Chapter 6 Lecture Notes

Greg Knapp

November 25, 2018

<u>Section 6.3:</u> Improper Integrals

- Question: How much area is under the curve of $f(x) = \frac{1}{x^2}$ to the right of $x = 1$?
- We don't really know how to answer this.
- A good question to ask is "is this even finite?"
- How do we answer this?
- Let's look at the area from $x = 1$ to $x = 2$.
  - Get .5
- What about from $x = 1$ to $x = 10$?
  - Get .9
- What about from $x = 1$ to $x = 100$?
- $x = 1$ to $x = 1000$?
- $x = 1$ to $x = \infty$?
  - Wait, this question doesn't make sense.
  - Infinity isn't a number, so it can't be equal to x.
- Whatever the question turns out to be, though, the answer had better be 1.
- So how should we talk about this area?
- Well, we started by taking $\int_1^2 \frac{1}{x^2}\,dx$ and increased the upper bound without limit.
- **Def:** For any function $f(x)$ defined on $[c, \infty)$, $\int_c^\infty f(x)\,dx = \lim_{b\to\infty} \int_c^b f(x)\,dx$.
- **Ex:** Compute $\int_1^\infty \frac{5}{x^3}\,dx$
- **Ex:** Compute $\int_1^\infty \frac{1}{x}\,dx$
- **Def:** If $\int_c^\infty f(x)\,dx$ exists and is finite, we say that the integral *converges*. Otherwise, the integral *diverges*.
- **Ex:** Compute $\int_7^\infty \frac{x}{(3x+1)^3}\,dx$
  - $u = 3x + 1$
  - Ends up being $\approx .007$
- What else can we do? Instead of integrating from a number to ∞, we could integrate from $-\infty$ to a number.

- **Def:** For any function $f(x)$ defined on $(-\infty, c]$, $\int_{-\infty}^{c} f(x)\,dx = \lim_{b\to-\infty} \int_{b}^{c} f(x)\,dx$.
- Before we do an example, let's note a helpful limit computation
  - For any positive constant n and any constant p (positive or negative), $\lim_{t\to\infty} t^p e^{-nt} = 0$
  - Hand wave why this is true
- **Ex:** Compute $\int_{-\infty}^{0} xe^{-2x}\,dx$
  - Use formula $\int xe^{kx}\,dx = \frac{e^{kx}}{k}(x - \frac{1}{k})$
  - Get $-\frac{1}{4}$
- The last thing that we might want to consider is the integral over the entire real line, i.e. from $-\infty$ to ∞.
- **Def:** For a function, $f(x)$, defined for every real number, and for any number, c, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$.
- Note: 0 is almost always the best choice for c.
- Also note that this lines up with our previous formula $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
- **Ex:** Compute $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$
  - Get 0
- **Ex:** Marginal sales for a company t years into the future are estimated to be given by $s(t) = \frac{160}{(t+1)^2}$ thousand units sold per year. Determine the net change in sales...
  1. ...over the next three years.
  2. ...in the long run.

  - Working through numbers gives 120 thousand units in first part
  - 160 thousand units in second part.
- **Ex:** When a political campaign puts out an ad, t weeks later, the advertisement is being seen at a rate of $600te^{-2t}$ thousand new viewers per week. How many new viewers should the political campaign expect to see the ad in the long run?
  - Get 150 thousand new viewers.
- **Ex:** Uday wishes to endow a scholarship at a local college with a gift that provides a continuous income stream at the rate of $25000 + 1200t$ dollars per year in perpetuity. Assuming the prevailing annual interest rate stays fixed at 5% compounded continuously, what donation is required to finance the endowment?
  - Should end up with 980,000, after recalling that $\frac{24}{.05} = 480$

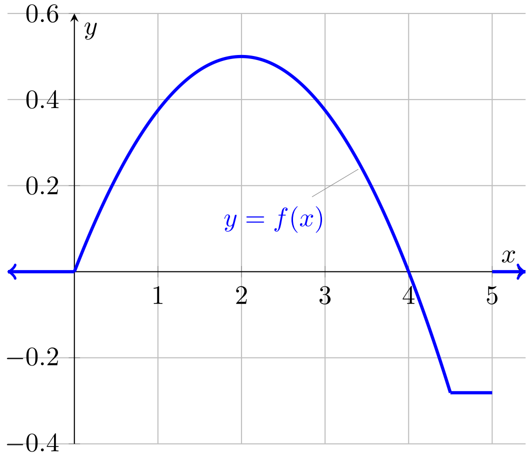
<u>Section 6.4:</u> Introduction to Continuous Probability

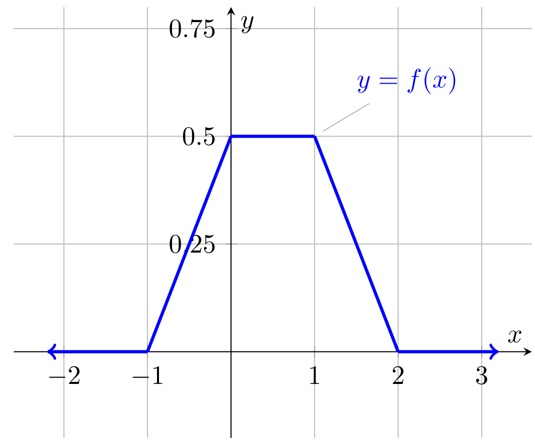
- At this point, you are probably familiar with discrete probability
- These are questions like "what is the probability of randomly drawing a full house from a deck of cards?"
- These questions require counting methods
  - First, count the number of ways of randomly drawing a full house
  - Then, count the number of different hands that you could draw

