Chapter 7 Lecture Notes

Greg Knapp

October 5, 2018

Section 7.1: Functions of several variables

- Recall that a function is a rule, together with a set of inputs (called the domain), and a set of outputs (called the range, or image), which assigns each input to a unique output.
- You've seen lots of different function types in the past, but in calculus, the functions that we're interested in studying are those which take numbers as input and have numbers as outputs.
- We give names to functions (frequently letters, like f) and we use function notation to express the rule which turns inputs into outputs: for example, $f(x) = x^2 - 1$
  - Dissect the different parts of the above notation: f is name, x is input, $x^2 - 1$ is output.
  - Explain how to read it: "f of x equals x squared minus one"
  - For a particular input, like $x = 2$, we can check to see what the output of the function is by replacing the x in the formula defining f by a 2: $f(2) = 2^2 - 1 = 3$
- Up to this point in your mathematical career, we've only looked at functions which take one number as input.
- The reason for doing this is that functions which take one number as input are simple: we have nice ways of graphing them, and we were able to do a lot of calculus with them.
- However, now that we know a lot about functions with one variable, we want to look at functions which take multiple numbers as input.
- There are going to be a lot of similarities between functions with one variable and functions with several variables, and I'll do my best to point them out as we go.
- **Def:** A function with two variables is a function which takes an ordered pair, (x, y), as input and has a single real number as output. The definition of a function with three, four, etc. variables is as you expect.
- What does ordered pair mean? It means that order matters: i.e. $(2, 1)$ is a different thing from $(1, 2)$.
- To start, we use similar notation: $f(x, y) = x^2 + y^2 - 100$ for example.
- This says that given a number for x and a number for y, square x, square y, add them, then subtract 100 from the result.
- For instance $f(6, 9) = 6^2 + 9^2 - 100 = 17$.
- Recall that, when defining a function with one variable by a formula, we cared about the mathematical domain of that function: i.e. the set of all numbers which we are allowed to plug in.
- For example, with $g(x) = \frac{x}{2x-1}$, we notice that we're not allowed to plug in anything which makes the denominator 0: so we can't have $2x - 1 = 0$, i.e. $x = \frac{1}{2}$.

- Alternatively, with $h(x) = \sqrt{3x+6}$, we know that we can't take the square root of a negative, so we must have $3x + 6 \geqslant 0$, so $x \geqslant -2$.
- Note the difference in reasoning here: "can't have" versus "must have."
- A similar process works for functions with multiple variables.
- For example, with $j(x, y) = \sqrt{2x - y}$, we see that the domain is the set of all (x, y) with $y \leqslant 2x$.
- What does this region look like in the xy-plane? Shade the region.
- With $k(x, y) = \frac{1}{xy}$, we have that the domain is the set of all (x, y) with $x \neq 0$ and $y \neq 0$.
- What does this region look like in the xy-plane?
- What about graphing?
- With a function of one variable, $f(x)$, we know that the graph is the set of all pairs of the form $(x, f(x))$ where x is in the domain of f.
- **Ex:** Graph $f(x) = \sqrt{x}$
- Notice that we think about the inputs as lying on a line and we think about the outputs as the height above that line.
- We're going to do a similar thing with graphing functions of two variables.
- The inputs live in the xy-plane and the outputs are going to be represented as a height above the plane.
- This means that we need to think about things in three dimensions.
- Draw three dimensional coordinates and explain how you find points like $(1, -1, 2)$
- **Def:** The *graph* of a function, f, of two variables is the set of all points $(x, y, f(x, y))$ where (x, y) lies in the domain of f.
- In particular, the graphs of functions with two variables will be surfaces, because we're assigning a height to each point in the xy-plane.
- Show Mathematica examples.
- What are some strategies that we have for visualizing the graph of a function of two variables?
- **Def:** Given a function, $f(x, y)$, the *trace* of f at $x = c$ is the graph in the yz-plane given by $z = f(c, y)$. Similarly, the trace of f at $y = c$ is the graph in the xz-plane given by $z = f(x, c)$
- **Def:** Given a function, $f(x, y)$, a level curve of f at $z = c$ is the curve in the xy-plane given by $c = f(x, y)$
- These have nice interpretations in terms of the 3D graph of $f(x, y)$.
- Show Mathematica examples.
- **Ex:** Consider the function $z = g(x, y) = yx^2 + y$. Sketch the following
  - The trace of $g(x, y)$ at $y = -1$
  - The trace of $g(x, y)$ at $x = -1$
  - The level curve of $g(x, y)$ with $z = -1$
- Notice that each of these curves lies in a different plane and the curves don't really have anything to do with one another.

