Key

Final Exam Review

General Information

The exam will take place on Thursday, June 14 at 2:45 pm in Deady 205 (in the usual classroom, but not the usual time). You will be permitted to use a *scientific calculator only*, though a calculator will not be necessary to solve many of the problems on the exam. You must bring your own calculator if you wish to use one. You will be permitted to use a piece of 8.5×11 inch sheet of paper with whatever notes you want on it. In addition to using this review guide, you should study the example problems we've done in class, homework problems, and previous quick hit problems. You should be prepared to answer questions about...

Topics

- Parent functions
  - Domain
  - Image
  - Long-term behavior
- Even and odd functions
- Vertical transformations
- Horizontal transformations
- Combinations of vertical and horizontal transformations
- Point-slope form of a line
- Transformations of graphs of equations
  - Transforming the circle
- Periodic functions
  - Period
  - Amplitude
  - Midline
  - Computations
  - Graphs
- Triangles
  - Right triangles

    * Using sine, cosine, and tangent
    * The Pythagorean Theorem
  - General triangles
    * Sum of interior angles
    * Law of sines and cosines

- Unit circle
- Trig functions
  - Definitions
  - Geometric meaning (e.g. cosine is x-coordinate, sine is y-coordinate, tangent is slope)
  - Domain, image, period, midline, amplitude
  - Pythagorean Identity
  - Special angles
  - Graphs (and all transformations)
- Inverse trig functions
  - When can they be used? When can't they?
- Radians
- Arc length
- Solving trigonometric equations
- Sinusoidal Functions
- Trigonometric Identities
- Vectors
  - Vector operations
  - Polar and rectangular/Cartesian forms
  - The angle between vectors
  - Force diagrams

Final Exam Review

Trigonometric Identities

Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$
$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Angle Sum Formulas

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Power Reduction Formulas

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$
$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

Final Exam Review

Practice Problems

True/False Questions:

1. True or False: The period of $\sin(2\theta)$ is π.

   period $= \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

2. True or False: The midline of $3(\cos(\theta)+2)$ is $y = 2$.

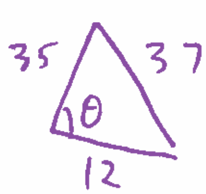
   $3(\cos(\theta)+2) = 3\cos(\theta) + 6 \to$ midline: $y = 6$

3. True or False: There is exactly one angle θ between 0 and 2π such that $3\sin^2(\theta) = 3$.

   $3\sin^2(\theta) = 3 \to \sin^2(\theta) = 1 \to \sin(\theta) = \pm 1$

   $\to \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

4. True or False: A triangle having sides of length 12, 35, and 37 has a 95° angle.



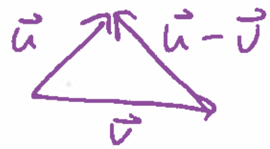
if there is a 95° angle, its across from the side length of 37 (since 37 is the longest side). ~~In that case~~ The law of cosines tells us that $37^2 = 35^2 + 12^2 - 2\cdot 35\cdot 12\cos\theta$

so $0 = \cos\theta$, implying that $\theta = 90°$

5. True or False: If the angle between two vectors $\vec{u}$ and $\vec{v}$ is $100°$, then $\vec{u} \cdot \vec{v} > 0$.

$\cos(100) < 0$ and since $\vec{u} \cdot \vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos(100)$,
$\vec{u} \cdot \vec{v} < 0$

6. True or False: If $\vec{u}$, $\vec{v}$, are nonzero vectors with $\vec{u} \neq \vec{v}$, then $\vec{u}$, $\vec{v}$, and $\vec{u} - \vec{v}$ form three sides of a triangle.



7. True or False: Any pair of vectors $\vec{u}$ and $\vec{v}$ satisfies $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$.

