Exam 1

Math 242, Fall 2018

Name: Key

Don't leave anything blank: If you don't know the entire answer, showing a formula or writing something illustrating that you understand any concept involved in the problem will allow me to give partial credit. I have to give you a 0 if you write nothing down.

Show your work: If you give me an answer without any kind of demonstration of how you got that answer, you will not receive credit for that part of the problem.

Check your answers: Take the time before you turn in your test to make sure you have read the directions correctly and in their entirety, that all your work shown is correct, and that you have clearly stated your answer (by boxing or circling it where appropriate).

Pace yourself: If you're stuck on a problem, move on and come back to it later. Don't risk forcing yourself to give partial answers if you run out of time near the end of the test. Do the easy ones first. There are 140 points on this exam. That means you should budget about 0.4 minute(s) for each point a problem is worth in order to complete the exam in time.

Reminder: There are to be no devices with internet access nor graphing calculators/devices which can compute derivatives or integrals used in conjunction with this test. If you use any such material, you will receive a zero on this assessment.

see HW 1, problem 2; W.W. 1a problem 2; W.W. 4a, problem 2
Lecture notes 7.1

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1. (12 points) Describe the domains of the following functions as regions in the xy-plane. Give your answer either by writing a complete sentence or by shading in the domain in the xy-plane.

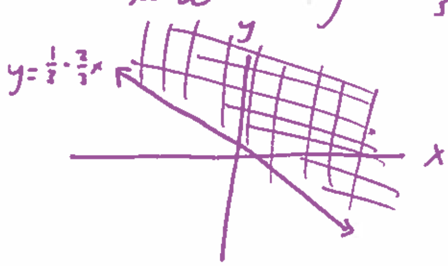
(a) $f(x, y) = \sqrt{2x + 3y - 1}$

Must have: $2x + 3y - 1 \geq 0$

$3y \geq 1 - 2x$

$y \geq \frac{1}{3} - \frac{2}{3}x$

The domain of $f(x,y)$ is the set of all points in the xy-plane above the line $y = \frac{1}{3} - \frac{2}{3}x$



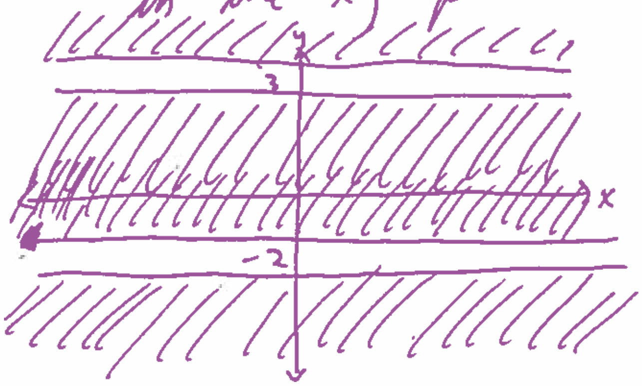
(b) $g(x, y) = \frac{4x - y}{y^2 - y - 6}$

Can't have: $y^2 - y - 6 = 0$

$(y - 3)(y + 2) = 0$

$y = -2, 3$

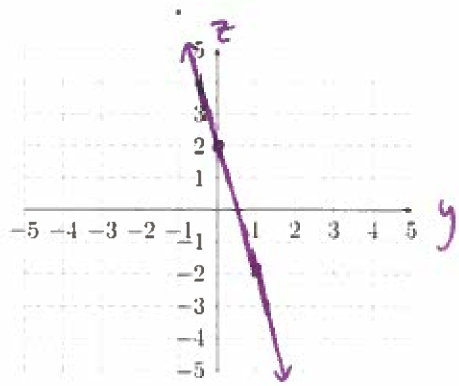
The domain of g is the set of all points in the xy plane with $y \neq -2$ and $y \neq 3$



see HW 1, # 3b; 7.1 Lecture Guide Ex 1; WW 4a #4; 7.1 lecture notes

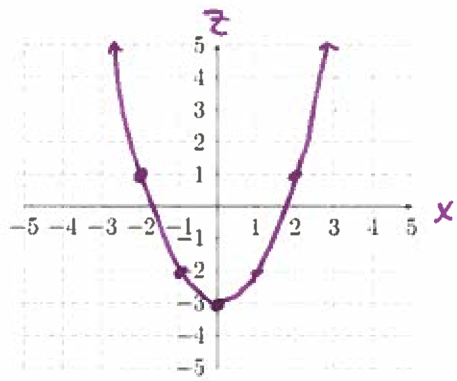
2. (24 points) Consider the function $z = h(x, y) = x^2 - 4y + 1$. Give graphs of the following. *Make sure to label your axes*:

