The Index Theorem

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1. Introduction

Let g be a Riemannian metric on a smooth compact manifold M of dimension m. We shall assume for the moment that the boundary of M is empty and postpone until Section 5 a discussion of the more general setting. If $x=(x_1,...,x_m)$ is a local system of coordinates on M, let:

$$g_{ij}:=g(\partial_i^x,\partial_j^x)$$

give the components of the metric tensor. Let D be an operator of *Laplace type* on a smooth vector bundle V over M. Adopt the Einstein convention and sum over repeated indices. Relative to a local coordinate frame for V, D has the form:

$$D=-\left\{g^{ij}\mathrm{Id}\,\partial_i^x\partial_j^x+A^k\partial_k^x+B\right\},$$

where A^k and B are endomorphisms (i.e. matrices) of V.

We shall assume that V is equipped with a positive definite inner product and that D is self-adjoint. There is then a complete orthonormal basis $\{\phi_i\}$ for $L^2(V)$, where $\phi_i\in C^\infty(V)$ and $D\phi_i=\lambda_i\phi_i$. The collection $\{\phi_i,\lambda_i\}$ is called a *discrete spectral resolution* of D. For example, if $D=-\partial_\theta^2$ on the circle, then the discrete spectral resolution is:

$$\{e^{\sqrt{-1}n\theta},n^2\}_{n\in\mathbb{Z}}.$$

If we order the eigenvalues $\lambda_1\le\lambda_2\le...$ and repeat each eigenvalue according to multiplicity, then there is the following estimate due to Weyl:

$$\lambda_n\sim n^{2/m}\quad\text{as}\quad n\to\infty.$$

We now suppose given a pair of vector bundles V_1 and V_2 over M and a k^{th} order partial differential elliptic operator

$$A:C^\infty(V_1)\to C^\infty(V_2).$$

Locally we decompose

$$A=\textstyle\sum_{|I|\le k}a_I\partial_x^I,$$

where $I=(i_1,...,i_m)$ is a multi-index and where

$$\partial_x^I=(\partial_1^x)^{i_1}...(\partial_m^x)^{i_m}.$$

The a_I are linear maps from V_1 to V_2. The *leading symbol* of A is then defined by setting:

$$\sigma_L(A)(x,\xi):=(\sqrt{-1})^k\textstyle\sum_{|I|=k}a_I(x)\xi^I,$$

where $\xi^I=(\xi_1)^{i_1}...(\xi_m)^{i_m}$ and where

$$\xi=(\xi_1,...,\xi_m)$$

are local fiber coordinates on the cotangent bundle. The leading symbol is an invariantly defined map

$$\sigma_L:T^*M\to\mathrm{End}\,(V_1,V_2).$$

For example, if $V_1 = V_2$ and if D is an operator of Laplace type, then the leading symbol is given by the metric tensor, i.e.

$$\sigma_L(D) = g^{ij}\xi_i\xi_j \mathrm{Id} = |\xi|^2 \mathrm{Id} .$$

If d is exterior differentiation, then the leading symbol is given by exterior multiplication, i.e.

$$\sigma_L(d)(\xi)\omega = \sqrt{-1}\,\xi \wedge \omega .$$

The operator A is said to be *elliptic* if $\sigma_L(A)$ is an isomorphism from V_1 to V_2 for any $\xi \neq 0$. If A is an elliptic partial differential operator, then

$$\begin{aligned} & \mathrm{index}\,(A) := \dim\ker(A) - \dim\mathrm{coker}\,(A) \\ = \; & \dim\ker(A^*A) - \dim\ker(AA^*) \end{aligned}$$

is well defined. As the index vanishes if m is odd, we shall suppose for the most part m is even.

If A_ε is a smooth 1 parameter family of such operators, then $\mathrm{index}\,(A_\varepsilon)$ is independent of ε. The index only depends on the homotopy class of the leading symbol of A within the class of invertible symbols; it does not depend on the underlying metric of the manifold and it does not depend on the fiber metrics chosen for V_1 and V_2.

The Atiyah-Singer index theorem expresses the index as the integral of suitably chosen polynomials in the curvature tensor for the classical elliptic complexes and, more generally, in terms of cohomological information for general elliptic complexes. We refer to Section 4.8 for further details.

We shall work primarily with complexes which are of *Dirac type*, i.e. complexes where A is a first order partial differential operator and where the associated second operators $D_1 := A^*A$ and $D_2 := AA^*$ are of Laplace type.

Here is a brief outline of this article. In Section 2, we describe the classical elliptic complexes (de Rham, signature, spin, Dolbeault, Yang-Mills). In Section 3, we introduce the characteristic classes. In Section 4, we give the relevant formula for the index of the classic elliptic complexes. In Section 5, we discuss manifolds with boundary and in Section 6 we discuss the equivariant index. Index theory is an enormous topic and we have chosen to emphasize the classical features as a complete treatment is beyond the scope of a short expository note such as this one. As some guide to various applications in mathematical physics see the list of references.

2. The classical elliptic complexes

2.1. The de Rham complex

Let $\Lambda^p M$ be the bundle of smooth p forms over M and let

$$\begin{aligned} & d : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M) \quad \text{and} \\ & \delta : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p-1} M) \end{aligned}$$

be the *exterior derivative* and dually the *interior derivative*, respectively. We set

$$\Delta := (d+\delta)^2 \text{ on } C^\infty(\Lambda M)$$

and decompose $\Delta = \oplus_p \Delta^p$, where Δ^p is an operator of Laplace type on $C^\infty(\Lambda^p M)$.

