Solutions

1. Show that the area of the region contained between the curves $y = 4-2x^2$ and $y = x^2-8$ is 32 square units.

   Intersection: $4 - 2x^2 = x^2 - 8$ at $x = \pm 2$.

   Then
   $$\int_{-2}^{2} \left[(4-2x^2) - (x^2-8)\right]\ dx = 32\ \checkmark.$$

2. A good model for the sale price of all unground beef in the United States t months after May, 2015, was
   $$P(t) = 4.7 - 0.037t + 0.0004t^2$$
   dollars per pound. What was the average price of unground beef during the 3-month period from July to October in 2015?

$$AV = \frac{1}{b-a}\int_a^b f(x)\ dx = \frac{1}{5-2}\int_2^5 P(t)\ dt \approx \frac{1}{3}(13.71) \approx \$4.57 \text{ per pound}$$

3. Consider the "Chotikapanich" model for a Lorenz curve, where

$$L(x) = \frac{1}{e^k - 1}\left(e^{kx} - 1\right),$$

where k is an (as yet) unknown constant.

(a) Check that, regardless of the value of k, for this formula we get $L(0) = 0$ and $L(1) = 1$.

(b) Write one or two sentences explaining why it is important, in the context of distribution of income, that any Lorenz curve should satisfy the equations $L(0) = 0$ and $L(1) = 1$.

(c) According to the Gini Index, which model shows greater income inequality for a country: a model with Lorenz curve given by the Chotikapanich model with $k = 2.7$, or by the Lorenz curve $L(x) = \frac{3}{4}x^3 + \frac{1}{4}x$?

(a)

$$L(0) = \frac{1}{e^k - 1}\left(e^{k(0)} - 1\right) = \frac{1}{e^k - 1}(1 - 1) = \boxed{0}.$$

$$L(1) = \frac{1}{e^k - 1}\left(e^{k(1)} - 1\right) = \frac{1}{e^k - 1}\left(e^k - 1\right) = \boxed{1}.$$

(b) $x = 0$ means 0% of the population, which should necessarily control 0% of the income, thus $L(0) = 0$ should be true.

$x = 1$ means 100% of the population, which should necessarily control 100% of the income, thus $L(1) = 1$ should be true.

(c) In the first case:

$$\begin{aligned} GI &= 2\int_0^1 [x - L(x)]\ dx \\ &= 2\int_0^1 \left[x - \frac{1}{e^{2.7} - 1}\left(e^{2.7x} - 1\right)\right]\ dx \\ &\approx 0.403 \end{aligned}$$

In the second case:

$$\begin{aligned} GI &= 2\int_0^1 [x - L(x)]\ dx \\ &= 2\int_0^1 \left[x - \left(\frac{3}{4}x^3 + \frac{1}{4}x\right)\right]\ dx \\ &\approx 0.375 \end{aligned}$$

The first model shows greater income inequality (larger GI)

4. What lump sum is required to be invested now at 4% annual interest (compounded continuously) in order to match the future value of a continuous income stream of $50 + 10t$ thousand dollars per year, at the same interest rate and over the next fifteen years? *Hint:* You may use the fact that $\int te^{kt}\,dt = \frac{e^{kt}}{k}(t - \frac{1}{k}) + C$ for any constant, k.

   The present value is such a lump sum, so

$$\begin{aligned}
PV &= \int_0^T f(t)e^{-rt}\,dt \\
&= \int_0^{15} (50 + 10t)e^{-0.04t}\,dt \\
&= \int_0^{15} 50e^{-0.04t}\,dt + \int_0^{15} 10te^{-0.04t}\,dt \\
&= \left[\frac{50}{-0.04}e^{-0.04t} + \frac{10}{-0.04}e^{-0.04t}\left(t - \frac{1}{-0.04}\right)\right]\Big|_0^{15} \\
&\approx -6174.13 - (-7500) \\
&= 1325.84 \text{ thousand dollars}
\end{aligned}$$