- **Ex:** LotSW Incorporated produces two top-sellling items: replicas of the sword Andúril and replicas of Jedi light sabres. Light sabre replicas cost \$25 each to produce while Andúril replicas cost \$35 each to produce. Light sabres sell for \$100 each while Andúril replicas sell for \$150 each. There is a fixed overhead cost of \$1500 associated with producing these items every month. Find the profit function and determine the profit obtained from producing and selling 50 swords and 75 light sabres in a given month.
  - Profit $P(a, \ell) = 75\ell + 115a - 1500$
  - $P(50, 75) = 9875$
- **Ex:** Recall that the present value of an investment, B, at interest rate, r, compounded continuously, for t years is given by the formula $P(B, r, t) = Be^{-rt}$. Compute $P(2000, .05, 3)$ and interpret the value in context.
  - $P(2000, .05, 3) = 1721.42$
  - If you want to have 2000 in 3 years, you should invest \$1721.42 now in an account with a 5% interest rate, compounded continuously.

Section 7.2: Partial Derivatives

- We're going to continue with the theme of observing similarities between one variable functions and multivariable functions.
- In particular, the most important thing that we cared about in business calc 1 was the slope of a function at a point.
- Recall that if we had some function, we determined the slope at that point with a limit computation: we computed the slope of the secant line between $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ and let $h \to 0$.
- Recall that this led to the following derivative rules:
  - The derivative of $f(x) = x^p$ is $f'(x) = px^{p-1}$ for all p.
    - **Ex:** For $h(x) = x^3$, we have $h'(x) = 3x^2$
  - For any functions f and g, the derivative of $(f + g)(x)$ is $(f + g)'(x) = f'(x) + g'(x)$.
    - **Ex:** For $h(x) = x^{31} + x^{-2}$, we have $h'(x) = 31x^{30} - 2x^{-3}$
  - For any constant c and function f, the derivative of $cf(x)$ is $cf'(x)$.
    - **Ex:** For $h(x) = 5x^4$, we have $h'(x) = 20x^3$
  - For any functions f and g, the derivative of $(fg)(x)$ is $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
    - **Ex:** For $h(x) = (2x - 1)(x^3 + 5x)$, we have $h'(x) = 2(x^3 + 5x) + (2x - 1)(3x^2 + 5)$
  - For any functions f and g, the derivative of $(\frac{f}{g})(x)$ is $(\frac{f}{g})'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
    - **Ex:** For $h(x) = \frac{2x-1}{x^3+5x}$, we have $h'(x) = \frac{(x^3+5x)(2)-(2x-1)(3x^2+5)}{(x^3+5x)^2}$
  - For $f(x) = e^x$ and $g(x) = \ln(x)$, we have $f'(x) = e^x$ and $g'(x) = \frac{1}{x}$
  - For function f and g, we have that the derivative of $(f \circ g)(x)$ is $(f \circ g)'(x) = f'(g(x))g'(x)$.
    - **Ex:** For $h(x) = \ln(x^2 + 1)$, we have $h'(x) = \frac{2x}{x^2+1}$.
- For functions of two variables, $f(x, y)$, however, we don't have this concept of slope at a point to work with.
- However, if we have a point (x_0, y_0), we do have a couple of lines that go through the point $(x_0, y_0, f(x_0, y_0))$: namely, the trace at $x = x_0$ and the trace at $y = y_0$.
- **Def:** The *partial derivative of f with respect to x at* (x_0, y_0) is the slope of the trace at $y = y_0$ at the point $x = x_0$. If we are letting z denote the output of f (i.e. $z = f(x, y)$), then we denote this partial derivative by $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial z}{\partial x}(x_0, y_0)$, or $f_x(x_0, y_0)$.

- **Def:** The *partial derivative of f with respect to y at (x_0, y_0)* is the slope of the trace at $x = x_0$ at the point $y = y_0$. If we are letting z denote the output of f (i.e. $z = f(x, y)$), then we denote this partial derivative by $\frac{\partial f}{\partial y}(x_0, y_0)$, $\frac{\partial z}{\partial y}(x_0, y_0)$, or $f_y(x_0, y_0)$.

- Break these definitions down. Recall that the trace of f at $y = y_0$ is a line in the xz-plane. Hence, we can talk about its slope at the point $x = x_0$. Similar story with y.