   (a) The trace of h at $x = 1$



$z = 1^2 - 4y + 1$
$= -4y + 2$

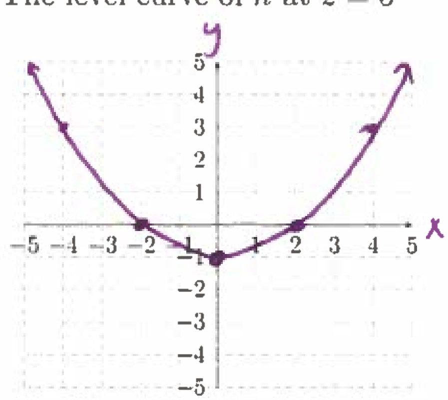
   (b) The trace of h at $y = 1$



$z = x^2 - 4(1) + 1$
$= x^2 - 3$

has vertex at $x = \frac{-0}{2(1)} = 0$
$z = 0^2 - 3 = -3$

   (c) The level curve of h at $z = 5$



$5 = x^2 - 4y + 1$
$4y = x^2 - 4$
$y = \frac{1}{4}x^2 - 1$
has vertex at $x = \frac{-0}{2(\frac{1}{4})} = 0$
$y = \frac{1}{4}0^2 - 1 = -1$

see any derivative problem ~~~~ involving product/chain rule

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3. (36 points) Compute the following derivatives of the function $r(x, y) = e^{x^2 - y^2}$

(a) $r_x(x, y) = 2x\, e^{x^2-y^2}$

(b) $r_y(x, y) = -2y\, e^{x^2-y^2}$

(c) $r_{xx}(x, y) = 2e^{x^2-y^2} + 2x e^{x^2-y^2} \cdot 2x$

(d) $r_{xy}(x, y) = -4xy\, e^{x^2-y^2}$

(e) $r_{yx}(x, y) = -4xy\, e^{x^2-y^2}$

(f) $r_{yy}(x, y) = -2e^{x^2-y^2} - 2y e^{x^2-y^2}(-2y)$

see HW 3, #1; 7.3 lecture notes; all of WW 3a, b

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4. (26 points) Find and classify the critical points of

$$m(x, y) = x^2 - 3xy + 3y^2 - 2x + 6y + 2018$$

$m_x(x,y) = 2x - 3y - 2$

$m_y(x,y) = -3x + 6y + 6$

① $2x - 3y - 2 = 0 \to 3y = 2x - 2 \to$ ③ $y = \frac{2}{3}x - \frac{2}{3}$

② $-3x + 6y + 6 = 0$

② becomes $-3x + 6\left(\frac{2}{3}x - \frac{2}{3}\right) + 6 = 0$

$\to -3x + 4x - 4 + 6 = 0$

$\to x + 2 = 0$

$\to x = -2$

③ gives $y = +\frac{2}{3}(-2) - \frac{2}{3} = -\frac{4}{3} - \frac{2}{3} = -\frac{6}{3} = -2$

The only critical point is $(-2,-2)$

$m_{xx}(x,y) = 2$, $m_{xy}(x,y) = -3$, $m_{yy}(x,y) = 6$

$D(x,y) = m_{xx} \cdot m_{yy} - m_{xy}^2 = 2 \cdot 6 - (-3)^2 = 12 - 9 = 3 > 0$

$m_{xx} < 0$

Therefore, $(-2,-2)$ is a relative minimum.

see HW 3, #2; 7.2 L.G. Ex 1,3,4; 7.3 L.G Ex 1; Quick Hit 7.2(2) #2; all of WW 3c;

5. Greg's Apple Juice Stand sells two different types of apple juice. One type is a bottled store brand, the other is freshly pressed. Greg can obtain the store brand at a cost of 15 cents per cup and the freshly pressed juice at a cost of 25 cents per cup. Greg estimates that if he sells the store brand for x cents per cup and the freshly pressed juice for y cents per cup, he can sell $30-2x+5y$ cups of the store brand and $150+x-4y$ cups of the freshly pressed juice.