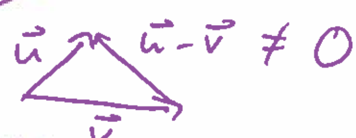
$\|\vec{i} + \vec{j}\| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 2 = \|\vec{i}\| + \|\vec{j}\|$
There are many examples where $\|\vec{u}+\vec{v}\| \neq \|\vec{u}\| + \|\vec{v}\|$

8. True or False: The angle $\theta = \dfrac{3\pi}{2} + n\pi$ is a valid solution to $\sin(\theta) = -1$ for any integer n.

$\sin\left(\frac{3\pi}{2} + (-1)\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1 \neq -1$

9. True or False: The vectors $a = -2t\vec{i} + \vec{j}$ and $\vec{b} = \vec{i} + 2t\vec{j}$ are perpendicular for any t.
$\vec{a} \cdot \vec{b} = -2t + 2t = 0$ for all t

10. True or False: If $\|\vec{u}\| = \|\vec{v}\|$, then $\|\vec{u} - \vec{v}\| = 0$.



11. True or False: For any angle θ, $\sin(\theta) = \sin(-\theta)$.

$\sin\left(\frac{\pi}{2}\right) = 1$
$\sin\left(-\frac{\pi}{2}\right) = -1$

Free Response

1. Consider the function $f(x) = 4\sin(2\pi(x-8)) - 1$.

   (a) State the parent function $p(x)$ and describe the transformations done to the graph of $p(x)$ to obtain the graph of $f(x)$, *in the correct order.*

   $p(x) = \sin(x)$

   1. vertical stretch by a factor of 4
   2. horizontal stretch by a factor of $\frac{1}{2\pi}$
   3. vertical shift down 1 unit
   4. horizontal shift right 8 units

   (b) Find *all* x such that $f(x) = -3$.

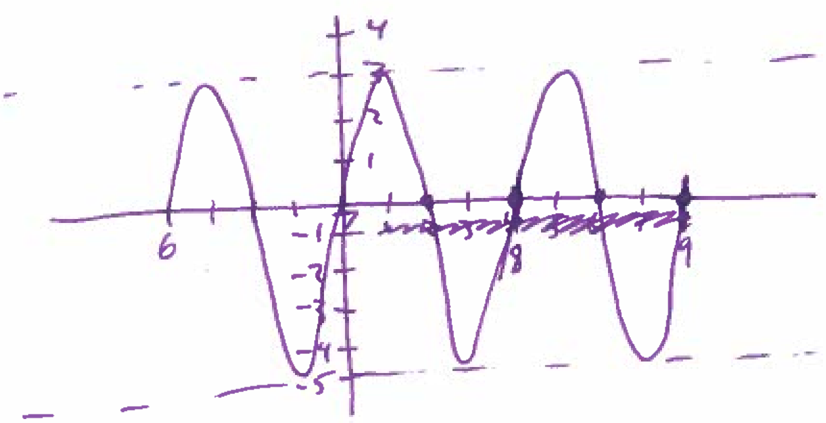
   $-3 = 4\sin(2\pi(x-8)) - 1 \to -2 = 4\sin(2\pi(x-8)) \to -\frac{1}{2} = \sin(2\pi(x$

   $\to 2\pi(x-8) = -\frac{\pi}{6} + 2\pi n$ or $2\pi(x-8) = \frac{7\pi}{6} + 2\pi n$

   $\to x - 8 = -\frac{1}{12} + n$ or $x - 8 = \frac{7}{12} + n$

   $\to x = \frac{95}{12} + n$ or $x = \frac{103}{12} + n$ for integers n

   (c) Sketch a graph of f.



2. Suppose the function $f(x)$ has domain $[-4, \infty)$ and image $[2, 5]$. What are the domain and image of $g(x) = 3f(-2x + 2) + 1$? vertical transformations are stretch by 3, shift up 1. so image gets stretched by 3, shifted up 1. $[2,5] \xrightarrow{\text{stretch}} [6, 15] \xrightarrow{\text{shift}} [7, 16]$

horizontal transformations are stretch by -2, shift right 1 (since $-2x+2 = -2(x-1)$), so the domain becomes $[-4,\infty) \xrightarrow{\text{stretch}} (-\infty, 8] \xrightarrow{\text{shift}} (-\infty, 9]$

3. Determine the parity of the following functions. Provide a justification for your answer.

(a) $g(x) = x^2 + \cos(x)$

$g(-x) = (-x)^2 + \cos(-x) = x^2 + \cos(x) = g(x)$

hence, $g(x)$ is even

(b) $h(x) = \csc(x)$.