We have $d^2 = 0$. The *de Rham* cohomology groups are given by taking the quotient of the closed forms by the exact forms:

$$H^p(M;\mathbb{R}) := \frac{\ker(d : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M))}{\mathrm{im}\,(d : C^\infty(\Lambda^{p-1} M) \to C^\infty(\Lambda^p M))} .$$

The *Hodge-de Rham theorem* identifies $H^p(M;\mathbb{R})$ with the kernel of the Laplacian

$$\ker(\Delta^p) = H^p(M;\mathbb{R})$$

and with the topological cohomology groups.

If ξ is a cotangent vector, let $\mathfrak{e}(\xi) : \omega \to \xi \wedge \omega$ be *exterior multiplication*. Let $\mathfrak{i}(\xi)$ be the dual operator, *interior multiplication*. If $\{e_i\}$ is a local orthonormal frame for TM, let $e^I = e^{i_1} \wedge \ldots \wedge e^{i_p}$, where $I = \{1 \le i_1 < \ldots < i_p \le m\}$. We then have

$$\mathfrak{e}(e^1)e^I = \begin{cases} 0 & \text{if} \quad i_1 = 1, \\ e^1 \wedge e^I & \text{if} \quad i_1 > 1, \end{cases}$$

$$\mathfrak{i}(e^1)e^I = \begin{cases} e^{i_2} \wedge \ldots \wedge e^{i_p} & \text{if} \quad i_1 = 1, \\ 0 & \text{if} \quad i_1 > 1 . \end{cases}$$

Define a *Clifford module structure* on ΛM by:

$$\gamma(\xi) := \mathfrak{e}(\xi) - \mathfrak{i}(\xi) .$$

If $\{e_i\}$ is a local orthonormal basis for TM, then

$$\gamma(e^i)\gamma(e^j) + \gamma(e^j)\gamma(e^i) = -2\delta_{ij}\mathrm{Id}$$

so the usual *Clifford commutation* rules are satisfied. Let ∇ be the Levi-Civita connection on M. We may then expand

$$\begin{aligned} d &= \quad \mathfrak{e}(e^i)\nabla_{e_i}, \\ \delta &= -\mathrm{i}(e^i)\nabla_{e_i}, \\ d+\delta &= \gamma(e^i)\nabla_{e_i}. \end{aligned}$$

The *de Rham* complex is then defined by taking

$$\begin{aligned} &\Lambda^{\mathrm{even}} M := \oplus_k \Lambda^{2k} M, \\ &\Lambda^{\mathrm{odd}} M := \oplus_k \Lambda^{2k+1} M, \\ &d+\delta : C^\infty(\Lambda^{\mathrm{even}} M) \to C^\infty(\Lambda^{\mathrm{odd}} M). \end{aligned}$$

2.2. The signature complex

The signature complex arises from a different decomposition of the exterior algebra. Let Clif M be the *Clifford algebra* of T^*M; this is the universal unital algebra generated by T^*M subject to the Clifford commutation relations given above:

$$\xi_1 * \xi_2 + \xi_2 * \xi_1 = -2g(\xi_1, \xi_2) \cdot \mathrm{Id}\ .$$

We suppose M is orientable and let

$$\mathrm{orn} = e_1 * ... * e_m \in \mathrm{Clif}\ M$$

be the orientation class. The map $\xi \to \gamma(\xi)$ extends to a unital algebra homomorphism

$$\gamma : \mathrm{Clif}\ M \to \mathrm{End}\,(\Lambda M);$$

$\gamma(\mathrm{orn})$ defines an endomorphism of ΛM which is, modulo suitable sign conventions, the Hodge $\star$ operator. If $m = 2k$ is even, then

$$(d+\delta)\gamma(\mathrm{orn}) = -\gamma(\mathrm{orn})(d+\delta).$$

Set

$$\Theta := (\sqrt{-1})^k \gamma(\mathrm{orn}).$$

As $\Theta^2 = \mathrm{Id}$, we can decompose

$$\Lambda M \otimes \mathbb{C} = \Lambda^+ M \oplus \Lambda^- M,$$

where $\Lambda^\pm M$ are the ± 1 eigenspaces of Θ. The signature complex is then given by:

$$(d+\delta) : C^\infty(\Lambda^+ M) \to C^\infty(\Lambda^- M).$$

2.3. Twisted signature complex

Let V be an auxiliary complex vector bundle over M which is equipped with a unitary connection ∇^V. We use the connection ∇^V on V and the Levi-Civita connection on TM to covariantly differentiate tensors of all types. The *twisted signature* complex is defined by setting:

$$\begin{aligned} &(d+\delta)_V := (\gamma(e^i) \otimes \mathrm{Id}\,)\nabla_{e_i} \\ &\qquad : C^\infty(\Lambda^+ M \otimes V) \to C^\infty(\Lambda^- M \otimes V). \end{aligned}$$

2.4. Yang-Mills complex

This complex in dimension 4 arises from yet another decomposition of the exterior algebra. We use the discussion of the previous section to decompose

$$\Lambda^2 M = \Lambda^{2,+} M \oplus \Lambda^{2,-} M$$

into the ± 1 eigenspaces of Θ. Let

$$\pi : \Lambda^2 M \to \Lambda^{2,-} M$$

be orthogonal projection. The Yang-Mills complex is the 3 term sequence

$$\begin{aligned} &d : C^\infty(\Lambda^0 M) \to C^\infty(\Lambda^1 M) \quad \text{and} \\ &\pi d : C^\infty(\Lambda^1 M) \to C^\infty(\Lambda^{2,-} M). \end{aligned}$$

We can wrap up this sequence to obtain an equivalent elliptic complex

$$(d+\delta) : C^\infty(\Lambda^{\mathrm{even}\,,-} M) \to C^\infty(\Lambda^{\mathrm{odd}\,,+} M).$$

As with the signature complex, this complex can be twisted by taking coefficients in an auxiliary vector bundle V. It is crucial to the study of 4 dimensional geometry using Yang-Mills theory.

