer index*

theorem (Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995)

[55] P. Gilkey: The heat content asymptotics for variable geometries *J. Phys. A* **32** (1999) 2825–2834

[56] P. Gilkey: *Asymptotic Formulas in Spectral Geometry* (Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 2004)

[57] P. Gilkey and K. Kirsten: Heat content asymptotics with transmittal and transmission boundary *J. London Math Soc.* **68** (2003) 431–443

[58] P. Gilkey, K. Kirsten and J. H. Park: Heat trace asymptotics of a time dependent process *J. Phys. A* **34** (2001) 1153–1168

[59] P. Gilkey, K. Kirsten and J. H. Park: Heat content asymptotics for oblique boundary conditions *Lett. Math. Phys.* **59** (2002) 269–276

[60] P. Gilkey, K. Kirsten and J. H. Park: Heat content asymptotics for spectral boundary conditions *Trends in Math.* **5** (2002) 49–58

[61] P. Gilkey, K. Kirsten and J. H. Park: Heat Content asymptotics for operators of Laplace type with spectral boundary conditions *Lett. Math. Phys.* **68** (2004) 67–76

[62] P. Gilkey, K. Kirsten and D. Vassilevich: Heat trace asymptotics with transmittal boundary conditions and quantum brane-world scenariø *Nuclear Phys. B* **601** (2001) 125–148

[63] P. Gilkey, K. Kirsten and D. Vassilevich: Heat trace asymptotics defined by transfer boundary conditions *Lett. Math. Phys.* **63** (2003) 29–37

[64] P. Gilkey, J. V. Leahy and J. H. Park: *Spectral Geometry, Riemannian Submersions, and the Gromov-Lawson Conjecture* (CRC Press, 1999)

[65] S. I. Goldberg and T. Ishihara: Riemannian submersions commuting with the Laplacian *J. Differential Geom.* **13** (1978) 139–144

[66] C. Gordon: Survey of isospectral manifolds *In: Handbook of differential geometry Vol I* (North-Holland, Amsterdam, 2000) 747–778

[67] C. Gordon and E. Wilson: Isospectral deformations of compact solvmanifolds *J. Differential Geom.* **19** (1984) 241–256

[68] P. Greiner: An asymptotic expansion for the heat equation *In: Global Analysis, Berkeley 1968, Proc. Symp. Pure Math.* (16, 1970) 133–137

[69] P. Greiner: An asymptotic expansion for the heat equation *Arch. Rat. Mech. Anal.* **41** (1971) 163–218

[70] C. Griffith and M. L. Lapidus: Computer graphics and the eigenfunctions for the Koch snowflake drum *In: Progress in inverse spectral geometry* (Trends Math., 1997) 95–113

[71] G. Grubb: Heat operator trace expansions and index for general Atiyah Patodi Singer boundary problems *Comm. in PDE* **17** (1992) 2031–2077

[72] G. Grubb: Complete heat trace, resolvent and zeta expansions for general Atiyah Patodi Singer problems *Nonlinear Diff. Eq. Appl.* **21** (1996) 158–167

[73] G. Grubb: *Functional calculus of pseudodifferential boundary problems* (Prog. in Math., Birkhaeuser, 1996)

[74] G. Grubb and R. Seeley: Weakly parametric pseudodifferential operators and Atiyah Patodi Singer boundary problems *Invent. Math.* **121** (1995) 481–529

[75] G. Grubb and R. Seeley: Zeta and eta functions for Atiyah-Patodi-Singer operators *J. Geom. Anal.* **6** (1996) 31–77

[76] P. Günther: *Huygen's principle and hyperbolic equations* (Perspectives in Mathe-

matics, 5. Academic Press, Inc., Boston, MA, 1988)

[77] V. P. Gusynin, E. V. Gorbar and V. V. Romankov: Heat kernel expansion for non minimal differential operators and manifolds with torsion *Nuclear Phys. B* **362** (1991) 449–471

