Chapter 1 Lecture Notes

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Section 1.1: Parent Functions

- One of the more challenging problem types in 111 is the problem of graphing functions.
- Without just testing a bunch of points and connecting some lines, how do you graph functions?
- Main idea of this section: there are some functions that are easy to graph (called parent functions) and most other functions that we're interested in graphing are going to be transformations of those parent functions.
- So let's first learn what the parent functions are and look like.
- Let's start with an easy graph: $f(x) = x$ (draw on board)
  - Domain
  - Image
  - Long term behavior
  - Zeroes
- Let's step it up a little and look at $f(x) = x^2$
  - Domain
  - Image
  - Long term behavior
  - Zeroes
- Repeat with x^3 and x^4
- What patterns do you notice (LTB and zeroes)?
- Let's generalize this to $f(x) = x^p$ for positive whole numbers p (recall that these are called "monomials")
- What about the negative whole numbers though?
- Go through same process where you cover...
  - Domain
  - Image
  - Long term behavior
  - Zeroes
- ...for $f(x) = \frac{1}{x^p}$ (recall that these are called "basic rational functions"), again starting with $p = 1$, doing examples through $p = 4$, and then generalizing

- So now that we've dealt with $f(x) = x^p$ for all whole numbers p (except 0, why have we skipped 0?), let's look at $f(x) = x^{1/p}$ for (positive) whole numbers p (recall these are called the "basic power functions").
- Start with $p = 1$ through $p = 4$ and don't forget to discuss
  - Domain
  - Image
  - Long term behavior
  - Zeroes
- If we have time at the end of the week, we'll talk about $f(x) = x^{1/-p}$ and $f(x) = x^{p/q}$ more generally, but for now, we'll call this sufficient.
- Next up: e^x and $\ln(x)$
- Then discuss even and odd functions, by using x^p as a motivating example for each type.
  - A function $f(x)$ is **even** if, for all x in the domain of f, $f(-x) = f(x)$
  - A function $f(x)$ is **odd** if, for all x in the domain of f, $f(-x) = -f(x)$
- How do we check if a given function is even or odd? If given a formula, compute $f(-x)$
- Example:
  - $f(x) = \frac{x^4}{x^2+1}$ is even
  - $f(x) = x^5 + x^3$ is odd
  - $f(x) = x + 1$ is neither
  - $f(x) = 0$ is both even and odd! (This is the only function with this property)
- If given a graph, what are we looking for?
- Lemma: Even functions are those functions which have reflective symmetry across the y-axis. Odd functions are those which have 180° rotational symmetry about the origin.
- Justify this
- Graph some wonky looking even and odd functions on the board

<u>Section 1.2</u>: Vertical Transformations

- There are three primary types of vertical transformations that we can do.
  - Translations
  - Reflections
  - Stretches
- Say we want to translate the classic parabola up one unit. How might we do that?
- Let's find some points first and see.
- Then talk about this for general functions. Shifting up 1 unit means adding 1 to all the y-values on the graph, so we'd better think about $f(x) + 1$.
- Generalize to $f(x) + k$ (mention shifting down)
- What about reflecting across the x-axis? Make our y-values negative!
- What about stretching by a factor of A? Multiply by A!

- Each transformation on its own is fairly straightforward
  - If we want to graph $x^3 + 1$, we shift the graph of x^3 up 1 unit
  - If we want to graph $-\ln(x)$, we reflect the graph of $\ln(x)$
  - If we want to graph $\frac{1}{3}\sqrt{x}$, we stretch the graph of $\sqrt{x}$ by a factor of $\frac{1}{3}$
- What about combining transformations?
- It's important to note that the order of transformations (sometimes) matters: consider the difference between starting with x^3, first translating and then stretching, or first stretching and then translating. Keep track of the "center" of the graph.
- Let's look at an example that we're all familiar with: the line
  - We know that a linear function can always be represented by $g(x) = mx+b$ for some real numbers m and b.
  - What does this mean in context? It means that we first stretch the parent function $f(x) = x$ by a factor of m and then we shift the line up/down by b
  - Hence, we can get every line by some combination of vertical tranformations of $f(x) = x$.
  - Can we get every parabola that way?