2.5. Dolbeault Complex

Let $z = (z_1, ..., z_k)$ be a local system of holomorphic coordinates on a complex manifold M, where $z_i = x_i + \sqrt{-1} y_i$. We define

$$\begin{aligned} &dz^i := dx^i + \sqrt{-1} dy^i \qquad d\bar{z}^i := dx^i - \sqrt{-1} dy^i, \\ &\partial_i^z = \tfrac{1}{2}(\partial_i^x - \sqrt{-1}\partial_i^y), \quad \bar{\partial}_i^z = \tfrac{1}{2}(\partial_i^x + \sqrt{-1}\partial_i^y) \end{aligned}$$

and decompose $d = \partial + \bar{\partial}$, where

$$\partial := \mathfrak{e}(dz^i)\partial_i^z \quad \text{and} \quad \bar{\partial} := \mathfrak{e}(d\bar{z}^i)\partial_i^{\bar{z}}$$

on the complexified exterior algebra. We let δ' be the adjoint of ∂ and δ'' be the adjoint of $\bar\partial$. Let

$$\begin{aligned} d\bar z^I &:= d\bar z^{i_1} \wedge ... \wedge d\bar z^{i_p}, \\ \Lambda^{(0,\text{even})} &:= \text{Span}\,\{d\bar z^I\}_{|I| \text{ is even}}, \\ \Lambda^{(0,\text{odd})} &:= \text{Span}\,\{d\bar z^I\}_{|I| \text{ is odd}}. \end{aligned}$$

The Dolbeault complex is then defined by

$$(\bar\partial + \delta'') : C^\infty(\Lambda^{(0,\text{even})}M) \to C^\infty(\Lambda^{(0,\text{odd})}M).$$

This complex can be twisted by taking coefficients in a holomorphic bundle V over M.

2.6. The spin complex

Let M be orientable. Let P_{SO} be the principle SO bundle of orthonormal frames for the tangent bundle. A spin structure s on M is a principle Spin bundle P_{Sp} together with a double cover $\rho : P_{\text{Sp}} \to P_{\text{SO}}$ which respects the usual double cover ρ : Spin $\to$ SO of the structure groups. Equivalently, a spin structure is a lifting of the transition functions from SO to Spin which preserves the cocycle condition. One says M is *spin* if it admits a spin structure.

A manifold is orientable if and only if the first Stiefel-Whitney class of M vanishes; an orientable manifold is spin if and only if the second Stiefel-Whitney class of M vanishes as well; these are $\mathbb{Z}_2$ valued cohomology classes. Inequivalent spin structures are parametrized by the cohomology group $H^1(M;\mathbb{Z}_2)$ or, equivalently, by real line bundles on M.

The spin representation $\mathcal{S}$ of Spin defines an associated spin bundle $\mathcal{S}M = \mathcal{S}(M,s)$. There is a natural Clifford action c of TM on $\mathcal{S}M$. The Levi-Civita connection lifts to define the spin connection on $\mathcal{S}$ and the Dirac operator is defined by

$$A(s) := c(dx^i)\nabla_{\partial_i^x} \quad \text{on} \quad C^\infty(\mathcal{S}M).$$

Let $m = 2k$ and let $\Theta := (\sqrt{-1})^k c(\text{orn})$. Since $c(\Theta)^2 = \text{Id}$, one can decompose

$$\mathcal{S}M = \mathcal{S}^+M \oplus \mathcal{S}^-M$$

as the direct sum of the half spin bundles to obtain the spin complex:

$$A(s) : C^\infty(\mathcal{S}^+M) \to C^\infty(\mathcal{S}^-M).$$

As with the signature complex, the spin complex can be twisted by taking coefficients in an auxiliary vector bundle V.

2.7. Relating the classic elliptic complexes

One has natural isomorphisms of virtual representations of the spinor group:

$$\begin{aligned} &\Lambda^+ - \Lambda^- \\ &\quad = (\mathcal{S}^+ - \mathcal{S}^-) \otimes (\mathcal{S}^+ + \mathcal{S}^-), \\ &\Lambda^{\text{even}} - \Lambda^{\text{odd}} \\ &\quad = (-1)^{m/2}(\mathcal{S}^+ - \mathcal{S}^-) \otimes (\mathcal{S}^+ - \mathcal{S}^-) \end{aligned}$$

which show that the signature complex and de Rham complexes are the spin complex with coefficients in the virtual bundles

$$\begin{aligned} &\mathcal{S}^+M + \mathcal{S}^-M, \quad \text{and} \\ &(-1)^{m/2}(\mathcal{S}^+M - \mathcal{S}^-M), \end{aligned}$$

respectively. If M is complex and spin, then the Dolbeault complex is the spin complex with coefficients in the square root of the canonical bundle.

One can consider complex spinors to define the group $\text{Spin}^c(m)$. Any spin manifold admits a Spin^c structure with trivial associated complex line bundle. Any complex manifold admits a Spin^c structure with associated complex line bundle given by the canonical bundle. Thus a complex manifold admits a Spin^c structure if and only if it is possible to take a square root of the canonical line bundle; inequivalent Spin structures are parametrized by inequivalent square roots. If M is orientable, then M admits a Spin^c structure if and only if the second Stiefel-Whitney class of M lifts from $H^2(M;\mathbb{Z}_2)$ to $H^2(M;\mathbb{Z})$; in the complex setting, this lifting is performed by the first Chern class. Inequivalent Spin^c structures are parametrized by $H^2(M;\mathbb{Z})$ or, equivalently, by complex line bundles over M.

