Solutions

1. Compute $\int 6x^{-2} - 12x^{-1} + 8\,dx$

   Applying the addition and constant multiple rules gives

$$\int 6x^{-2} - 12x^{-1} + 8\,dx = 6\int x^{-2}\,dx - 12\int x^{-1}\,dx + \int 8\,dx$$

   and applying the power, natural log, and constant rules respectively gives

$$\begin{aligned}\int 6x^{-2} - 12x^{-1} + 8\,dx &= 6\int x^{-2}\,dx - 12\int x^{-1}\,dx + \int 8\,dx \\ &= -6x^{-1} - 12\ln(|x|) + 8x + C\end{aligned}$$

2. Find the particular solution to the differential equation $\frac{dy}{dx} = e^{5x} + 3\sqrt{x}$, where $y = \frac{6}{5}$ when $x = 0$.

$$\begin{aligned}\frac{dy}{dx} &= e^{5x} + 3\sqrt{x} \\ \int dy &= \int \left(e^{5x} + 3x^{1/2}\right) dx \\ y &= \frac{1}{5}e^{5x} + 2x^{3/2} + C\end{aligned}$$

   Now we use $x = 0$ and $y = \frac{6}{5}$ to find the value of C:

$$\begin{aligned}y &= \frac{1}{5}e^{5x} + 2x^{3/2} + C \\ \frac{6}{5} &= \frac{1}{5}e^{5(0)} + 2(0)^{3/2} + C \\ \frac{6}{5} &= \frac{1}{5} + C \\ C &= 1.\end{aligned}$$

   So then

$$\boxed{y = \frac{1}{5}e^{5x} + 2x^{3/2} + 1}$$

3. Find a general solution to the differential equation $3t \cdot \dfrac{dx}{dt} = x^{-2}$.

   We multiply the dt term over and then isolate x terms with the dx differential, as well as the t terms with the dt differential, then integrate:

$$\begin{aligned} 3t \cdot \frac{dx}{dt} &= x^{-2} \\ \frac{3}{x^{-2}}\, dx &= \frac{1}{t}\, dt \\ \int 3x^2\, dx &= \int \frac{1}{t}\, dt \\ x^3 &= \ln|t| + C \\ x &= \boxed{(\ln|t| + C)^{1/3}} \end{aligned}$$

4. Verify that $y = e^{-3x} + e^x$ is a solution to the differential equation $y'' = 3y - 2y'$.

   To be a solution to the equation, we need to know that replacing the expression for y into the equation should result in both sides being equal. The equation involves both y' and y'', so let's first compute those for the given solution:

$$\begin{aligned} y' &= -3e^{-3x} + e^x \\ y'' &= 9e^{-3x} + e^x \end{aligned}$$

   Now we compute the left hand side and the right hand side of the given differential equation:

$$\begin{aligned} &\text{LHS: } y'' = 9e^{-3x} + e^x \\ &\text{RHS: } 3y - 3y' = 3(e^{-3x} + e^x) - 2(-3e^{-3x} + e^x) \\ &\qquad\qquad\quad = 9e^{-3x} + e^x \end{aligned}$$

   Noting that the left hand side is equal to the right hand side, we conclude that $y = e^{-3x} + e^x$ is a solution to the given differential equation.

5. According to Robert Solow's economic theory, when a portion of all output is reinvested in capital, the rate of change in capital stock, K (in thousands of dollars), can be written in terms of capital stock and time t years from now by the differential equation

$$\frac{dK}{dt} = Se^{bt}K^{1-a},$$

where a, b, and S are all positive constants. Given $K(0) = 1600$, $a = 0.5$, $b = 0.02$, and $S = 2$, find **and interpret including units** the value of $K(5)$. [Hint: in your solution choose the less negative option for the constant]

With constants plugged in, our equation looks like

$$\frac{dK}{dt} = 2e^{0.02t}K^{1-0.5},$$

which we can solve by separating variables:

$$\begin{aligned}
\frac{dK}{dt} &= 2e^{0.02t}K^{0.5} \\
dK &= 2e^{0.02t}K^{0.5}\ dt \\
\frac{1}{K^{0.5}}\ dK &= 2e^{0.02t}\ dt \\
\int K^{-0.5}\ dK &= \int 2e^{0.02t}\ dt \\
\frac{K^{0.5}}{0.5} &= \frac{2}{0.02}e^{0.02t} + C_1 \\
K^{0.5} &= 50e^{0.02t} + C, \text{ where } C = 0.5C_1 \\
K &= \left(50e^{0.02t} + C\right)^2
\end{aligned}$$

Now we can use the initial value $K(0) = 1600$ to find the value of C:

$$\begin{aligned}
K &= \left(50e^{0.02t} + C\right)^2 \\
1600 &= \left(50e^{0.02(0)} + C\right)^2 \\
\pm 40 &= 50 + C \\
C &= -90 \text{ or } -10
\end{aligned}$$

We are instructed to use the least negative option, so with $C = -10$, we get

$$K = \left(50e^{0.02t} - 10\right)^2.$$

Then, finally, we get

$$K(5) = \left(50e^{0.02(5)} - 10\right)^2 \approx 2048.34.$$

Five years from now, capital stock will be about 2048.34 thousand dollars.