[78] J. Hadamard: *Le problème de Cauchy et les èquations aux dérivées partielles linéaires hyperboliques* (Hermann et Cie, Paris, 1932)

[79] A. Ikeda: Riemannian manifolds p isospectral but not $p+1$ isospectral *In: Geometry of manifolds* (Coll. Pap. 35th Symp. Differ. Geom., Matsumoto/Japan 1988, Perspect. Math. 8, 1989) 383–417

[80] D. Jakobson, M. Levitin, N. Nadirashvili and I. Polterovich: Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond *J. Comput. Appl. Math.* **194** (2006) 141–155

[81] M. Kac: Can one hear the shape of a drum? *Amer. Math. Monthly* **783** (1966) 1–23

[82] G. Kennedy, R. Critchley and J. S. Dowker: Finite temperature field theory with boundaries: stress tensor and surface action renormalization *Ann. Physics* **125** (1980) 346–400

[83] K. Kirsten: The a_5 heat kernel coefficient on a manifold with boundary *Class. Quantum Grav.* **15** (1998) L5–L12

[84] K. Kirsten: *Spectral functions in mathematics and physics* (Chapman and Hall/CRC, Boca Raton, FL, 2001)

[85] M. Lapidus and M. Pang: Eigenfunctions of the Koch snowflake domain *Comm. Math. Phys.* **172(2)** (1995) 359–376

[86] A. Lichnerowicz: Spineurs harmoniques *C. R. Acad. Sci. Paris Sér. I Math.* **257** (1963) 7–9

[87] D. M. McAvity: Heat kernel asymptotics for mixed boundary conditions *Classical Quantum Gravity* **9** (1992) 1983–1997

[88] D. M. McAvity: Surface energy from heat content asymptotics *J. Phys. A* **26** (1993) 823–830

[89] P. McDonald and R. Meyers: Diffusions on graphs, Poisson problems and spectral geometry *Trans. Amer. Math. Soc.* **354** (2002) 5111–5136

[90] H. P. McKean and I. M. Singer: Curvature and the eigenvalues of the Laplacian *J. Differential Geom.* **1** (1967) 43–69

[91] R. B. Melrose: *The Atiyah-Patodi-Singer Index Theorem* (Research Notes in Mathematics, 4, A K Peters, Ltd., Wellesley, MA, 1993)

[92] J. Milnor: Eigenvalues of the Laplace operator on certain manifolds *Proc. Nat. Acad. Sci. U.S.A.* **51** (1964) 542

[93] S. Minakshisundaram: Eigenfunctions on Riemannian manifolds *J. Indian Math. Soc.* **17** (1953) 158–165

[94] S. Minakshisundaram and Å. Pleijel: Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds *Canad. J. Math.* **1** (1949) 242–256

[95] I. G. Moss: Heat kernel expansions for distributional backgrounds *Phys. Lett. B* **491** (2000) 203–206

[96] I. Moss and J. S. Dowker: The correct B_4 coefficient *Phys. Lett. B* **229** (1989) 261–263

[97] Y. Muto: Riemannian submersions and critical Riemannian metrics *J. Math. Soc. Japan* **29** (1977) 493–511