$h(-x) = \csc(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin(x)} = -\csc(x) = -h(x)$

hence, $h(x)$ is odd

(c) $f(x) = (x^2 - x^5)e^{x^2}$

$f(-x) = ((-x)^2 - (-x)^5)e^{(-x)^2} = (x^2 + x^5)e^{x^2} \neq f(x)$

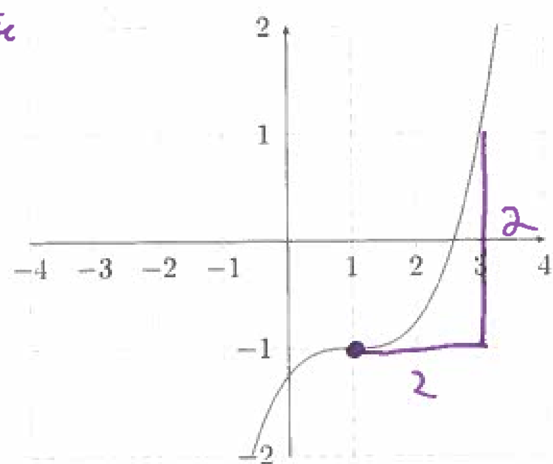
$\neq -f(x)$

hence, $f(x)$ is neither even, nor odd

4. Below is the graph of a function, $f(x)$, which is a transformation of the parent function, $p(x)$. Find a formula for $f(x)$.

(a) $p(x) = x^3$

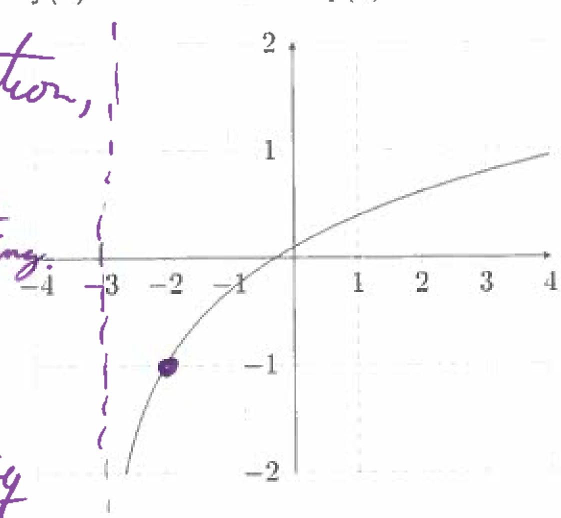
going right 2 from the "focal point" goes up 2 on the graph of f. On the graph of p(x), going right 2 from the "focal point" goes up 8 units. Hence, our vertical stretch is by a factor of $\frac{1}{4}$



$f(x) = \frac{1}{4}(x-1)^3 - 1$

(b) $p(x) = \ln(x)$ *Hint:* $f(x)$ is a translation of $p(x)$.

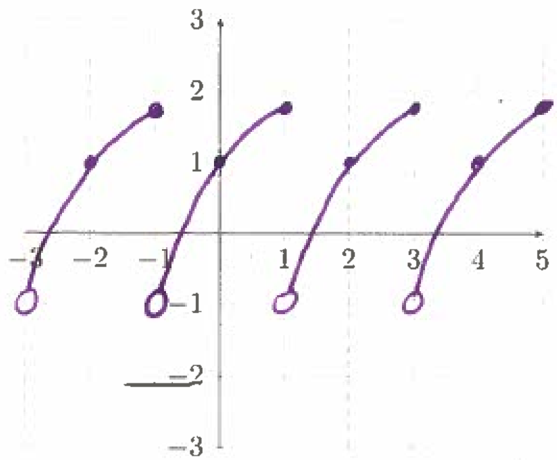
since f(x) is a translation, we don't have to worry about ~~stretching~~. Since the asymptote occurs at $x=-3$ and there are no stretches, the y-value at $x=-2$ must be the vertical shift