<u>Section 1.3</u>: Horizontal Transformations

- To get certain parabolas, we have to have horizontal transformations as well
- We again, have three types of horizontal transformations
  - Translations
  - Reflections
  - Stretches
- Let's shift $f(x) = x^2$ one unit to the right
- Generalize to $f(x - h)$
- Let's stretch $f(x) = x^2$ by a factor of 3
- Generalize to $f(Bx)$
- Let's reflect $f(x) = x^3$ across the y-axis
- Generalize to $f(-x)$
- Again, each transformation individually is fairly straightforward
  - If you want to graph $f(x) = \ln(x + 1)$, shift graph of $\ln(x)$ to the *left* 1 unit
  - If you want to graph $f(x) = \sqrt{-x} = (-x)^{1/2}$, reflect graph of $\sqrt{x}$ across y-axis
  - If you want to graph $f(x) = (5x)^2$, stretch the x-values by a factor of $\frac{1}{5}$
- Notice with this last example, we could rewrite $f(x) = (5x^2) = 25x^2$
  - This tells us that *for this particular function*, a horizontal stretch by a factor of $\frac{1}{5}$ is the same as a vertical stretch by a factor of 5.
  - So in this case, there are multiple different ways to think about the same transformation
  - What about the function $f(x) = \ln(5x)$?
  - Rewrite this as $f(x) = \ln(5) + \ln(x)$, so this is a vertical translation by $\ln(5)$
  - So stretches don't always translate to stretches

  - Sometimes, stretches don't translate into anything (draw something spiky)

- As with vertical transformations, order of horizontal transformations matters:
  - The function $\ln(3x-1)$ is different from $\ln(3(x-1))$
  - Note that, unexpectedly, $\ln(3x-1)$ is shift by 1, then stretch by $\frac{1}{3}$, whereas $\ln(3(x-1))$ is stretch by $\frac{1}{3}$ then shift by 1. This looks like the *opposite* of what order of operations suggests.

- How to remember which transformations are vertical and which are horizontal: remember that vertical transformations affect the y-values and horizontal transformations affect the x-values. So vertical transformations should occur on the "outside" of functions while horizontal transformations should occur on the "inside"

<u>Section 1.4</u>: General Transformations

- First, summarize all vertical and horizontal transformations.
- It turns out that we can do as many of these transformations as we want at a time
- We call a function g a *transformation* of another function f if there are numbers A, B, k, and h, so that for all x, $g(x) = Af(B(x-h)) + k$.
- If you want to graph a transformation, it's important to first write it in the form of a general transformation (with A, B, k, and h in the appropriate locations), and then graph in the following order
  1. Vertical stretch by A
  2. Vertical shift by k
  3. Horizontal stretch by $\frac{1}{B}$
  4. Horizontal shift by h.
- Note: the book says that you have to graph things in this exact order. This is not true! There are many orders that you can do these transformations in. The important thing is that you do vertical stretches before vertical shifts and horizontal stretches before horizontal shifts. Any ordering which has that property is fine.
- Example 1 (already in transformation form): graph $g(x) = 2e^{-(x-3)} + 4$
- Example 2 (not in transformation form): graph $g(x) = (2x-1)^{1/3}$
- How do we reverse this process? If you are given a graph, can you come up with an equation for the function that describes the graph?
  - In general, this is easier said than done. But with a little bit of guidance (e.g. if I also give you the parent function), you can do it!
  - Example, graph $g(x) = -3(x+1)^2 + 5$ and figure out the equation, given that the parent function is $f(x) = x^2$ (not x^4 or x^6 etc.)

<u>Section 1.5</u>: Justification (Beyond Functions)

- Functions definitely aren't the only thing that we can graph
- For example, we can graph circles, but those definitely aren't functions (why?)
- How do we describe a circle of radius 1?
  - justify the equation for this circle
  - How do we tell if a given point is on the circle? (e.g. (.5, .5))