   (a) (4 points) Give the function $P(x,y)$ which describes Greg's profit.

$$P(x,y) = (x-15)(30-2x+5y) + (y-25)(150+x-4y)$$
$$= 30x - 2x^2 + 5xy - 450 + 30x - 75y + 150y + xy - 4y^2 - 3750 - 25x + 100y$$
$$= -2x^2 + 6xy - 4y^2 + 35x + 175y - 4200$$

   (b) (10 points) Compute $P(20,30)$ and give an interpretation of this number in context.

$$P(20,30) = (5)(30-2(20)+5(30)) + (5)(150+20-4(30))$$
$$= 5 \cdot 140 + 5(50)$$
$$= 950$$

When Greg sells the store brand for 20 cents per cup and the freshly pressed juice for 30 cents per cup, he will make a profit of \$9.50

(c) (12 points) Compute $P_y(20, 30)$ and give an interpretation of this number in context.

$$P_y(x,y) = (x-15)(5) + (1)(150 + x - 4y) + (y-25)(-4)$$

$$P_y(20,30) = (5)(5) + 150 + 20 - 120 - 20$$
$$= 55$$

When Greg sells the store brand for 20 cents per cup and sells the freshly pressed juice for 30 cents per cup, his profit is increasing at a rate of 55 cents per cent increase in the price of the freshly pressed juice.

(d) (16 points) Greg now has a new business model and his profit is now given by

$$P(x, y) = 300 + 15x - 2x^2 + 28y - 4y^2$$

Suppose further that Greg knows that the price of the store brand will be $x = 20+t$ cents in t months and the price of the freshly pressed juice will be $y = 30 + t^2$ cents in t months. At what rate is Greg's profit changing with respect to time in 1 month? Include units.

$$P_x(x,y) = 15 - 4x \qquad P_y(x,y) = 28 - 8y$$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2t$$

$$\frac{dP}{dt} = (15-4x)(1) + (28-8y)(2t)$$

when $t = 1$, $x = 20+1 = 21$, $y = 30 + 1^2 = 31$,

so $\left.\frac{dP}{dt}\right|_{t=1} = (15 - 4\cdot 21) + (28 - 8\cdot 31)(2)$

$= -509$ cents per month

Bonus: (14 possible points) Find and classify the critical points of $q(x, y) = (x^2 - 3)(y+1) + \frac{1}{2}y^2$

$q_x(x,y) = 2x(y+1) = 0$ ①

$q_y(x,y) = (x^2-3) + y = 0$ ②

①

$x = 0$

② $0^2 - 3 + y = 0$

$y = 3$

$y + 1 = 0$

$y = -1$

② $x^2 - 3 - 1 = 0$

$x^2 = 4$

$x = \pm 2$

CP: $(0,3)$, $(2,-1)$, $(-2,-1)$

$q_{xx}(x,y) = 2(y+1)$ $\qquad$ $q_{xy}(x,y) = 2x$

$q_{yy}(x,y) = 1$

$D(0,3) = (2(3+1))(1) - 0^2 = 8 > 0$

$q_{xx}(0,3) = 8 > 0$, so $(0,3)$ is a relative min.

$D(2,-1) = (2(-1+1))(1) - (2\cdot 2)^2 = -16 < 0$

so $(2,-1)$ is a saddle point

$D(-2,-1) = (2(-1+1))(1) - (2\cdot -2)^2 = -16 < 0$

so $(-2,-1)$ is a saddle point

The Second Derivative Test: For a function, $f(x, y)$, with critical point (x_0, y_0), consider the quantity

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

If $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.
If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a relative maximum.
If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a relative minimum.
Finally, if $D(x_0, y_0) = 0$, the test is inconclusive.

The Multivariable Chain Rule: For function $f(x, y)$, if x and y are both functions of t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$