3. Characteristic classes

3.1. The Euler form

Let ∇ be the Levi-Civita connection on M. Let

$$R(x,y) := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]}$$

be the curvature operator. Let $\{e_1, ..., e_m\}$ be a local orthonormal frame for TM and let

$$R_{ijkl} := g(R(e_i,e_j)e_k, e_l)$$

give the components of the curvature relative to a local orthonormal frame. Let

$$\varepsilon^{I,J} := g(e^{i_1} \wedge ... \wedge e^{i_m}, e^{j_1} \wedge ... \wedge e^{j_m})$$

be the totally anti-symmetric tensor; this is the sign of the permutation which sends $i_\nu \to j_\nu$. Let $m = 2\bar{m}$. The *Euler form* is given by setting:

$$\mathcal{E}_m := \tfrac{1}{8^{\bar{m}}\pi^{\bar{m}}\bar{m}!}\varepsilon^{I,J}R_{i_1i_2j_1j_2}...R_{i_{m-1}i_mj_{m-1}j_m}\,.$$

Let $\rho_{ij} := R_{ikkj}$ and $\tau := \rho_{ii}$ be the *Ricci tensor* and the *scalar curvature*, respectively. Then:

$$\mathcal{E}_2 = \tfrac{1}{4\pi}\tau \quad \text{and} \quad \mathcal{E}_4 = \tfrac{1}{32\pi^2}\{\tau^2 - 4|\rho|^2 + |R|^2\}\,.$$

3.2. The Pontrjagin forms

Since $R(x,y) = -R(y,x)$, we can regard R as a 2-form valued endomorphism of the tangent bundle. We define the *Pontrjagin forms* $p_i \in C^\infty(\Lambda^{4i}M)$ by expanding

$$\det(I + \tfrac{1}{2\pi}R) = 1 + p_1 + p_2 + ...\,.$$

These differential forms are closed and the corresponding cohomology classes

$$P_i = [p_i] \in H^{4i}(M;\mathbb{R})$$

in the de Rham cohomology are independent of the particular Riemannian metric on M which was chosen.

The $\hat{A}$ *genus* and the *Hirzebruch L polynomial* are expressed in terms of these classes using the *splitting principle*. Let A be a skew-symmetric matrix. One sets

$$p(A) := \det(I + A) = 1 + p_1(A) + p_2(A) + ...\,.$$

As A is skew-symmetric, it decomposes as the direct sum of 2×2 blocks of the form

$$\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}.$$

We then have that

$$p(A) = \prod_\nu\{1 + \lambda_\nu^2\} \quad \text{so}$$
$$p_i(A) = s_i(\lambda_1^2, \lambda_2^2, ...),$$

where s_i is the i^{th} symmetric function;

$$p_1 = \sum_i \lambda_i^2, \quad p_2 = \sum_{i<j}\lambda_i^2\lambda_j^2,$$

and so forth. Let

$$\begin{aligned} L(\vec{\lambda}): &= \prod_\nu \frac{\lambda_\nu}{\tanh(\lambda_\nu)} \\ &= 1 + L_1(\vec{\lambda}) + L_2(\vec{\lambda}) + ..., \\ \hat{A}(\vec{\lambda}): &= \prod_\nu \frac{\lambda_\nu}{2\sinh(\frac{1}{2}\lambda_\nu)} \\ &= 1 + \hat{A}_1(\vec{\lambda}) + \hat{A}_2(\vec{\lambda}) + ...\,. \end{aligned}$$

As L_i and $\hat{A}_i$ are even symmetric functions of $\vec{\lambda}$, one can write $L_i = L_i(p_1(A), ..., p_k(A))$. For example

$$\begin{aligned} L &= 1 + \tfrac{1}{3}p_1 + \tfrac{1}{45}(7p_2 - p_1^2) + ..., \\ \hat{A} &= 1 - \tfrac{1}{24}p_1 + \tfrac{1}{5760}(7p_1^2 - 4p_2) + ...\,. \end{aligned}$$

Substituting $\frac{1}{2\pi}R$ for A then permits one to define the Hirzebruch polynomial $L(R)$ and the $\hat{A}$ genus $\hat{A}(R)$.

3.3. The Chern forms

Let V be a k-dimensional complex vector bundle over M. Let ∇ be a Hermitian connection on V and let Ω be the associated curvature endomorphism. The *Chern forms* $c_i \in C^\infty(\Lambda^{2i}M)$ are defined by expanding

$$\det(I + \tfrac{\sqrt{-1}}{2\pi}\Omega) = 1 + c_1 + c_2 + ...\,.$$

As with the Hirzebruch polynomial and the $\hat{A}$ genus, the Chern character and Todd genus are expressed in terms of the generating functions:

$$\mathrm{Td}\,(\vec{\lambda}) = \prod_\nu \frac{\lambda_\nu}{1 - e^{-\lambda_\nu}} \quad \text{and}$$
$$\mathrm{ch}\,(\vec{\lambda}) = \sum_\nu \frac{\lambda_\nu}{\nu!}\,.$$

One has

$$\begin{aligned} \mathrm{Td} &= 1 + \mathrm{Td}_1 + \mathrm{Td}_2 + ... \\ &= 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) + ... \\ \mathrm{Ch} &= ch_0 + ch_1 + ch_2 + ... \\ &= k + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + ...\,. \end{aligned}$$

4. The index theorem

4.1. The Gauss-Bonnet theorem

We return to the de Rham complex. Let

$$\chi(M)=\textstyle\sum_p(-1)^p\dim H^p(M;\mathbb{R})$$

be the *Euler-Poincaré* characteristic; $\chi(M)=0$ if m is odd. Let M have a simplicial structure with $n(k)$ cells of degree k; $n(0)$ is the number of vertices, $n(1)$ is the number of edges, $n(2)$ is the number of triangles, etc. Then

$$\chi(M)=\textstyle\sum_k(-1)^k n(k),$$

so the Euler-Poincaré characteristic is a combinatorial invariant. By the Hodge-de Rham theorem,

$$\begin{aligned}&\operatorname{index}(d+\delta)\\ =\;&\dim\ker(\Delta^{\text{even}})-\dim\ker(\Delta^{\text{odd}})\\ =\;&\chi(M)\,.\end{aligned}$$