- You can read $\frac{\partial z}{\partial x}$ as "del f del x" or "partial f partial x." My college roommate read it as "six f six x" because the del looks like a backwards six.

- Note: these are formal definitions and are not going to be how we actually compute partial derivatives. However, they are very helpful for thinking about what the partial derivatives *mean*, which is far more important than knowing how to compute them (and actually, it informs you how to compute them).

- However, we'll use them in this coming example to motivate our future method of doing computations.

- **Ex:** Compute $\frac{\partial f}{\partial x}(1, -2)$, $\frac{\partial f}{\partial x}(-2, 3)$, $\frac{\partial f}{\partial y}(1, -2)$, and $\frac{\partial f}{\partial y}(-2, 3)$ for $f(x, y) = y^2(3x - 1)$

  - When computing $\frac{\partial f}{\partial x}(1, -2)$, we first find the trace at $y = -2$. This is the line $z = 4(3x - 1) = 12x - 4$. To find the slope of that line, we take a derivative with respect to x and get 12. Then, we plug in the point $(1, -2)$ and get 12 for the partial derivative.
  - When computing $\frac{\partial f}{\partial x}(-2, 3)$, we first find the trace at $y = 3$. This is the line $z = 9(3x-1) = 27x-9$. Hence, the partial derivative is 27.
  - When computing $\frac{\partial f}{\partial y}(1, -2)$, we first find the trace at $x = 1$. This is the line $z = 2y^2$. Then, we compute the slope by finding the derivative $4y$ and plugging in $y = -2$ gives a slope of -8.
  - When computing $\frac{\partial f}{\partial y}(-2, 3)$, we first find the trace at $x = -2$. This is the line $z = -7y^2$, which has slope $-14y$, so at $y = 3$, we have slope -52.

- What steps did we follow in all of these computations?

  1. Identify the variable with respect to which you are differentiating.
  2. Hold the other variable(s) constant.
  3. Take a derivative with respect to the variable in 1.
  4. Plug in the point.

- If we simply omit the very last step, we can find a general formula for our partial derivatives which works at every point. This is how we're going to actually compute our partial derivatives.

- **Ex:** Consider the function $g(x, y) = xy + 2x^3 - 3\sqrt{y}$. Compute $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

- **Ex:** Consider the function $h(x, y) = (xy^2 + 3)(y - x)$. Compute $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

- **Ex:** Consider the function $j(x, y) = \log(xy - 3y^4)$. Compute $\frac{\partial j}{\partial x}$ and $\frac{\partial j}{\partial y}$.

- **Ex:** A particular manufacturer's productivity (in millions of units produced) is modeled well by the function $P(K, L) = 0.3(0.4K^{-0.5} + 0.6L^{-0.5})^{-2}$ where K is millions of dollars of capital investment, and L is thousands of worker-hours every month. Find and interpret the value of $P_K(10, 6)$.

  - $P_K(K, L) = -0.6(0.4K^{-0.5} + 0.6L^{-0.5})^{-3}(-0.2K^{-1.5})$ and so $P_K(10, 6) \approx 0.074$
  - Interpretation: Production is increasing at a rate of 0.074 million units per million dollars of capital investment when \$10 million of capital and 6 thousand worker-hours of labor per month are used.

- Let's look at an application of partial derivatives: substitute and complementary goods.

- **Def:** Let product 1 be sold at a price of a dollars per unit and product 2 be sold at a price of b dollars per unit. Suppose that the demand for product 1 is $A(a, b)$ units and the demand for product 2 is $B(a, b)$ units. We say that product 1 and 2 are *substitute* commodities if $\frac{\partial A}{\partial b} > 0$ and $\frac{\partial B}{\partial a} > 0$ for all $a, b > 0$. Product 1 and 2 are *complementary* commodities if $\frac{\partial A}{\partial b} < 0$ and $\frac{\partial B}{\partial a} < 0$ for all $a, b > 0$.

- Intuitively, goods are substitute if an increase in demand for one causes a decrease in demand for the other.

- For example, any competing goods will be substitutes. Dell laptops versus Lenovo laptops for instance. An increase in demand for Dells should decrease demand for Lenovos.

- Goods are complementary if an increase in demand for one causes an increase in demand for the other.

- For instance, tortilla chips and salsa are complementary goods.