- Circles aren't the only non-functions we can graph. Most things that are equations involving only x, y, and numbers are things that we can graph, even if they're not usually very easy to graph.
- **Def:** the *graph* of an equation is the set of all points (x, y) which satisfy that equation (i.e.), when you plug them into the equation, you get something true back out.
- Example: is the point $(2, -\sqrt{5})$ on the graph of $y^2 = x^3 - 2x + 1$? What about $(1, \sqrt{3})$?
- How do we transform the graphs of these equations?
  - Horizontal transformations are going to work the exact same way.
  - Replacing x by $x - h$ shifts the graph to the right h units
  - Replacing x by Bx stretches the graph horizontally by a factor of $\frac{1}{B}$
  - What about vertical transformations?
  - Earlier, we didn't have a y to deal with, at least not explicitly. But what we were really graphing was $y = Af(B(x-h)) + k$ (stretch by A, then shift k), so we could rewrite this as $\frac{1}{A}(y - k) = f(B(x - h))$.
  - This tells us that replacing y by $y - k$ shifts up by k units
  - Replacing y by $\frac{1}{A}y$ stretches by a factor of A. Equivalently, replacing y by Ay gives a stretch by a factor of $\frac{1}{A}$.
  - And this is great because our vertical and horizontal transformations are completely symmetric. Subtracting means shift right/up and multiplying means stretching by the reciprocal.
- **Ex:** Graph the equation $(2(x-1))^2 + (3(y+2))^2 = 1$. (after doing the graph) How is this different from the graph of $(-2(x-1))^2 + (3(y+2))^2 = 1$?
- Let's look at our favorite functions: lines.
- All lines have the same parent function: $p(x) = x$. And we know that we can get every line by doing a vertical stretch, then a vertical shift of $p(x)$. But also note that we can get every line by doing a vertical stretch, then a vertical shift, then a horizontal shift, where we are moving the origin to any point on the line.
- So if (x_0, y_0) is a point on a line with slope m, the equation of that line can be written as $y - y_0 = m(x - x_0)$. This is called *point-slope* form and it's handy in calculus.
- Example: the line that passes through $(1, -3)$ with slope -4 has equation $y + 3 = -4(x - 1)$.
- Example: the line that passes through $(2, 5)$ and $(-3, 4)$ has slope $\frac{4-5}{-3-2} = \frac{1}{5}$, so it has equation $y - 5 = \frac{1}{5}(x - 2)$ or $y - 4 = \frac{1}{5}(x + 3)$.

<u>Section 1.6</u>: Periodic Functions

- Motivating example: consider a ferris wheel with radius 80 meters, positioned so that it is 100 meters off the ground. The radius spins so that each second, it rotates exactly 1 degree counterclockwise. Consider $h(t)$, the function which gives your position as a function of t, which describes the number of seconds since you have boarded the ferris wheel. Let's plot some points on the graph of $h(t)$.
- Hey, notice how this graph repeats itself every 360 seconds?
- **Def:** We call this being *periodic*. More generally, a function f is *periodic* if there exists a number p so that $f(x + p) = f(x)$ for all x in the domain of f. The *period* of f is the smallest p with this property.
- Notice that $f(x + p)$ is one of our transformations (which one?). So periodic functions with period p are the functions such that when you shift the graph to the left p units, you get the exact same graph.
- Examples: are the following functions periodic? Draw graphs which are...

- ...periodic that aren't sin or cosine (maybe a wiggle at the top)
- ...periodic with asymptotes
- ...not periodic, but repetitive

- What's the deal with this "smallest" p? Why did we say that? Use one of the above graphs to show that period could be p, $2p$, $3p$, etc. if we don't define it to be the smallest one
- Note that a periodic function is *completely* determined by its output values on any interval of length p (i.e. $[a, a+p)$ for some a). After all, if I want to know what $f(x)$ is for any x, I can simply move x some number of periods so that it fits in that interval.
- **Example:** Consider the function f, which has period 5. On the interval $[-2, 3)$, $f(x) = -x^2 + 1$. Find $f(1)$. Find $f(4)$. Find $f(-10)$. Graph f. Find all x values so that $f(x) = \frac{1}{2}$. Find all x values so that $f(x) = -\frac{7}{2}$.
- Let's go back to our ferris wheel height function. What are some other interesting features of this graph?
- There is an obvious midline. What is the height of the midline of the function?
- **Def:** This motivates the following definitions. If $f(x)$ is a periodic function with maximum value M and minimum value m, then the *midline* of f is the line given by $y = \frac{M+m}{2}$. The *amplitude* of f is the quanitity $\frac{1}{2}(M - m)$.
- Note that the amplitude is the distance from the midline to either the max or the min.
- Note also that the midline and amplitude don't have to exist. If f has no min or no max, then they don't exist.
- **Example:** Draw a graph of a periodic function. Find the period, midline, and amplitude of the function.
- Perhaps it's worth observing that none of our parent functions are periodic. This should be a hint that we're going to have some new parent functions in the nearish future.

<u>Notes</u>

- Make sure to touch on point-slope form of a line (but this should be in 1.4)
- Idea: assign challenge problem, but say you can either get it right, or you can write a couple sentences explaining what you tried and why you aren't sure how to do it.
- Idea: give students access to webwork to use for practice problems and steal a test question from the webwork
- On exam, don't ask students to graph transformations of x^2 or anything easy to compute by hand. Either do exponential, log, fractional power, or very large integer power of x.