$f(x) = \ln(x+3) - 1$

5. Consider the function $f(x)$ which is periodic with period 2. On the interval $(-1, 1]$, $f(x)$ is defined by the equation $f(x) = 2\sqrt{x+1} - 1$.

(a) Graph $f(x)$ on the axes below.



(b) Compute $f(1)$

(c) Compute $f(12)$

(d) Compute $f(-7)$

(e) Find all solutions to $f(x) =$ 1

b) 1 is in $(-1, 1]$, so $f(1) = 2\sqrt{1+1} - 1 = 2\sqrt{2} - 1$

c) $f(12) = f(10) = f(8) = \ldots = f(0) = 1$

d) $f(-7) = f(-5) = f(-3) = f(-1) = f(1) = 2\sqrt{2} - 1$

e) $1 = 2\sqrt{x+1} - 1 \rightarrow 2 = 2\sqrt{x+1} \rightarrow 1 = \sqrt{x+1} \rightarrow 1 = x+1$

$\rightarrow x = 0$. is the only solution in $(-1, 1]$.

Hence all solutions have the form $x = n$ for integers

6. Is the point $\left(-\frac{6}{7}, \frac{\sqrt{13}}{7}\right)$ on the unit circle? Provide a justification for your answer.

$$\left(-\frac{6}{7}\right)^2 + \left(\frac{\sqrt{13}}{7}\right)^2 = \frac{36}{49} + \frac{13}{49} = \frac{49}{49} = 1$$

Since $\left(-\frac{6}{7}, \frac{\sqrt{13}}{7}\right)$ satisfies $x^2+y^2=1$, the point is on the unit circle

7. Suppose θ is an angle in the third quadrant so that $\sin(\theta) = 2/9$. Find the *exact* value of $\sec(\theta)$.

$$\sin^2\theta + \cos^2\theta = 1 \rightarrow \frac{4}{81} + \cos^2\theta = 1 \rightarrow \cos^2\theta = \frac{77}{81}$$

$\rightarrow \cos\theta = \frac{-\sqrt{77}}{81}$ since θ lies in the third quadrant

Hence, $\sec\theta = \frac{1}{\cos\theta} = \frac{-81}{\sqrt{77}}$

8. Prove the following trigonometric identity:

$$\sin^2\frac{\theta}{2} = \frac{\tan\theta - \sin\theta}{2\tan\theta}.$$

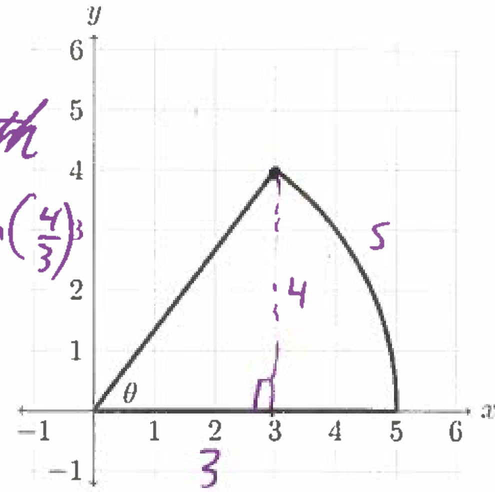
$$\frac{\tan\theta - \sin\theta}{2\tan\theta} = \frac{\frac{\sin\theta}{\cos\theta} - \sin\theta}{2\,\frac{\sin\theta}{\cos\theta}} = \left(\frac{1}{2}\right)\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos\theta} - \sin\theta\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sin\theta}\right)\left(\sin\theta - \sin\theta\cos\theta\right) = \frac{1-\cos\theta}{2} = \sin^2\left(\frac{\theta}{2}\right)$$