[98] Y. Muto: Some eigenforms of the Laplace-Beltrami operators in a Riemannian

submersion *J. Korean Math. Soc.* **15** (1978) 39–57
[99] Y. Muto: Riemannian submersion and the Laplace-Beltrami operator *Kodai Math. J.* **1** (1978) 329–338
[100] J. Neuberger, N. Sieben and J. Swift: Computing eigenfunctions on the Koch snowflake: a new grid and symmetry *J. Comput. Appl. Math.* **191** (2006) 126–142
[101] B. Ørsted and A. Pierzchalski: The Ahlfors Laplacian on a Riemannian manifold with boundary *Michigan Math. J.* **43** (1996) 99–122
[102] B. Osgood, R. Phillips and P. Sarnak: Compact isospectral sets of surfaces *J. Funct. Anal.* **80** (1988) 212–234
[103] J. H. Park: The spectral geometry of Riemannian submersions for manifolds with boundary *Rocky Mountain J. Math.* **30** (2000) 353–369
[104] J. H. Park: The spectral geometry of Einstein manifolds with boundary *J. Korean Math. Soc.* **41** (2004) 875–882
[105] V. K. Patodi: Curvature and the fundamental solution of the heat operator *J. Indian Math. Soc.* **34** (1970) 269–285
[106] V. K. Patodi: Curvature and the eigenforms of the Laplace operators *J. Differential Geom.* **5** (1971) 233–249
[107] V. K. Patodi: An analytic proof of the Riemann Roch Hirzebruch theorem for Kaehler manifolds *J. Differential Geom.* **5** (1971) 251–283
[108] C. G. Phillips and K. M. Jansons: The short-time transient of diffusion outside a conducting body *Proc. London Math. Soc.* **428** (1990) 431–449
[109] I. Polterovich: A commutator method for computation of heat invariants *Indag. Math. (N.S.)* **11** (2000) 139–149
[110] I. Polterovich: Heat invariants of Riemannian manifolds *Israel J. Math.* **119** (2000) 239–252
[111] T. Sakai: On eigen-values of Laplacian and curvature of Riemannian manifolds *Tohoku Math. J.* **23** (1971) 589–603
[112] A. Savo: Une méthode de symétrization et quelques applications *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996) 861–864
[113] A. Savo: Uniform estimates and the whole asymptotic series of the heat content on manifolds *Geom. Dedicata* **73** (1998) 181–214
[114] A. Savo: Heat content and mean curvature *Rend. Mat. Appl.* **18** (1998) 197–219
[115] A. Savo: On the asymptotic series of the heat content *Contemp. Math.* **288** (2001) 428–432
[116] R. Seeley: *Topics in pseudo-differential operators* (1969 Pseudo-Diff. Operators (C.I.M.E., Stresa, 1968), 167–305. Edizioni Cremonese, Rome; republished in Pitman Research Notes in Mathematics Series 359. Longman, Harlow, 1996)
[117] R. Seeley: Complex powers of an elliptic operator *In: Singular Integrals* ((Proc. Sympos. Pure Math., Chicago, Ill.), Amer. Math. Soc., Providence, R.I, 1966) 288–307
[118] R. Seeley: The resolvent of an elliptic boundary value problem *Amer. J. Math.* **91** (1969) 889–920
[119] R. Seeley: Analytic extension of the trace associated with elliptic boundary problems *Amer. J. Math.* **91** (1969) 963–983
[120] K. Stewartson and R. T. Waechter: On hearing the shape of a drum: further results *Proc. Cambridge Philos. Soc.* **69** (1971) 353–363

[121] T. Sunada: Riemannian coverings and isospectral manifolds *Ann. of Math.* **121** (1985) 169–186

[122] V. Tanno: A characterization of the canonical spheres by the spectrum *Math. Z.* **175** (1980) 267–274

[123] H. Urakawa: Bounded domains which are isospectral but not congruent *Ann. Scient. École Norm. Sup.* **15** (1982) 441–456

[124] A. E. van de Ven: Index-free heat kernel coefficients *Classical Quantum Gravity* **15** (1998) 2311–2344

[125] M. Vigneras: Variete Riemanniennes isospectrales et non isometriques *Ann. of Math.* **112** (1980) 21–32

[126] B. Watson: Manifold maps commuting with the Laplacian *J. Diff. Geo.* **8** (1973) 85–94

[127] H. Weyl: Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalten elastischen Körpers *Rendiconti del Circolo Mat. di Palermo* **39** (1915) 1–49

PETER B. GILKEY

Mathematics Department, University of Oregon, Eugene Or 97403 USA

`http://www.uoregon.edu/~gilkey`

E-mail: `gilkey@uoregon.edu`.