- **Ex:** Local demand for grapefruit is given by $f(p, n) = 10 + \frac{5}{p+2} + 3e^{0.4n}$ while demand for oranges is $g(p, n) = 7 - \frac{4}{p+6} - 2n$, where each demand is given in thousands of units per month at p dollars per pound for grapefruit and n dollars per pound for oranges. Are grapefruit and oranges substitute, complementary, or neither?

- They end up as substitutes.

- Another item of interest is that of marginal analysis.

- Recall from 241 that the derivative of a 1-variable function does a good job estimating the net increase of that function if the input is increased by one unit.

- **Ex:** Given $f(x) = \ln(x)$, estimate $f(2)$, using the fact that $f(1) = 0$.

- We have the same idea with partial derivatives: the partial derivative f_x approximates the net increase in f if x is increased by 1. The partial derivative f_y approximates the net increase in f if y is increased by 1.

- **Ex:** The output at a factory is often thought of as a function of the amount of capital, K, and the size of the labor force, L, Output functions of the form

  $$Q(K, L) = AK^{\alpha}L^{\beta}$$

  where A, α, β are constants with $\alpha + \beta = 1$ have proven to be especially useful and are known as Cobb-Douglas production functions.

  Suppose that, at a particular factory, its output is given by the function $Q(K, L) = 60K^{0.3}L^{0.7}$ thousand units, where K represents capital investment in millions of dollars, and L represents thousands of workers. Suppose in addition, that the factory has \$3 million in capital investment and employs 5 thousand workers. Estimate the increase in output if another million dollars is invested in capital. Estimate the increase in output if another thousand workers are hired. Which would be a more effective way of increasing output?

- Back to theory. For a two variable function, $f(x, y)$ notice that $\frac{\partial f}{\partial x}$ is a function of both x and y. So $\frac{\partial f}{\partial x}$ has partial derivatives with respect to x and y. Same with $\frac{\partial f}{\partial x}$.

- We write these different second derivatives as $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2}{f}\partial x \partial y = f_{yx}$, $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$, and $\frac{\partial^2 f}{\partial y^2} = f_{yy}$.

- It turns out that we're always going to have $f_{xy} = f_{yx}$ (this is called the equality of mixed partials).

- **Ex:** Find all of the second order partial derivatives of $f(x, y) = xe^{5y^2}$

- No applications for these yet, but we're getting there.

- In some cases, we may be interested in a function of two variables, where both variables are themselves functions of a third variable. For instance, suppose you manage a store that has two brands, A and B, of the same product. Brand A costs x dollars per unit and brand B costs y dollars per unit. You know that demand for brand A is given by $Q(x, y) = 300 - 20x^2 + 30y$ units and you also know that the price of brand A will be $x = 2 + 0.05t$ and the price of brand B will be $y = 2 + 0.1\sqrt{t}$ in t months. At what rate is demand for brand A changing with respect to time?
- One way in which we could solve this is simple substitution. Find the demand for A as a function of time, then take the normal derivative.
- However, there is going to be a more general method for doing this. The chain rule for multivariable functions states that if z is a function of x and y and if x and y are both functions of t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- We can now compute the rate of change of demand in this new way.
- Derivative practice. If time, here are some examples to have students work through on their own.
  - **Ex:** Find second order partials of $f(x, y) = 3x^2 - 2e^y$
  - **Ex:** Find second order partials of $g(x, y) = \ln(3x - 1) + 5y^2$
  - **Ex:** What is the reason that the mixed partials for both functions came out to be 0?
  - **Ex:** Find second order partials of $h(x, y) = 2x\ln(4xy + 2)$
  - **Ex:** Suppose $j(x, y) = x^4 - 2xy^3 + 3x^2 + 1$. In addition, suppose that x and y are functions of t: $x(t) = 2t^3$ and $y(t) = 5t$. What is $\frac{dj}{dt}$ in terms of t?

<u>Section 7.3</u>: Optimizing Functions of Two Variables

- One of the first items of interest after you learned how to compute derivatives was how to optimize functions of one variable.
- This means that given some function, $f(x)$, how do you find its maxima and minima?
- First, we should think about what it means to be a local max/min.
- We say that f has a local max at x_0 if for all x "near" x_0, $f(x) < f(x_0)$. There is a similar definition for min.
- What does the word "near" mean? It means that there exists an open interval (a, b) with $a < x_0 < b$ so that for all x in (a, b), $f(x) < f(x_0)$.
- So how to we find maxes and mins? At each of them, notice that the slope of f is 0.
- So we find all places where the derivative is 0.
- The x-values for which $f'(x) = 0$ are called critical points.
- However, we run into a slight problem. We might find points that aren't maxes or mins (draw pictures).
- So when we find all of the critical points, how could we tell the difference between these types of points?
- The second derivative test aided us: if the slope of the function is increasing, you have a local minimum; if the slope is decreasing, you have a local maximum; if the slope is neither increasing nor decreasing, you have neither.
- Equivalently, if $f''(x_0) > 0$, you have a local min; if $f''(x_0) < 0$, you have a local max; if $f''(x_0) = 0$, you have neither.