9. For the figure below, find the angle θ and the length of the arc.

$\tan\theta = \frac{4}{3} \rightarrow \theta = \arctan\left(\frac{4}{3}\right)$

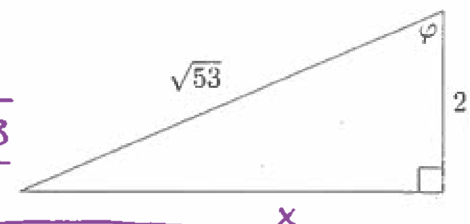
the radius is 5, so the length of the arc is $s = r\theta = 5\arctan\left(\frac{4}{3}\right)$



10. Using the triangle below, find the *exact* value of $\csc(\varphi)$.

$x^2 + 2^2 = (\sqrt{53})^2 \rightarrow x^2 = 49 \rightarrow x = 7$

$\csc(\varphi) = \frac{1}{\sin(\varphi)} = \frac{1}{(7/\sqrt{53})} = \frac{\sqrt{53}}{7}$



11. Find the *exact* value of the following.

(a) $\cos(75°)$

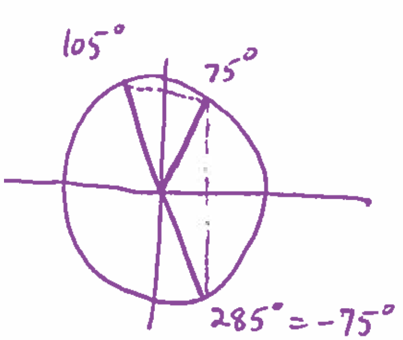
$\cos^2(75) = \frac{1+\cos(150)}{2} = \frac{1-\frac{\sqrt{3}}{2}}{2} = \frac{2-\sqrt{3}}{4}$

$\rightarrow \cos(75) = \sqrt{\frac{2-\sqrt{3}}{4}}$

(b) $\sin(105°)$

$\sin(105°) = \sin(75°) = -\sin(-75°)$

$= \sqrt{\frac{4}{2+\sqrt{3}}}$



(c) $\csc(285°)$

$\cos^2(285) + \sin^2(285) = 1$

$\cos^2(-75) + \sin^2(285) = 1$

$\cos^2(75) + \sin^2(285) = 1$

$\frac{2-\sqrt{3}}{4} + \sin^2(285) = 1$

$\sin^2(285) = \frac{2+\sqrt{3}}{4} \rightarrow \sin(285) = -\sqrt{\frac{2+\sqrt{3}}{4}}$

$\rightarrow \csc(285) = -\sqrt{\frac{4}{2+\sqrt{3}}}$

(d) $\cot\left(\frac{3\pi}{4}\right)$

$\cot\left(\frac{3\pi}{4}\right) = \frac{1}{\tan\left(\frac{3\pi}{4}\right)} = -1$

12. Find a sinusoidal function, $f(x)$, which has the following...

(a) ...Amplitude 4, Period 3, Maximum 9, $f(1) = 5$ and f is increasing at $x = 1$

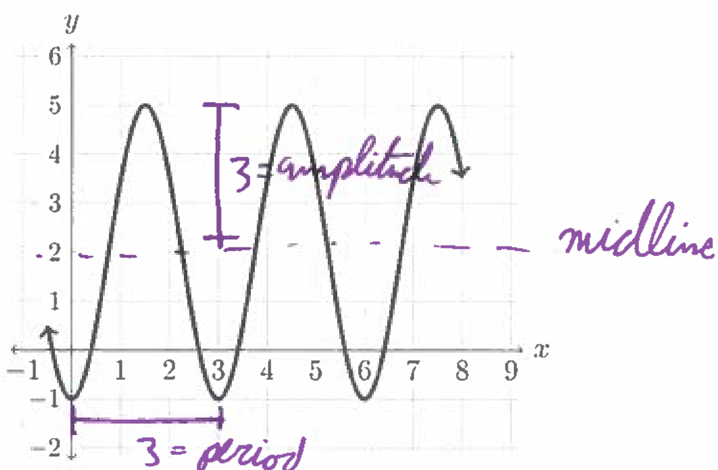
$f(x) = 4\sin\left(\frac{2\pi}{3}(x-h)\right) + 5$

