Solutions

1. Show that $\displaystyle\int_0^\infty \frac{4t}{(2t^2+1)^3}\,dt = 0.5.$

   $u = 2t^2 + 1$ and $du = 4t\;dt$, so

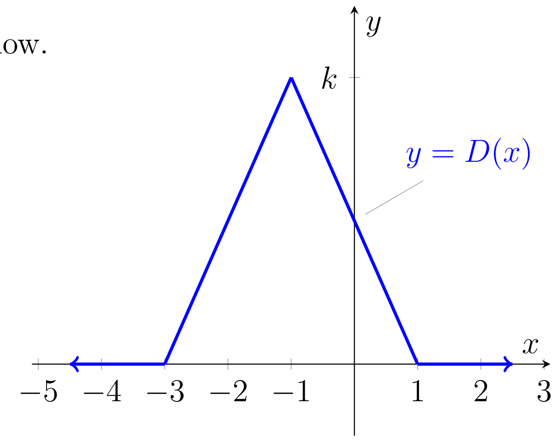
$$
\begin{aligned}
\int_0^\infty \frac{4t}{(2t^2+1)^3}\,dt &= \lim_{b\to\infty} \int_{t=0}^{t=b} \frac{1}{u^3}\,du \\
&= \lim_{b\to\infty} \left[-1/2\cdot u^{-2}\right]\Big|_{t=0}^{t=b} \\
&= \lim_{b\to\infty} \left[-1/2(2t^2+1)^{-2}\right]\Big|_0^b \\
&= \lim_{b\to\infty} \left[\cancelto{0}{-1/2(2b^2+1)^{-2}} - -1/2(2(0)^2+1)^{-2}\right] \\
&= \boxed{1/2}.
\end{aligned}
$$

2. Determine whether or not $f(x)$ is a probability density function, where

$$
f(x) = \begin{cases} 3 - x^2 & \text{, if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.
$$

   $\displaystyle\int_{-\infty}^{\infty} f(x)\;dx = \int_0^1 (3 - x^2)\;dx = \left[3x - \frac{x^3}{3}\right]\Big|_0^1 = 8/3$. This is not equal to 1, so the function is not a probability density function

3. Consider the function $D(x)$ shown the graph below.



(a) For what value of k is $D(x)$ a probability density function for a random variable X?
Solve $\frac{1}{2}(4)(k) = 1$, so $k = 0.5$.

(b) Assuming the value of k from part (a), determine $P(-1 < X < 1)$.
$P(-1 < X < 1) = \frac{1}{2}(2)(0.5) = \boxed{0.5}$.

4. The length of time spent waiting for a red stoplight to change is a uniform random variable X on the interval $[0, 80]$ seconds. Write a formula for the probability density function $U(x)$. How often would you expect to wait less than 10 seconds for the stoplight to change?

$$U(x) = \begin{cases} \frac{1}{80} & \text{, if } 0 \leq x \leq 80 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X < 10) = \int_0^{10} \frac{1}{80}\, dx = \left[\frac{x}{80}\right]\Big|_0^{10} = \left[\frac{10}{80} - \frac{0}{80}\right] = 12.5\%.$$

5. The lifespan of an office copy machine is a random variable T having an expected value of 1.25 years and is described by an exponential probability density function. How likely is it that a copy machine's lifespan will be at least 3 years?

$$P(T > 3) = \int_3^\infty ke^{-kx}\, dx$$

We know $k = 1/EV = 1/1.25 = 0.8$, so

$$P(T > 3) = \int_3^\infty 0.8e^{-0.8x}\, dx = \lim_{b\to\infty} \left[-e^{-0.8x}\right]\Big|_3^b = \lim_{b\to\infty} \left[\cancel{-e^{-0.8b}} \xrightarrow{} 0 - -e^{-0.8\cdot 3}\right] \approx .09$$

6. Find the expected value of the random variable Y described by density function

$$f(y) = \begin{cases} \dfrac{8}{y^3} & \text{, if } y > 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} EV &= \int_{-\infty}^\infty y \cdot f(y)\, dy \\ &= \int_2^\infty y \cdot \frac{8}{y^3}\, dy \\ &= \lim_{b\to\infty} \left[-8y^{-1}\right]\Big|_2^b \\ &= \lim_{b\to\infty} \left[\cancel{-8b^{-1}} \xrightarrow{} 0 - (-8(2)^{-1})\right] \\ &= \boxed{4} \end{aligned}$$