Chapter 3 Lecture Notes

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Section 3.1: Radians and arc length

- Open with the twizzler activity!
- **Def:** The *radian* measure of an angle, θ, is the length of the arc on the unit circle cut out by the angle θ.
- Note that one radian corresponds to an arc length of 1 on the unit circle.
- It would be great if we had an easy way of converting between degrees and radians.
- How many radians are in 360 degrees? Convert to a proportion. Then, we get the following formula: if θ is an angle with degree measure D and radian measure R, then $R = \frac{\pi}{180}D$. Equivalently, $D = \frac{180}{\pi}R$.
- Let's redraw the unit circle with degrees and radians.
- **Exs:** Compute the following values: $\cos(\frac{\pi}{3})$, $\sin(\frac{5\pi}{6})$, $\tan(\frac{5\pi}{4})$, $\sin(-\frac{\pi}{6})$.
- We can use radians in a very similar way to how we use degrees in triangles.
- **Ex:** Given a right triangle with base angle $\frac{5\pi}{12}$ and base side 3, find the other angle and side lengths.
  - Solution: other angle is $\frac{\pi}{12}$, hypotenuse is 11.6, final leg is 11.2.
- Since radians are defined with reference to arc length, it makes sense that they should have a nice relation to arc length. The following lemma gives us that relationship.
- **Lemma:** Given a circle of radius r and an angle θ, the arc length cut out by that circle is $r\theta$.
- First, why should this be true? Large angles should have large arc lengths. Large circles should have large arc lengths.
- Second, note that this explains why we don't use units for radians.
  - If we did, we would be measuring arc lengths with a different unit than radii
  - It's also really cool that this tells us that radians are the "natural" way of measuring angle length.
  - The situation isn't like centimeters versus inches where there's nothing special about how long either one is.
- Third, give a "formal proof".
  - An angle of θ (in radians) is $\frac{\theta}{2\pi}$ of a circle. But the circumference of a circle is $2\pi r$. So the arc length is θr.
- **Ex:** Do example 3.1.15 in the book (on page 272–273)

Section 3.2: Non-right Triangles

- There's the natural question: why are we doing everything with right triangles?

- Part of the answer is that they are the best to work with. They're not as rigid as equilateral triangles, but there's a lot more information that we can get out of them than we could get out of triangles without any structure.
- That said, we can get some information about non-right triangles with our trig functions.
- The way we get this information is with the law of cosines and the law of sines.
- Let's first find out what they say:
- **Thm** (The Law of Cosines): For a triangle with side lengths a, b, and c, and angle C opposite to c, we have the following relationship: $c^2 = a^2 + b^2 - 2ab\cos(C)$.
- Thing to notice before we prove this: this is kind of like the Pythagorean Theorem. The difference is the $2ab\cos(C)$ term. This kind of measures how far away from being right a triangle is.
- **Pf:** Draw a triangle (vertex B on top). Drop a perpendicular from B, labeling the height h. Split up the side b into x and $b - x$ (x should be on the C vertex side of things). Solve for $\cos(C)$. Use Pythagorean theorem twice, one of which should be $x^2 + h^2 = a^2$ (b^2 if you put A on top). Substitute things and voila, the law of cosines pops out.
- Cool, so how do we use this?
- How can we use it? Well, we have 4 unknowns in the statement of the law of cosines. So in order to use it to solve for something, you'd better know three pieces of information.
- Because some of those things are side lengths, there are really two cases in which you might want to use it: first, is if you know all three side lengths and you want to find out an angle. Second is if you know two side lengths and the angle between them, you can find the third side length.
- **Ex:** Find the angles in a triangle with side lengths $4, 6, 7$ (assume that all angles are acute). End up with $\arccos(\frac{69}{84})$, $\arccos(\frac{3}{48})$, and $\arccos(\frac{29}{56})$, or π minus any two of the others.
- **Ex:** Find the possible values for x in the following triangle: (triangle has side lengths $x, 10$, and 9 and angle between x and 10 is $\frac{\pi}{3}$).
- **Soln:** $x = 7.45$ or $x = 2.55$.
- Next up is the law of sines:
- **Thm** (Law of Sines): In a triangle with sides a, b, c and angles A, B, C, we have the following relationship $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{C}$.
- Something to notice is that the way I've written it is mostly for compactness. You really only ever use one equality at a time.
- **Pf:** Drop a perpendicular from one vertex, compute sines of relevant angles, ta-da.
- How can we use this? If we know two sides and either angle not between them, we can find the angle between the side lengths. If we know two angles and a side, we can find another side.
- **Ex:** Find the angles and side lengths of the following triangle: side lengths 5 and 6, angle $\frac{\pi}{6}$ not between those sides.
- **Soln:** Angle opposite 5 is $\arcsin(5/12)$, third angle is $\pi - \frac{\pi}{6} - \arcsin(5/12) \approx 2.2$, last side is ≈ 9.78
- **Ex:** Two trains depart from Portland at constant speeds, one traveling due north, the other at an angle $\frac{\pi}{4}$ east of due north. The train traveling north travels at 60 miles per hour. After two hours, the conductor of the northbound train observes that the other train is $\frac{\pi}{6}$ south of due east from the northbound train's current position. How far apart are the two trains after two hours? *Challenge:* After three hours?

