Key

Final Exam Review

General Information

The exam will take place on Friday, December 7 at 10:15 am in Deady 106 (in the usual classroom, but not the usual time). You will be permitted to use a *scientific calculator which cannot compute derivatives or integrals*, though a calculator will not be necessary to solve many of the problems on the exam. In particular, the TI 36X Pro is not allowed (though this is certainly not the only such calculator). You must bring your own calculator if you wish to use one. In addition to using this review guide, you should study the example problems we've done in class, homework problems, and previous quick hit problems. You should be prepared to answer questions about...

Topics

- Indefinite integral techniques
  - Basic integral rules (power rule, exponential rule, sum rule, constant multiple rule,etc.)
  - Simplifying the integrand
  - u-substitution
- Differential equations
  - Solving separable differential equations
  - Checking if a given function is a solution to any given differential equation.
- Definite integrals
  - Definition of definite integral
  - Fundamental theorem of calculus
  - Computing definite integrals by computing areas
- Applications
  - Area between two curves
  - Average value (both the formula and the geometric interpretation)
  - Net excess profit
  - Lorenz curves and Gini index
  - Continuous income streams and future/present value
- Improper Integrals
  - Using limits

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- Continuous Probability
  - Terminology (sample space, continuous random variable, probability density function, etc.)
  - Characteristic properties of probability density functions
  - Uniform distributions
  - Exponential distributions
  - Expected value

Formula Sheet

The following formulas will be provided for you on the exam:

Given two investment plans whose profits are described by the functions $P_1(t)$ and $P_2(t)$ with $P_1'(t) \geq P_2'(t)$ on the interval $[a, b]$, the *net excess profit* of the first plan over the second plan from $t = a$ to $t = b$ is

$$\int_a^b P_1'(t) - P_2'(t)\, dt$$

Given a Lorenz curve, $L(x)$, for some society, the *Gini Index* for that society is

$$2\int_0^1 x - L(x)\, dx$$

The *average value* of a function, $f(x)$, on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

The *future value* after T years of an income stream into which money is being deposited at a rate given by $f(t)$ which earns interest compounded continuously at an annual rate, r, is

$$e^{rT}\int_0^T f(t)e^{-rt}\, dt$$

The *present value* of an income stream into which money is being deposited for T years at a rate given by $f(t)$ which earns interest compounded continuously at an annual rate, r, is

$$\int_0^T f(t)e^{-rt}\, dt$$

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If $f(x)$ is a probability density function for a random variable, X, with a *uniform distribution*, then

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

If $f(x)$ is a probability density function for a random variable, X, with an *exponential distribution*, then for some constant, k,

$$f(x) = \begin{cases} ke^{-kx} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

If $f(x)$ is a probability density function for a random variable, X, then the *expected value* of X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

Practice Problems

Note: These examples, while good review, are *not* characteristic of every question I could ask on the final exam.

1. Compute the following integrals

(a) $\displaystyle\int \frac{e^{3x} - e^{-x} + 1}{e^x}\,dx = \int \frac{e^{3x}}{e^x} - \frac{e^{-x}}{e^x} + \frac{1}{e^x}\,dx$

$$= \int e^{2x} - e^{-2x} + e^{-x}\,dx$$

$$= \frac{e^{2x}}{2} + \frac{e^{-2x}}{2} - e^{-x} + C$$

(b) $\displaystyle\int \frac{\ln(x)^3}{x}\,dx = \int \frac{u^3}{x}\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{\ln(x)^4}{4} + C$

$(u = \ln(x)\ ;\ du = \frac{1}{x}\,dx)$

2. Give an example of two distinct functions $f(x)$ and $g(x)$ so that $f'(x) = g'(x) = x^{-4} + e^{5x}$

$$f(x) = \frac{x^{-3}}{-3} + \frac{e^{5x}}{5}$$

$$g(x) = \frac{x^{-3}}{-3} + \frac{e^{5x}}{5} + 1$$

3. A manufacturer has found that the marginal cost of a certain product is $4q^2 - 20q + 300$ dollars per unit when q units have been produced. The total cost of producing ~~the first~~ 2 units is \$800. What is the total cost of producing exactly 5 units? What is the cost of producing the first 5 units? Why are these two numbers different?

