Solutions

1. Consider the function $f(x,y) = 24x + 18y - 3x^2 - 9y^2 + 16$.

   (a) Show calculations to verify that $f(x,y)$ has a critical point at $(4,1)$.
   We find critical points by solving both $f_x = 0$ and $f_y = 0$:

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \implies \begin{cases} 24 - 6x = 0 \\ 18 - 18y = 0 \end{cases}$$
$$\implies \begin{cases} x = 4 \\ y = 1 \end{cases}$$

   So $(4,1)$ is a critical point.

(b) Show calculations to verify that $f(x,y)$ has a relative maximum at $(4,1)$.

We know from part (a) that $(4,1)$ is a critical point, so we just need to check that it corresponds to a maximum using the second partials test. We need to know the value of $D(4,1)$, so first we find the second partial derivatives of f:

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(f_x) \\ &= \frac{\partial}{\partial x}(24-6x) \\ &= -6. \end{aligned}$$

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(f_y) \\ &= \frac{\partial}{\partial y}(18-18y) \\ &= -18. \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) \\ &= \frac{\partial}{\partial y}(24-6x) \\ &= 0. \end{aligned}$$

Then we calculate

$$\begin{aligned} D(4,1) &= f_{xx}(4,1)\cdot f_{yy}(4,1) - [f_{xy}(4,1)]^2 \\ &= (-6)\cdot(-18) - [0]^2 \\ &= 108. \end{aligned}$$

So we know that $D(4,1) > 0$. That paired with the fact that $f_{xx}(4,1) < 0$ tells us that $(4,1)$ is the location of a maximum for f.

2. The online retailer, ADress, is offering two summer dresses, the Misty Breeze and the Tahitian Dream. Each Misty Breeze costs \$41 to manufacture and publicize, while each Tahitian Dream costs \$46. The manager of the company estimates that if each Misty Breeze is priced at x dollars and each Tahitian Dream at y dollars, then $101 - 5x + 4y$ Misty breeze dresses and $56 + 6x - 7y$ Tahitian Dream dresses will be sold on average every day.

   (a) Express the total daily profit from the sale of the dresses as a function of x and y.

   The profit is the difference of revenue and cost, and the revenue and cost are each of the form "unit price/cost times number of units produced/sold", so we get

$$\begin{aligned}\text{Profit} &= \text{Revenue} - \text{Cost}\\ P(x,y) &= R_1 + R_2 - (C_1 + C_2)\\ &= x \cdot (101 - 5x + 4y) + y \cdot (56 + 6x - 7y) - [41 \cdot (101 - 5x + 4y) + 46 \cdot (56 + 6x - 7y)]\\ &= -5x^2 + 10xy + 30x - 7y^2 + 214y - 6717.\end{aligned}$$

   (b) What pricing for each dress maximizes the total daily profit? What is the maximum profit from the sale of the dresses?

   We need to find a maximum for the profit function from part (a). First we find critical points, then evaluate using the discriminant D.

$$\begin{cases} P_x = -10x + 10y + 30 = 0 \\ P_y = 10x - 14y + 214 = 0 \end{cases} \implies y = x - 3$$

$$\begin{aligned} 10x - 14(x-3) + 214 &= 0\\ -4x + 256 &= 0\\ x &= 64.\end{aligned}$$

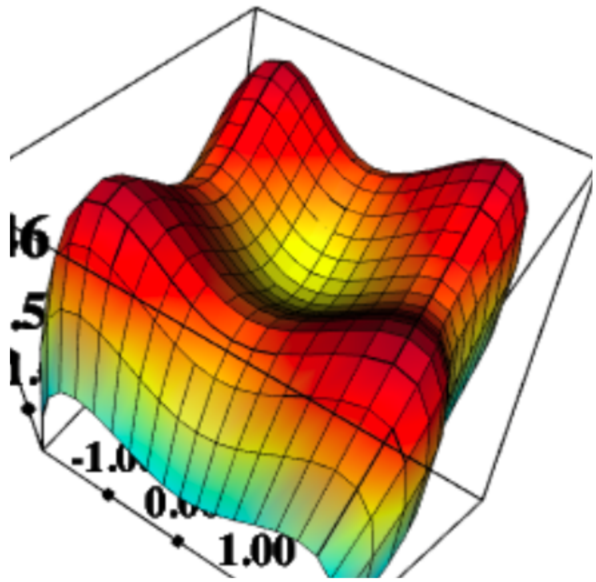
   Then we get

$$y = x - 3 = 64 - 3 = 61.$$

   This is the only critical point, so assuming that there is a maximum at all, it occurs when charging \$64 per Tahitian Dream and \$61 per Misty Breeze for a maximum of \$770 daily profit.

   Note: We can verify that because $P_{xx} = -10$, $P_{yy} = -14$, and $P_{xy} = 10$, that $D = (-10) \cdot (-14) - (10)^2 = 40 > 0$, and since $P_{xx} < 0$ any critical point is a relative maximum.

3. Estimate the number of (a) relative maxima, (b) relative minima, and (c) saddle points of $g(x, y)$ from its graph provided.



There are four peaks in the graph, one valley, and four points where the slope is flat at a point and yet are neither maxima nor minima (in between each of the peaks). Thus there are (a) four maxima, (b) one minimum, and (c) four saddle points.

4. Suppose we know that a function $f(x, y)$ has its only critical point at $(2, -1)$, and that furthermore $f_{xx}(2, -1) = 4$ and $f_{xy}(2, -1) = 2$. What need be true about f_{yy} so that f has a saddle point at $(2, -1)$? Is it possible for f to have a relative maximum at $(2, -1)$? What about a relative minimum?

   In order that f has a saddle point at $(2, -1)$, we need $D(2, -1) < 0$. Specifically, this implies a range of values for $f_{yy}(2, -1)$:

$$\begin{aligned} D(2,1) = f_{xx}(2,-1) \cdot f_{yy}(2,-1) - [f_{xy}(2,-1)]^2 &< 0 \\ 4 \cdot f_{yy}(2,-1) - [2]^2 &< 0 \\ 4 \cdot f_{yy}(2,-1) &< 4 \\ f_{yy}(2,-1) &< 1 \end{aligned}$$

   So if $f_{yy}(2, -1)$ is less than 1, f will have a saddle point at $(2, 1)$.

   For relative maximum, similarly we need $D(2, -1) > 0$, but also $f_{xx}(2, -1) < 0$. We know f_{xx} at this point is 4 (not a negative number), so it isn't possible for $(2, -1)$ to be the location of a relative maximum of f.

   For relative minimum, we need $D(2, -1) > 0$, but also $f_{xx}(2, -1) > 0$. Clearly f_{xx} at this point is already positive (it is equal to 4), so we just need to make sure that $D(2, -1)$ is positive. Following the work from the saddle point calculation, but with the inequality reversed, we see that $f_{yy}(2, -1) > 1$ would guarantee a relative minimum at $(2, -1)$.