The Chern-Gauss-Bonnet theorem expresses this invariant in terms of curvature

$$\chi(M)=\textstyle\int_M\mathcal{E}_m dx,$$

where $\mathcal{E}_m$ is the Euler form given above. If one twists the de Rham complex to take coefficients in an auxiliary vector bundle V, then no new information results since:

$$\operatorname{index}\{d+\delta\}_V=\chi(M)\cdot\dim(V)\,.$$

4.2. The Hirzebruch signature theorem

Let $\operatorname{sign}(M)$ be the index of the signature complex on a manifold of dimension $4k$; the index vanishes in dimensions $m\equiv 2 \bmod 4$. Let $\star$ be the Hodge duality operator. As $\star\Delta^p\star^{-1}=\Delta^{m-p}$, $\star$ preserves the eigenspaces of the Laplacian. In particular, $\star$ induces an isomorphism

$$\begin{aligned}&\star: H^p(M;\mathbb{R})=\ker(\Delta^p)\rightarrow\\ &H^{m-p}(M;\mathbb{R})=\ker(\Delta^{m-p})\end{aligned}$$

which implements Poincaré duality. In dimension $2k$, $\star^2=\mathrm{Id}$. Decompose

$$H^{2k}(M;\mathbb{R})=H^{2k,+}(M;\mathbb{R})\oplus H^{2k,-}(M;\mathbb{R})$$

into the ±1 eigenspaces of $\star$; these may be identified with $\ker(\Delta^{2k,\pm})$ acting on $C^\infty(\Lambda^{2k,\pm}M)$. As the contributions to the signature away from the middle dimension cancel,

$$\begin{aligned}&\operatorname{sign}(M)\\ =\;&\dim H^{2k,+}(M;\mathbb{R})-\dim H^{2k,-}(M;\mathbb{R})\,.\end{aligned}$$

As with the de Rham complex, there is a topological description of this invariant. If α and β are closed $2k$ forms, one sets

$$\langle\alpha,\beta\rangle:=\textstyle\int_M\alpha\wedge\beta\,.$$

One can use Stoke's theorem to see this induces a symmetric bilinear form on the de Rham cohomology groups $H^{2k}(M;\mathbb{R})$. Poincaré duality then shows this symmetric bilinear form is non-degenerate so this is a form of type (p,q); $\operatorname{sign}(M)$ is the signature of this quadratic form:

$$\operatorname{sign}(M)=q-p\,.$$

The Hirzebruch signature formula expresses $\operatorname{sign}(M)$ in terms of curvature; if L is the Hirzebruch polynomial described above and if $m=4k$, then

$$\operatorname{sign}(M)=\textstyle\int_M L_k\,.$$

Let V be an auxiliary coefficient bundle. Taking coefficients in V then yields the formula

$$\begin{aligned}&\operatorname{sign}_V(M)\\ =\;&\textstyle\sum_{4i+2j=m}2^j\int_M L_i\wedge ch_j(V)\,.\end{aligned}$$

4.3. The index of the Yang-Mills complex

Let YM_V be the Yang-Mills complex with coefficients in an auxiliary vector bundle V, then the index can be evaluated using the formulae given above as

$$\begin{aligned}&\operatorname{index}\{\mathrm{YM}_V\}\\ =\;&\tfrac12\{\dim(V)\chi(M)-\operatorname{sign}(M,V)\}\\ =\;&\tfrac12\textstyle\int_M\{\dim V\ \mathcal{E}_4-\dim V L_1-4\mathrm{ch}_2(V)\}\,.\end{aligned}$$

4.4. The index of the Dolbeault complex

If V is a holomorphic bundle over a complex manifold M, then

$$\begin{aligned}&\operatorname{index}\{(\bar\partial+\delta'')_V\}\\ =\;&\textstyle\sum_{2i+2j=m}\int_M\mathrm{Td}_i(M)\wedge ch_j(V)\,.\end{aligned}$$

The index of the untwisted Dolbeault complex is called the *arithmetic genus* and denoted by $ag(M)$.

4.5. The index of the spin complex

If M is a spin manifold and if A_V is the Dirac operator with coefficients in an auxiliary coefficient bundle, then

$$\begin{aligned} &\text{index}\,\{A_V\} \\ = \quad &\textstyle\sum_{4i+2j=m}\int_M \hat{A}_i(M)\wedge ch_j(M)\,. \end{aligned}$$

The index of the spin complex is called the $\hat{A}$ genus and is denoted by $\hat{A}(M)$. If M is a Spin^c manifold, the appropriate formula becomes

$$\begin{aligned} &\text{index}\,\{A_V^c\} \\ = \quad &\textstyle\sum_{4i+2j+2k=m}\int_M \hat{A}_i(M)\wedge ch_j(M)\wedge\theta^k, \end{aligned}$$

where $\theta = \frac{1}{2}c_1(L)$ and where L is the complex line bundle associated to the Spin^c structure.