$$C'(q) = 4q^2 - 20q + 300$$

$$C(q) = \int 4q^2 - 20q + 300 \, dq = \frac{4q^3}{3} - 10q^2 + 300q + K$$

$$800 = C(2) = \frac{4 \cdot 2^3}{3} - 10 \cdot 2^2 + 300 \cdot 2 + K \approx 570.67 + K$$

$$\rightarrow K \approx 229.33$$

$$\rightarrow C(q) = \frac{4q^3}{3} - 10q^2 + 300q + 229.33$$

Total cost of producing 5 units: $C(5) \approx 1646$ dollars

Cost of producing the first 5 units: $C(5) - C(0) \approx 1416.67$ dollars

These numbers are different because the cost of producing the first 5 units does not include fixed costs, whereas total cost does

4. Is $y = e^{2x} + e^{-3x} + 2x - 1$ a solution to the following differential equation?

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = -12x + 8$$

$\frac{dy}{dx} = 2e^{2x} - 3e^{-3x} + 2$

$\frac{d^2y}{dx^2} = 4e^{2x} + 9e^{-3x}$

LHS: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = (4e^{2x} + 9e^{-3x}) + (2e^{2x} - 3e^{-3x} + 2) - 6(e^{2x} + e^{-3x} + 2x - 1)$

$= (4 + 2 - 6)e^{2x} + (9 - 3 - 6)e^{-3x} + 2 - 12x + 6$

$= -12x + 8$

RHS: $-12x + 8$

Yes $y = e^{2x} + e^{-3x} + 2x - 1$ is a solution to the given differential equation

5. Find the solution, $y(x)$, with $y(0) = 3$ to the differential equation

$$\frac{dy}{dx} = \frac{1}{y^2x + y^2} = \frac{1}{y^2(x+1)}$$

$\rightarrow y^2 \frac{dy}{dx} = \frac{1}{x+1} \rightarrow \int y^2 dy = \int \frac{1}{x+1} dx$

$\rightarrow \frac{y^3}{3} = \ln(|x+1|) + C \rightarrow \frac{3^3}{3} = \ln(|0+1|) + C$

$\rightarrow 9 = \ln(1) + C = C$

$\rightarrow \boxed{\frac{y^3}{3} = \ln(|x+1|) + 9}$

6. Compute the following integrals

(a) $\int_2^5 2t^4 - t^{-5}\,dt = \frac{2t^5}{5} - \frac{t^{-4}}{-4}\Big|_2^5 \approx 1237.2$

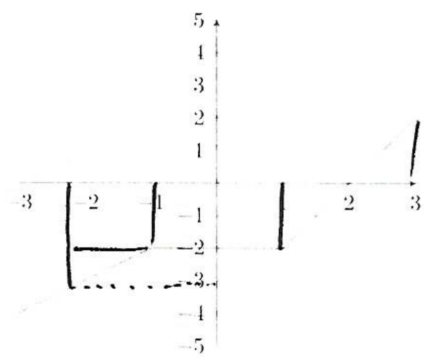
(b) $\int_{-3}^{-1} (2s+1)e^{s^2+s-3}\,ds = \int_{s=-3}^{s=-1} e^u\,du = e^u\Big|_{s=-3}^{s=-1} = e^{s^2+s-3}\Big|_{-3}^{-1}$

$(u = s^2+s-3;\ du = 2s+1\,ds)$

$= e^{-3} - e^{3}$

≈ -20.04

7. The following is the graph of a function, $f(x)$.



Compute the following integrals:

(a) $\int_{-2}^{0} f(x)\,dx = -\left(\frac{1}{2}\right)(1)(1) - (1)(1) - (1)(1) = -\frac{5}{2}$

(b) $\int_{1}^{3} f(x)\,dx = -\left(\frac{1}{2}\right)(1)(1) + \left(\frac{1}{2}\right)(1)(1) = 0$

(c) $\int_{-1}^{2} f(x) + x\,dx = \int_{-1}^{2} f(x)\,dx + \int_{-1}^{2} x\,dx = -(2)(1) - \left(\frac{1}{2}\right)(1)(1) + \left(\frac{x^2}{2}\Big|_{-1}^{2}\right.$

$= -\frac{5}{2} + \left(\frac{2^2}{2} - \frac{(-1)^2}{2}\right)$

$= -1$

8. Shares of a stock are traded at a rate approximated by $S'(t) = 3\sqrt{t}$ thousand shares per day, on the tth day of the year. How many shares are traded from day 31 to day 59?