- How do we generalize this notion to functions with multiple variables?
- **Def:** A function, $f(x, y)$ has a local maximum at (x_0, y_0) if there exists a small disk around (x_0, y_0) so that for all points (x, y) in that disk, we have $f(x, y) < f(x_0, y_0)$. A local minimum is defined similarly.
- Show Mathematica example.
- What is true about all local maxes and mins?
- Look at the traces! We have that the slopes of each trace have to be 0 at local maxes and mins.
- But this means that if $(x_0, y_0, f(x_0, y_0))$ is a max or min, we have $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.
- This means that when looking for maxes and mins, we had better find places where the partial derivatives are *both* 0.
- **Def:** Given a function, $f(x, y)$, a *critical point* of f is a point, (x_0, y_0), where $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.
- Let's practice finding critical points.
- **Ex:** Find all critical points of $f(x, y) = x^2 - (x + 1)y$
- **Ex:** Find all critical points of $g(x, y) = 3x^2 - 5y^2 + 5xy - 4x + y - 1$
- **Ex:** Find all critical points of $h(x, y) = \frac{1}{2}x^2y + \frac{1}{3}y^3 - \frac{1}{2}y^2 - 6y - 1$
- Are all critical points maxes or mins?
- Just as in 1 variable case, no! Show Mathematica example of saddle.
- How do we distinguish between maxes and mins and saddles? We have a multivariable second-derivative test.
- Suppose that (x_0, y_0) is a critical point of $f(x, y)$. Compute $D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$. If $D(x_0, y_0) > 0$, you have a relative extrema. If $f_{xx} > 0$, it's a min, if $f_{xx} < 0$, it's a max. If $D(x_0, y_0) < 0$, you have a saddle. If $D(x_0, y_0) = 0$, the test is inconclusive.
- **Ex:** Find and classify the critical points of $f(x, y) = xy + \frac{4}{x} + \frac{2}{y}$
  - Get rel. min
- **Ex:** Find and classify all critical points of $g(x, y) = xy^2 - 6x^2 - 3y^2$
  - Get rel. max and two saddles
- **Ex:** Jamaal manages a grocery store that carries two brands of cat food, a local brand obtained at the cost of 30 cents per can and a national brand obtained for 40 cents per can. He estimates that if the local brand is sold for x cents per can and the national brand for y cents per can, then approximately $70 - 5x + 4y$ cans of the local brand and $80 + 6x - 7y$ cans of the national brand will be sold each day. How should Jamaal price each brand to maximize total daily profit from the sale of cat food?
- Thinking back to our one-variable cousins, after we figured out how critical points worked, we sought to find the absolute max and min on a closed interval.
- How did we do this?
  1. Find critical points in the interval, compute y values at those points.
  2. Compute y values at endpoints of interval.
  3. Largest y value is max
  4. Smallest y value is min

- We want to generalize this process to multivariable functions.
- What is the equivalent of a closed interval? A closed region (i.e. contains its boundary)
- We're going to have the similar steps to the problem "find the absolute extrema of $f(x, y)$ in the region R"
  1. Find critical points in R, compute z values at those points.
  2. Find equations representing the boundary curves and substitute each of these into the equation for f.
  3. Find the critical points of f subject to these conditions and compute z values at those points and endpoints (what you did in calc 1).
  4. Largest z value is max
  5. Smallest z value is min
- **Ex:** Find the absolute extrema of the function $f(x, y) = 4xy - x^2 - 4y + 9$ on the triangular region R with vertices $(0, 0), (8, 0)$, and $(0, 16)$.
- **Ex:** Find the absolute extrema of the function $h(x, y) = \frac{1}{2}x^2y + \frac{1}{3}y^3 - \frac{1}{2}y^2 - 6y - 1$ on the rectangular region R given by the inequalities $-4 \leqslant x \leqslant 4$ and $-2 \leqslant y \leqslant 2$.