- **Soln:** Approximately 87.8 miles after two hours. Other side is ≈ 107.59 miles. So after three hours, have triangle with north train 180 miles, northeast train ≈ 161.4 miles, still angle of $\frac{\pi}{4}$. Law of cosines or similar triangles gives distance being ≈ 131.8.

Section 3.3: Trigonometric Equations

- We have looked at the problem "find all solutions to $\sin(\theta) = \frac{\sqrt{3}}{2}$" a couple of times now. Do it again.
- How exactly did we get the θ values between 0 and 2π? We had $\arcsin(\sqrt{3}/2)$ and $\pi - \arcsin(\sqrt{3}/2)$
- Note that we needed to find all solutions in an interval of length 2π because sin has period 2π. If we had only looked for solutions between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (which is where arcsin spits out values), we wouldn't have found everything.
- This generalizes: any time you want to solve an equation of the form $\sin(\theta) = y$ for some y in $[-1, 1]$ the solutions are all numbers of the form $\theta_1 + 2\pi n$ and $\theta_2 + 2\pi n$ for integers n, where $\theta_1 = \arcsin(y)$ and $\theta_2 = \pi - \arcsin(y)$.
- What about "find all solutions to $\cos(\theta) = -\frac{\sqrt{2}}{2}$"?
- How did we find our θ values? We found $\arccos(-\sqrt{2}/2)$ and $-\arccos(-\sqrt{2}/2)$.
- This also generalizes: any time you want to solve $\cos(\theta) = x$ for some x in $[-1, 1]$, the solutions all have the form $\theta_1 + 2\pi n$ and $\theta_2 + 2\pi n$ for integers n, where $\theta_1 = \arccos(x)$ and $\theta_2 = -\arccos(x)$.
- What about tangent? "Find all solutions to $\tan(\theta) = \sqrt{3}$"
- Note that because tangent has period π, we only need to find the solutions in an interval of length π, say from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. This means that we only need to look at $\arctan(m)$
- This also generalizes: any time you want to solve $\tan(\theta) = m$ for some number m, the solutions all have the form $\arctan(m) + \pi n$ for some integer n.
- Note that tangent is a little different than sin and cosine. Tangent has period π, rather than 2π which makes a big difference when solving problems like this.
- This is different than what your book says.
- Note that each of the above cases is different from the others. I don't really want you memorizing and using these formulas, I'd rather have you solving the problems conceptually. I do want you to know that these formulas exist, however, which is why I presented them. I will never force you to use them, however.
- **Ex:** Find all solutions to $5\cos(\theta) + 2 = -\frac{\sqrt{75}+4}{2}$
  - Make sure to think about this one algebraically and with graph transformations
- **Ex:** Find all solutions to $\tan(4\theta) = \frac{\sqrt{3}}{3}$
- **Ex:** Consider our old Ferris wheel setup, but this time, we change the rate at which our Ferris wheel rotates and we change the starting position. Suppose that we have a Ferris wheel with radius 80 meters, centered 100 meters above the ground, where you start at the lowest point on the Ferris wheel, and the Ferris wheel spins at a constant rate so that you make a counterclockwise rotation once every two minutes. Describe your height above the ground as a function of the number of seconds after you begin the ride. Assuming the ride goes on forever, at what times will you be 140 meters above the ground?
  - We already saw that if θ is the angle you make with the horizontal, your height is given by $80\sin(\theta) + 100$. But we want height to be a function of t, not θ. So our goal is to write θ in terms of t.
  - Since the Ferris wheel spins at a constant rate, the angle, θ is a linear function of t.