$$S(59) - S(31) = \int_{31}^{59} S'(t)\,dt = \int_{31}^{59} 3\sqrt{t}\,dt$$

$$= 3\frac{t^{3/2}}{(3/2)}\Big|_{31}^{59} = 2\cdot 59^{3/2} - 2\cdot 31^{3/2}$$

$$\approx 561 \text{ shares}$$

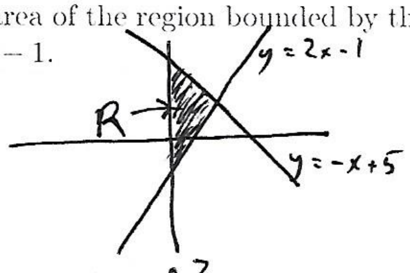
9. Compute the net excess profit between the first and second year for two investments whose profit functions are $A(t)$ and $B(t)$ respectively, with $A'(t) = \frac{10}{t}$ and $B'(t) = \frac{10}{t^2}$ thousand dollars per year.

When $t \geq 1$, $\frac{10}{t^2} \leq \frac{10}{t}$ (because squaring a number larger than 1 makes it larger, so the fraction gets smaller).

Hence, net excess profit is

$$\int_1^2 \frac{10}{t} - \frac{10}{t^2}\,dt = 10\ln(|t|) + \frac{10}{t}\Big|_1^2 \approx 1.93 \text{ thousand dollars}$$

10. Find the area of the region bounded by the y-axis and the curves $f(x) = -x + 5$ and $g(x) = 2x - 1$.



$2x - 1 = -x + 5$

$\rightarrow 3x = 6 \rightarrow x = 2$

$$\text{area}(R) = \int_0^2 (-x+5) - (2x-1)\,dx$$

$$= \int_0^2 -3x + 6\,dx = -\frac{3x^2}{2} + 6x\Big|_0^2$$

$$= 6$$

11. Let $L(x)$ be a Lorenz curve for some society.

(a) On what interval is $L(x)$ defined?

$[0, 1]$

(b) What is $L(0)$? Why?

$L(0) = 0$ because 0% of any society earns 0% of its income.

(c) What is $L(1)$? Why?

$L(1) = 1$ because 100% of any society earns 100% of its income

(d) Can there be an interval on which $L(x)$ is decreasing? Why or why not?

No such interval can exist because if you enlarge a set of people (corresponding to enlarging x), then they'll make at least as much money as the smaller set of people

(e) What is $L(x)$ for a society with perfect equality?

$L(x) = x$

(f) Can there be an x with $L(x) > x$? Why or why not?

No such x can exist because $100x$% of the population cannot earn more than $100x$% of the wealth

12. What is the Gini Index for a society with Lorenz curve $L(x) = \frac{1}{5}x^2 + \frac{1}{5}x$?

$$G.I. = 2\int_0^1 x - \left(\frac{4}{5}x^2 + \frac{1}{5}x\right)dx = 2\int_0^1 -\frac{4}{5}x^2 + \frac{4}{5}x\,dx$$

$$= 2\left(-\frac{4}{15}x^3 + \frac{2}{5}x^2\Big|_0^1\right) = \frac{4}{15} \approx .27$$

13. The rate of production of an oil field is given by $R(t) = \frac{10t}{t^2+1}$ hundred thousand barrels per year, t years from initial extraction. What is the ~~averate~~ average rate of yearly production over the 20-year lifetime of the oil field?

Average of $R(t)$ is $\frac{1}{20-0}\int_0^{20} \frac{10t}{t^2+1}dt = \frac{1}{20}\int_{t=0}^{t=20} \frac{10t}{u}dt$

$(u = t^2+1\ ;\ du = 2t\,dt)$

$$\frac{1}{20}\int_{t=0}^{t=20} \frac{5}{u}du = \frac{5}{20}\ln(|u|)\Big|_{t=0}^{t=20} = \frac{1}{4}\ln(|t^2+1|)\Big|_0^{20} \approx 1.5$$ hundred thousand barrels per year

14. Find the future value of a continuous income stream into which money is being added at a rate of $3e^{-.1t}$ thousand dollars per year and which earns interest compounded continuously at an annual interest rate of 5%. for 6 years

$$FV = e^{.05\cdot 6}\int_0^6 3e^{-.1t}\cdot e^{-.05t}dt = 3e^{.3}\int_0^6 e^{-.15t}dt$$

$$= 3e^{.3}\left(\frac{e^{-.15t}}{-.15}\Big|_0^6\right) \approx 16 \text{ thousand dollars}$$

15. You wish to provide a gift to the UO business school to fund a professor's position for the next 10 years. Assuming that the current interest rate is 3% and you want the account to continuously pay out \$100,000 per year, how much do you need to donate?