$5 = f(1) = 4\sin\left(\frac{2\pi}{3}(1-h)\right) + 5 \to 0 = 4\sin\left(\frac{2\pi}{3}(1-h)\right)$

$\to 0 = \sin\left(\frac{2\pi}{3}(1-h)\right) \to \frac{2\pi}{3}(1-h) = 0$ (since this is the angle at which $\sin\theta$ is increasing) $\to 1-h=0 \to h=1$

$f(x) = 4\sin\left(\frac{2\pi}{3}(x-1)\right) + 5$

(b) ...graph given below



$f(x) = 3\sin\left(\frac{2\pi}{3}(x-h)\right) + 2$

$-1 = f(0) = 3\sin\left(\frac{2\pi}{3}(0-h)\right) + 2$

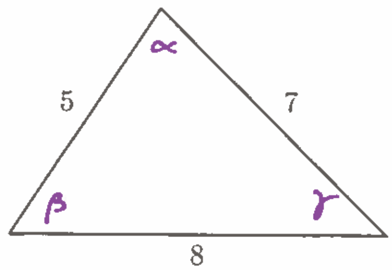
$-3 = 3\sin\left(\frac{2\pi}{3}(-h)\right)$

$-1 = \sin\left(\frac{2\pi}{3}(-h)\right)$

$\frac{3\pi}{2} = \frac{2\pi}{3}(-h) \to h = -\frac{9}{4}$

$f(x) = 3\sin\left(\frac{2\pi}{3}\left(x+\frac{9}{4}\right)\right) + 2$

13. Complete the triangle. Round all answers to two decimal places.



$$8^2 = 5^2 + 7^2 - 2\cdot 5\cdot 7\cos(\alpha)$$

$$-10 = -70\cos(\alpha)$$

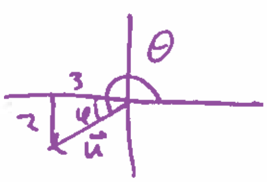
$$\frac{1}{7} = \cos(\alpha) \to \alpha = \arccos\left(\frac{1}{7}\right) \approx 1.43$$

$$\frac{\sin(\beta)}{7} = \frac{\sin(\alpha)}{8} \to \sin\beta \approx .87 \to \beta \approx \arcsin(.87) \approx 1.05$$

$$\gamma = \pi - \alpha - \beta \approx .66$$

14. Consider the vector $\vec{u} = -3\vec{i} - 2\vec{j}$. Find the polar form of $\vec{u}$ (i.e. find $\|\vec{u}\|$ and find the angle that $\vec{u}$ makes with the horizontal).

$$\|\vec{u}\| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$$



$$\tan(\varphi) = \frac{2}{3} \to \varphi = \arctan\left(\frac{2}{3}\right)$$

$$\to \theta = \pi + \varphi = \pi + \arctan\left(\frac{2}{3}\right)$$

15. Consider the vectors $\vec{v} = 3\vec{i} - 2\vec{j}$ and $\vec{w} = -\vec{i} + 4\vec{j}$. Compute the following

(a) $\vec{v} \cdot \vec{w}$ $= (3)(-1) + (-2)(4) = -11$

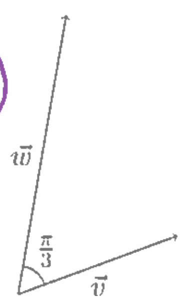
(b) $\vec{v} \cdot 2\vec{w}$ $= 2(\vec{v} \cdot \vec{w}) = -22$