4.6. Properties

The classic elliptic complexes defined above are multiplicative with respect to Cartesian product. Suppose that M_1 and M_2 are Riemannian manifolds with the appropriate structures. For the signature complex, we suppose M_1 and M_2 are oriented; for the Dolbeault complex, we suppose M_1 and M_2 are holomorphic; for the spin complex, we suppose M_1 and M_2 are spin. We take the twisting coefficient bundle to be trivial in the interests of simplicity. One then has

$$\begin{aligned} &\chi(M_1\times M_2)=\chi(M_1)\chi(M_2),\\ &\text{sign}\,(M_1\times M_2)=\text{sign}\,(M_1)\text{sign}\,(M_2),\\ &\text{ag}\,(M_1\times M_2)=\text{ag}\,(M_1)\text{ag}\,(M_2),\\ &\hat{A}(M_1\times M_2)=\hat{A}(M_1)\hat{A}(M_2)\,. \end{aligned}$$

These complexes behave well under finite coverings. Let $F\to M_2\to M_1$ be a finite covering projection with $|F|$ sheets. Then

$$\begin{aligned} &\chi(M_2)=|F|\chi(M_1),\\ &\text{sign}\,(M_2)=|F|\text{sign}\,(M_1),\\ &\text{ag}\,(M_2)=|F|\text{ag}\,(M_1),\\ &\hat{A}(M_2)=|F|\hat{A}(M_1)\,. \end{aligned}$$

The connected sum $M_1\# M_2$ is defined by punching out small disks about points P_i in M_i and then joining along the spherical boundaries that remain. It is necessary, of course, to smooth out the resulting corners. Note that if M_1 and M_2 are complex manifolds, then $M_1\# M_2$ is no longer a complex manifold in general. Since

$$\chi(S^m)=2,\quad \text{sign}\,(S^m)=0,\quad\text{and}\quad \hat{A}(S^m)=0,$$

the following additivity results follow from the integral formulae given above:

$$\begin{aligned} &\chi(M_1\# M_2)=\chi(M_1)+\chi(M_2)-2,\\ &\text{sign}\,(M_1\# M_2)=\text{sign}\,(M_1)+\text{sign}\,(M_2),\\ &\hat{A}(M_1\# M_2)=\hat{A}(M_1)+\hat{A}(M_2)\,. \end{aligned}$$

4.7. Examples and applications

Let S^m be the standard sphere and let $\mathbb{CP}^j$ be the complex projective plane. One then has

$$\begin{array}{ll} \chi(S^4)=2, & \text{sign}\,(S^4)=0,\\ \chi(S^2\times S^2)=4, & \text{sign}\,(S^2\times S^2)=0,\\ \chi(\mathbb{CP}^2)=3, & \text{sign}\,(\mathbb{CP}^2)=1\,. \end{array}$$

In dimension 4, the Riemann-Roch formula yields

$$ag(M^4)=\tfrac{1}{4}\{\chi(M)+\text{sign}\,(M)\}\,.$$

This would yield $ag(S^4)=\frac{1}{2}$; since $\frac{1}{2}$ is not an integer, this shows S^4 does not admit a complex structure; a similar argument shows S^n does not admit a complex structure for $n\neq 2,6$ and it is not known whether S^6 admits a holomorphic structure; it does admit an almost complex structure.

If we set $M=\mathbb{CP}^2\#\mathbb{CP}^2$, then

$$ag(M)=\tfrac{1}{4}(3+3-2+1+1)=\tfrac{3}{2}$$

and thus $\mathbb{CP}^2\#\mathbb{CP}^2$ does not admit a complex structure. These examples are typical of the use of the index theorem to prove the non-existence of certain structures.

4.8. The general index theorem

Let $S(T^*M)$ be the sphere bundle of unit cotangent vectors and let $D(T^*M)$ be the disk bundle of cotangent vectors of length at most 1. Let

$$P: C^\infty(V_1)\to C^\infty(V_2)$$

be an elliptic pseudo-differential operator. The leading symbol $p:=\sigma_L(P)$ induces a smooth map

$$p: S(T^*M)\to \text{End}\,(V_1,V_2)\,.$$

We form $\Sigma(M)$ by gluing 2 copies of $D(M)$ together along their common boundary $S(M)$ and we define a bundle $\Sigma(p, V_1, V_2)$ over $\Sigma(M)$ by gluing V_1 to V_2 over $S(M)$ using the *clutching function* p. The Atiyah-Singer index theorem expresses the index of P in terms of cohomological data involving the Chern class of the symbol bundle and the characteristic classes of the tangent bundle of M. If $\Sigma(M)$ is given a suitable orientation, then

$$\begin{aligned}&\operatorname{index}(P) = \textstyle\sum_{2i+4j=2m} \int_{\Sigma(M)}\\ &\quad \operatorname{ch}_i(\Sigma(p, V_1, V_2)) \wedge \operatorname{Td}_j(M)\,.\end{aligned}$$

It specializes to the results given above for the classical elliptic complexes. Conversely, by using K-theoretic methods, the index theorem in full generality can be derived from the special case of the twisted signature complex.

5. Manifolds with boundary

If the boundary of M is non-empty, we must impose suitable boundary conditions.

5.1. Local boundary conditions

Choose local coordinates $x = (x^1, ..., x^m)$ near the boundary of M so that x^m is the geodesic distance to the boundary. On the boundary, we can decompose a differential form $\omega \in C^\infty(\Lambda M)$ in the form $\omega = \omega_1 + dx^m \wedge \omega_2$, where ω_1 and ω_2 are tangential differential forms. *Absolute* and *relative* boundary conditions are defined by setting:

$$\mathcal{B}_a\omega := \omega_2|_{\partial M} \quad \text{and} \quad \mathcal{B}_r\omega := \omega_1|_{\partial M}\,.$$

Let $(d+\delta)_a$ and $(d+\delta)_r$ be the associated realizations. These operators preserves the grading of the exterior algebra $\Lambda M = \Lambda^{\text{even}} M \oplus \Lambda^{\text{odd}} M$ and define elliptic complexes:

$$\begin{aligned}&(d+\delta)_a : C^\infty(\Lambda^{\text{even}} M) \to C^\infty(\Lambda^{\text{odd}} M),\\ &(d+\delta)_r : C^\infty(\Lambda^{\text{even}} M) \to C^\infty(\Lambda^{\text{odd}} M)\,.\end{aligned}$$