- We know that when $t = 0$, $\theta = -\frac{\pi}{2}$ and when $t = 120$, $\theta = \frac{3\pi}{2}$, so the slope of θ is $\frac{\frac{3\pi}{2} - \frac{-\pi}{2}}{120-0} = \frac{2\pi}{120} = \frac{\pi}{60}$.
- Since we have the intercept at $-\frac{\pi}{2}$, we have that $\theta = \frac{\pi}{60}t - \frac{\pi}{2} = \frac{\pi}{60}(t - 30)$
- Hence, our height function is $H(t) = 80\sin\left(\frac{\pi}{60}(t - 30)\right) + 100$.
- To solve for the times when you are 140 meters off the ground, we set $H(t) = 140$

<u>Section 3.4</u>: Sinusoidal Functions

- At the end of the last section, we saw the function $H(t) = 80\sin\left(\pi\left(t - \frac{1}{2}\right)\right) + 100$ (I think)
- This was your height on a Ferris wheel with radius 80, height 100, and which rotated once every 2 minutes.
- Here we finally got to see all of the transformations applied to one of our trig functions.
- What are the midline, period, and amplitude of $H(t)$?
  - Midline: $y = 100$
  - Period: 2
  - Amplitude: 80
- This motivates the following definition
- **Def:** A function $f(x)$ is *sinusoidal* if it has the form $f(x) = A\sin(B(x - h)) + k$ for some numbers A, B, h, and k.
- It's good to know that we can always assume that $A > 0$ and $B > 0$. This is because horizontal reflections and vertical reflections are just translations of sin. E.g. $-\sin(x) = \sin(-x) = \sin(x - \frac{\pi}{2})$.
- What sorts of things are sinusoidal?
  - There are the obvious ones: e.g. $2\sin(3(x - 1)) + 4$
  - But there are also less obvious ones: $\cos(x)$ is sinusoidal because $\cos(x) = \sin(x + \frac{\pi}{2})$
- What are the elementary properties of $A\sin(B(x - h)) + k$?
  - Midline: $y = k$
  - Period: $\frac{2\pi}{B}$
  - Amplitude: A
  - Horizontal shift: h
- **Ex:** Graph $3\sin\left(\frac{\pi}{4}(x - 1)\right) - 2$
- **Ex:** Find a formula for a sinusoidal function, $f(x)$, which has the following properties
  - Amplitude: 4
  - Midline: $y = 1$
  - Period: $\frac{\pi}{3}$
  - $f(3) = 1$ and f is increasing at $x = 3$.
- Note that there are many correct answers to this question, because of periodicity. List them.
- Rather than graphically, how else could we have solved this problem?
- Algebraically, we could have solved $4\sin(\frac{\pi}{3}(3 - h)) + 1 = 1$. But we run into this problem where we have to appeal to the graph to solve this thing anyways, so starting with the graph is the best bet for solving this.

- **Ex:** Find a formula for a sinusoidal function, $f(x)$ with the following graph (graph the following properties, noting that there is a shift of a quarter unit to the right)
  - Maximum: -1
  - Minimum: -5
  - Period: 3
  - $f(1) = -1$

<u>Section 3.5:</u> Relationships and Graphs

- I know I said that we weren't going to do any identities, but I was wrong. They are important for calculus!
- So we need to introduce a couple of new trig functions before we do identities:
- **Defs:** The *secant* function is defined by $\sec(\theta) = \frac{1}{\cos(\theta)}$. The *cosecant* function is defined by $\csc(\theta) = \frac{1}{\sin(\theta)}$ and the *cotangent* function is defined by $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$.
- **Lemma:** Here is a list of important identities:
  - $\sin^2(\theta) + \cos^2(\theta) = 1$
  - $1 + \cot^2(\theta) = \csc^2(\theta)$
  - $\tan^2(\theta) + 1 = \sec^2(\theta)$
  - $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
  - $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$
  - $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
  - $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
  - $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
  - $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$
- **Ex:** Find an exact value for $\cos(\frac{5\pi}{12})$
- **Ex:** Show that $\frac{\sin(\alpha+\beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha) + \tan(\beta)$
- **Ex:** Show that $\sin(2\theta) = \frac{2\cot(\theta)}{1+\cot^2(\theta)}$
- **Ex:** Write $\cos(\theta) + \cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$ in terms only of $\cos(\theta)$.
- **Ex:** Write $\cos^2(\theta)\sin^2(\theta)$ without using any powers of trigonometric functions.