$$PV = \int_0^{10} 100000\,e^{-.03t}dt = \frac{100000\,e^{-.03t}}{-.03}\Big|_0^{10}$$

$\approx 864,000$ dollars

16. Does the following integral converge or diverge? $\int_5^\infty \frac{3}{\sqrt{t}}\,dt$

$$\int_5^\infty \frac{3}{\sqrt{t}}\,dt = \lim_{b\to\infty} \int_5^b 3t^{-1/2}\,dt = \lim_{b\to\infty} \frac{3t^{1/2}}{(1/2)}\Big|_5^b$$

$$= \lim_{b\to\infty} 6\sqrt{b} - 6\sqrt{5} = \infty$$

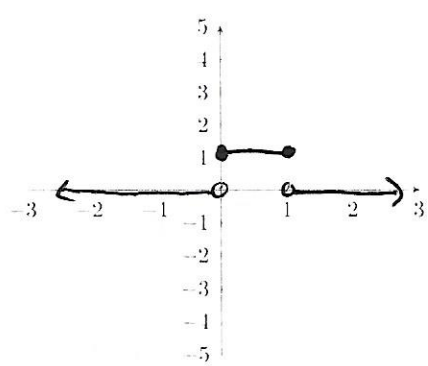
Therefore the integral diverges

17. Consider the experiment in which you randomly select a student at UO and measure their height. What is the sample space? What is the continuous random variable?

The sample space is the set of students at UO

The continuous random variable is the function that takes a student at UO as input and outputs that student's height

18. Draw a probability density function on the following axes:



19. The number of grams of sugar in a randomly selected bottle of Treetop apple juice is uniformly distributed and can range from 300 grams to 310 grams.

(a) Write a formula for the probability density function of this experiment.

$$f(x) = \begin{cases} \frac{1}{310-300} & 300 \leq x \leq 310 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{10} & 300 \leq x \leq 310 \\ 0 & \text{else} \end{cases}$$

(b) What is the probability that a randomly selected bottle of Treetop apple juice is between 301 and 304 grams?

$$P(301 \leq X \leq 304) = \int_{301}^{304} f(x)\,dx = \int_{301}^{304} \frac{1}{10}\,dx = \frac{1}{10}x \Big|_{301}^{304}$$

$$= .3$$

20. The price of a random pair of earbuds on Amazon is exponentially distributed with probability density function $f(x) = \begin{cases} .2e^{-.2x} & x \geq 0 \\ 0 & \text{else} \end{cases}$ What is the probability that a randomly selected pair of earbuds costs more than \$20?

$$P(X \geq 20) = \int_{20}^{\infty} .2e^{-.2x}\,dx = \lim_{b\to\infty} \int_{20}^{b} .2e^{-.2x}\,dx$$

$$= \lim_{b\to\infty} \frac{.2e^{-.2x}}{-.2}\Big|_{20}^{b} = \lim_{b\to\infty} -e^{-.2x}\Big|_{20}^{b}$$

$$= \lim_{b\to\infty} -e^{-.2b} + e^{-.2\cdot 20} = e^{-.2\cdot 20} \approx .02$$

21. For what value(s) of c is the function

$$f(x) = \begin{cases} \cancel{0.5 - x^2} \quad c - .5x^2 & -1 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

a probability density function?

$$1 = \int_{-1}^{1} c - .5x^2 \, dx = cx - \frac{.5x^3}{3} \Big|_{-1}^{1}$$

$$= \left(c - \frac{1}{6}\right) - \left(-c + \frac{1}{6}\right) = 2c - \frac{2}{3}$$

$$\to \frac{5}{3} = 2c \to \boxed{c = \frac{5}{6}}$$

Check that graph is above axis when $c = \frac{5}{6}$

22. What is the expected value of a random variable, X, with probability density function equal to [redacted] on the interval [redacted]?

$\cancel{E(X) = \int_0^{1.5} x \cdot \frac{1}{x} \, dx = \int_0^{1.5} \frac{1}{x} \, dx = \ln|x|}$

pdf: $\frac{2}{x^2}$ on $[1, 2]$

$$E(X) = \int_1^2 x \cdot \frac{2}{x^2} \, dx = \int_1^2 \frac{2}{x} \, dx = 2 \ln(|x|) \Big|_1^2$$

$$= 2\ln(2) - 2\ln(1) \approx \boxed{1.39}$$