We consider a collection

$$J = \{1 \le j_1 < ... < j_p < m\}$$

of tangential indices and let

$$dx^J = dx^{j_1} \wedge ... \wedge dx^{j_p}\,.$$

The associated absolute boundary conditions for the Laplacian are defined by

$$\begin{aligned}&\tilde{\mathcal{B}}_a(\phi_J dx^J + \psi_J dx^m \wedge dx^J)\\ =\;& (\psi_J|_{\partial M} dx^J) \oplus (\partial_m^x \phi_J|_{\partial M}) dx^J\,.\end{aligned}$$

If $\star$ is the Hodge operator, then one sets dually:

$$\tilde{\mathcal{B}}_r(\omega) = \tilde{\mathcal{B}}_a(\star\omega)\,.$$

Let Δ_a^p and Δ_r^p be the associated realizations of the Laplacian with these boundary conditions. The Hodge-de Rham theorem extends to this setting to yield isomorphisms

$$\begin{aligned}&\ker(\Delta_a^p) = H^p(M;\mathbb{R}) \quad \text{and}\\ &\ker(\Delta_r^p) = H^p(M, \partial M;\mathbb{R})\,.\end{aligned}$$

The Hodge $\star$ operator intertwines Δ_a^p and Δ_r^{m-p} and implements the Poincaré duality isomorphism $H^p(M;\mathbb{R}) = H^{m-p}(M, \partial M;\mathbb{R})$. This also shows

$$\begin{aligned}&\operatorname{index}(d+\delta)_a\\ =\;& \textstyle\sum_p (-1)^p \dim H^p(M;\mathbb{R}) = \chi(M),\end{aligned}$$

and

$$\begin{aligned}&\operatorname{index}(d+\delta)_r\\ =\;& \textstyle\sum_p (-1)^p \dim H^p(M, \partial M;\mathbb{R})\\ =\;& \chi(M, \partial M) = \chi(M) - \chi(\partial M)\,.\end{aligned}$$

Let $\mathcal{E}_m$ be the Euler form if m is even. We set $\mathcal{E}_m = 0$ if m is odd. Let L be the second fundamental form. Let $A = (a_1, ..., a_{m-1})$ and $B = (b_1, ..., b_{m-1})$ be collections of distinct indices ranging from 1 to $m-1$. Set

$$\begin{aligned}\mathcal{L}_{m-1} :\; =\;& \textstyle\sum_k \frac{1}{\pi^k 8^k k!(m-1-2k)!\operatorname{vol}(S^{m-1-2k})}\\ &\cdot\varepsilon^{A,B} R_{a_1 a_2 b_2 b_1} ... R_{a_{2k-1} a_{2k} b_{2k} b_{2k-1}}\\ &\cdot L_{a_{2k+1} b_{2k+1}} ... L_{a_{m-1} b_{m-1}}\,.\end{aligned}$$

The Chern-Gauss-Bonnet theorem generalizes to this setting to yield

$$\begin{aligned}&\chi(M) = \operatorname{index}(d+\delta)_a\\ =\;& \textstyle\int_M \mathcal{E}_m dx + \int_{\partial M} \mathcal{L}_{m-1} dy\,.\end{aligned}$$

For example:

$$\chi(M^2) = \tfrac{1}{4\pi}\left\{\textstyle\int_{M^2} \tau dx + 2\int_{\partial M^2} L_{aa} dy\right\},$$

$$\chi(M^3) = \tfrac{1}{8\pi}\int_{\partial M^3}\{R_{abba} + L_{aa}L_{bb} - L_{ab}L_{ab}\}dy,$$
$$\chi(M^4) = \tfrac{1}{32\pi^2}\int_{M^4}\{\tau^2 - 4|\rho|^2 + |R|^2\}dx + \tfrac{1}{24\pi^2}\int_{\partial M^4}\{3\tau L_{aa} + 6R_{amam}L_{bb} + 6R_{acbc}L_{ab} + 2L_{aa}L_{bb}L_{cc} - 6L_{ab}L_{ab}L_{cc} + 4L_{ab}L_{bc}L_{ac}\}\,dy.$$

The interior integral vanishes if m is odd. The boundary integral can be non-zero in any dimensions. Thus, in particular, the index of this elliptic complex can be non-zero even if m is odd; $\chi(D^m) = 1$ for any m. The index of $(d+\delta)_r$ is computed similarly.

5.2. Spectral boundary conditions

In contrast to the de Rham complex, there do not exist local boundary conditions for the signature, spin, and Dolbeault complexes. To simplify the discussion, we shall assume the metric is product near the boundary; there are appropriate compensating terms involving the second fundamental form in the more general setting. Let $A : C^\infty(V_1) \to C^\infty(V_2)$ denote either the twisted signature or twisted spin complexes; there are some additional difficulties for the Dolbeault complex. Near the boundary, we can express

$$A = \sigma(\partial_m^x + A_T),$$

where A_T is a self-adjoint tangential operator of Dirac type on $V_1|_{\partial M}$ and where σ is a unitary bundle isomorphism from $V_1|_{\partial M}$ to $V_2|_{\partial M}$. Let $\{\phi_i, \lambda_i\}$ be the discrete spectral resolution of A_T. One defines

$$\eta(A_T, s) = \textstyle\sum_{\lambda_k \neq 0} \operatorname{sgn}(\lambda_k)|\lambda_k|^{-s}$$

as a measure of the spectral asymmetry of A_T. This is well defined for $Re(s) >> 1$ and has a meromorphic extension to the complex plane $\mathbb{C}$. It turns out that 0 is a regular value and one defines

$$\eta(A_T) := \tfrac{1}{2}\{\eta(A_T, s) + \dim\ker(A_T)\}|_{s=0}\,.$$

We impose *spectral boundary conditions*. Let $\Pi_{\geq}$ be orthogonal projection in $L^2(V_1|_{\partial M})$ on the span of the eigensections of A_T corresponding to non-negative eigenvalues and let $A_{\geq}$ be the associated realization defined by this boundary condition.