(c) $(-5\vec{v} + 13\vec{w}) \cdot \vec{v}$

$$= -5\vec{v}\cdot\vec{v} + 13\vec{w}\cdot\vec{v}$$
$$= -5\|\vec{v}\|^2 + 13(-11)$$
$$= -5(13) + 13(-11)$$
$$= -208$$

16. Consider the vectors $\vec{v}$ and $\vec{w}$ drawn below, with $\|\vec{v}\| = 3$ and $\|\vec{w}\| = 5$. Compute $\vec{v} \cdot \vec{w}$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\left(\frac{\pi}{3}\right)$$
$$= 3 \cdot 5 \cdot \frac{1}{2}$$
$$= \frac{15}{2}$$



17. Find all vectors perpendicular to $4\vec{i} - 7\vec{j}$

$(x\vec{i} + y\vec{j}) \cdot (4\vec{i} - 7\vec{j}) = 0$

$\to 4x - 7y = 0 \to y = \frac{4x}{7}$

So, all vectors perpendicular to $4\vec{i} - 7\vec{j}$ have the form $x\vec{i} + \frac{4x}{7}\vec{j}$ for real numbers x

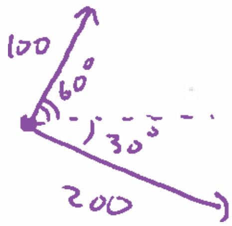
18. Find all values of t such that $\vec{a} = (6t)\vec{i} + (t+6)\vec{j}$ is perpendicular to $\vec{b} = t\vec{i} - 6\vec{j}$.

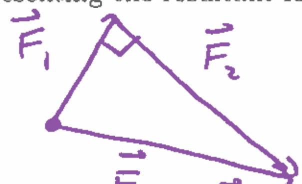
$$0 = \vec{a} \cdot \vec{b} = (6t)t + (t+6)(-6)$$
$$= 6t^2 - 6t - 36$$

$\to 0 = t^2 - t - 6 = (t-3)(t+2)$

$\to t = 3$ or $t = -2$

19. Two tugboats are pulling a ship. One tugboat is pulling 30° South of East with a force of 200 N and the other tugboat pulls 60° North of East with a force of 100N. Draw a force diagram with vectors representing the forces of the two tugboats. In addition, draw the vector representing the resultant force. What is its magnitude?





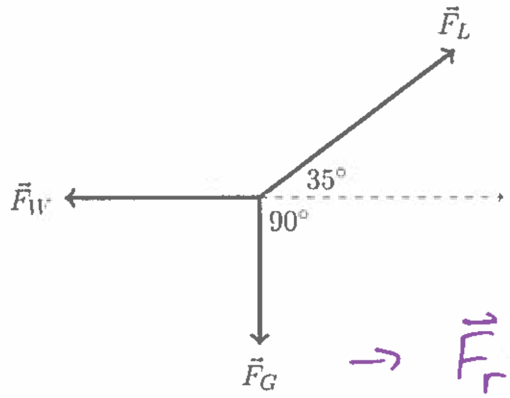
$\vec{F}_r = \vec{F}_1 + \vec{F}_2$

$\|\vec{F}_r\|^2 = \|\vec{F}_1\|^2 + \|\vec{F}_2\|^2$

$= 100^2 + 200^2$

$\rightarrow \|\vec{F}_r\| = \sqrt{50000}$

20. Consider the three forces drawn below, and suppose that $\|\vec{F}_G\| = 3N$, $\|\vec{F}_W\| = 4N$, and $\|\vec{F}_L\| = 5N$. Find the resultant force on the object.



$\vec{F}_w = -4\vec{i}$

$\vec{F}_G = -3\vec{j}$

$\vec{F}_L = 5\cos(35)\vec{i} + 5\sin(35)\vec{j}$

$\rightarrow \vec{F}_r = \vec{F}_w + \vec{F}_G + \vec{F}_L$

$= (5\cos(35) - 4)\vec{i} + (5\sin(35) - 3)\vec{j}$