  - Then, divide

- Consider this new question, however.
- You purchase a lightbulb at the store. What is the probability that it lasts for longer than 2 years?
- We certainly can't compute this probability by counting things.
- After all, there are infinitely many times that it could last for which are longer than 2 years and infinitely many times that it could last shorter than 2 years.
- How do we account for this?
- We need to introduce some terms, first, then we'll come back to this.
- **Def:** Given an experiment, the *sample space* for that experiment is the collection of all possible outcomes. A *continuous random variable* for that experiment is a function which assigns a number to each element of the sample space.
- **Ex:** In our motiviating example, the experiment is "pick a lightbulb." The sample space (we'll call it S) is the set of lightbulbs at the store. Our random variable is the function (we'll call it $X : S \to (-\infty, \infty)$) that takes a particular lightbulb as input and spits out the lifespan of that lightbulb.
- With this language out of the way, let's think about what questions would be reasonable to ask.
- We could ask "what is the probability that a random lightbulb has a lifespan of exactly two years?"
- This question doesn't really make sense, though. Or, if it does, the answer had better be 0. There's no way the lightbulb will last for exactly two years.
- A better question would be something like "what is the probability that the lightbulb will last for between 1 and 3 years?" Or "what is the probability that the lightbulb will last more than 2 years?"
- To denote the probability that the lightbulb lasts for between 1 and 3 years, we might write $P(1 \leqslant X \leqslant 3)$ and to denote the probability that the lightbulb lasts for more than 2 years, we write $P(2 \leqslant X)$.
- But we still have no way of answering our question without a little bit more information.
- This is going to be a magic black box for us. This stuff would be covered in a stats course, but we don't have time for that, so we're just going to take these things for granted.
- **Def:** A *probability density function* $f(x)$ for a random variable X has the property that $P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$
- **Ex:** Suppose that the probability density function for our random variable X representing the lifespan of a lightbulb at the store is $f(x) = \begin{cases} e^{-x} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$. What is the probability that a randomly chosen lightbulb lasts between 1 and 3 years? What about the probability that it lasts for more than 2 years?
  - $P(1 \leqslant X \leqslant 3) \approx .32$
  - $P(2 \leqslant X) \approx .14$
- Important: $f(x)$ does not tell you what the probabilities are! You have to do an integral to compute any probabilities at all.
- What properties should a probability density function have?
- We know that probabilities are always at least 0.
- If a function ever dips below 0, then some integral will be negative, which means that function cannot be a probability density function.

- So if $f(x)$ is a probability density function, then $f(x) \geqslant 0$ for all x.
- Furthermore, we had better have $P(-\infty < X < \infty) = 1$. In other words, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
- It turns out that these two properties characterize probability density functions.
- **Ex:** Could the following two graphs be graphs of a probability density function?

1. 

2. 

- What are some important types of probability density functions?
- The most basic is a uniform probability density function. This is a function which is constant on some interval $[a, b]$ and is 0 outside of that interval.
- Draw graph.
- What does the value of that constant have to be? $\frac{1}{b-a}$
- **Ex:** A random number generator (RNG) chooses real numbers between 0 and 100 according to a uniform distribution.
  1. What is the probability density function for this RNG?
  2. What is the probability that a randomly chosen number is between 50 and 60?
  3. What is the probability that a randomly chosen number is between 10 and 20?
- Another important type of probability density function is the *exponential density function.* A random variable with this type of density function is said to be *exponentially distributed*.
- This function has the form $f(x) = \begin{cases} ke^{-kx} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$ for a constant $k > 0$.
- **Ex:** Let X be a random variable that measures the duration of cell phone class in a particular city and assume that X has an exponential distribution with density function

$$\begin{cases} 0.5e^{-0.5t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

where t denotes the duration (in minutes) of a randomly selected call.

1. Find the probability that a randomly selected call will last between 2 and 3 minutes.
2. Find the probability that a randomly selected call will last less than 2 minutes.

- Our final notion is that of expected value.
- Given a probability density function $f(x)$ for a random variable X, the *expected value* of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

- Without going into too much detail, this should remind you of the formula for expected value in discrete probability, where you add up the values multiplied by the probabilities.
- Important note: the expected value is an X value and as such, takes on whatever units X has.
- **Ex:** Find the expected value of the RNG which is selecting values between 0 and 100. What if if were computing values between 100 and 300?
- Fact: the expected value of a uniform distribution on $[a, b]$ is $\frac{a+b}{2}$
- Prove if time
- New fact: the expected value of the exponential distribution $f(x) = \begin{cases} ke^{-kx} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$ is $\frac{1}{k}$
- Prove if time
- Interpretation note: the expected value is the average value that you would get if you repeated your experiment a large number of times. It is *not* "the value you should expect to get" nor is it "the most likely outcome."
- **Ex:** What is the expected value of a random variable X with probability density function $\frac{1}{x}$ on the interval $[1, e]$?
  - ≈ 1.7