One can use the Atiyah-Patodi-Singer index theorem to generalize the relations given above to this setting. Let f_A be the local integral given above that involves the Hirzebruch L polynomial for the signature complex or the $\hat{A}$ genus for the spin complex. One then has:

$$\operatorname{index}(A_{\geq}) = \eta(A_T) + \textstyle\int_M f_A\,.$$

There are suitable correction formulae involving integrals of polynomials in the second fundamental form and in the curvature tensor if the structures are not product near the boundary.

6. Equivariant problems

6.1. The classical Lefschetz formula

Let M be a compact Riemannian manifold without boundary. Let T be a smooth map from M to M. Then pullback T^* induces an action on $C^\infty(\Lambda^p M)$ which commutes with the exterior derivative d and hence an action on the de Rham cohomology groups $H^p(M;\mathbb{R})$. The Lefschetz number of T is then given by

$$\mathcal{L}(T) = \textstyle\sum_p (-1)^p \mathrm{Tr}\,\{T^* \text{ on } H^p(M;\mathbb{R})\}\,.$$

To illustrate the Lefschetz number, let $M = \mathbb{T}^2$ be the 2 dimensional torus. Let $e^1 := dx^1$, let $e^2 := dx^2$, and let $e^{12} := dx^1 \wedge dx^2$. Then:

$$H^0(\mathbb{T}^2;\mathbb{R}) = 1\cdot\mathbb{R},$$
$$H^1(\mathbb{T}^2;\mathbb{R}) = e^1\cdot\mathbb{R} + e^2\cdot\mathbb{R},$$
$$H^2(\mathbb{T}^2;\mathbb{R}) = e^{12}\cdot\mathbb{R}.$$

Let $T(x_1, x_2) = (n_{11}x_1 + n_{12}x_2, n_{21}x_1 + n_{22}x_2)$. Then

$$T^*(1) = 1,$$
$$T^*(e^1) = n_{11}e^1 + n_{12}e^2,$$
$$T^*(e^2) = n_{21}e^1 + n_{22}e^2,$$
$$T^*(e^{12}) = (n_{11}n_{22} - n_{12}n_{21})e^{12}$$

and consequently the Lefschetz number becomes

$$\mathcal{L}(T) = \det(I - T^*) = 1 - (n_{11} + n_{22}) + (n_{11}n_{22} - n_{12}n_{21})\,.$$

The *classical Lefschetz fixed point formula* expresses $\mathcal{L}$ in terms of data for the fixed point set $\mathcal{F}(T)$ and is an example of the equivariant index theorem. One assumes that the fixed point set of T consists of smooth submanifolds $N_1, ..., N_k$ and that the induced map dT_ν on the normal bundles of these manifolds is non-degenerate. This means that $\det(I - dT_\nu) \neq 0$, i.e. that there are no infinitesimal normal directions which are left fixed. One then has

$$\mathcal{L}(T) = \textstyle\sum_i \operatorname{sign}(\det(I - dT_\nu))\chi(N_i)\,.$$

6.2. The Lefschetz formula for the other classical elliptic complexes

Let T be an orientation preserving isometry of M. When dealing with the spin complex, we suppose that T preserves the spin structure; when dealing with the Dolbeault complex, we suppose that T preserves the holomorphic structure. If

$$A : C^\infty(V_1) \to C^\infty(V_2)$$

is one of the classic elliptic complexes, then by assumption T^* commutes with A and hence preserves the eigenspaces of the associated Laplacians. The Lefschetz number is defined by setting

$$\begin{aligned}\mathcal{L}_A(T) : \quad &= \quad \operatorname{Tr}(T^* \text{ on } \ker(A^*A)) \\ &- \quad \operatorname{Tr}(T^* \text{ on } \ker(AA^*))\,.\end{aligned}$$

Setting $T = \text{Id}$ one recovers the standard index.

To simplify the discussion, we shall assume henceforth that T is an orientation preserving isometry of M with only isolated fixed points. Let $\{\theta_1, ..., \theta_{m/2}\}$ be the rotation angles of dT at a fixed point x of T. Set

$$\lambda_j := \cos(\theta_j) + \sqrt{-1}\sin(\theta_j)\,.$$

We take the sum over the isolated fixed points x and take the product over the rotation angles $1 \leq j \leq m/2$ to express:

$$\mathcal{L}_{\text{sign}}(T) = \sum_x \prod_j \left\{-\sqrt{-1}\cot(\tfrac{\theta_j}{2})\right\},$$

$$\mathcal{L}_{\text{spin}}(T) = \sum_x \prod_j \left\{-\tfrac{1}{2}\sqrt{-1}\csc(\tfrac{\theta_j}{2})\right\},$$

$$\mathcal{L}_{\text{Dolb}}(T) = \sum_x \prod_j (1 - \bar\lambda_j)^{-1}\,.$$

In considering the spin complex, we assume T preserves the spin structure. This permits us to lift dT from $SO(m)$ to $Spin(m)$ and defines liftings of the rotation angles θ_i from $[0, 2\pi]$ to $[0, 4\pi]$ in such a way that the formula given above for the spin complex is well defined. In considering the Dolbeault complex, we assume T preserves a complex structure so the formula given above for the Dolbeault complex involving the complex eigenvalues λ_j is well defined.

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