Bounds on the Number of Solutions to Thue's Inequality

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  - $s = 3$
  - $H = 10$
- Let $h \in \mathbb{Z}_{>0}$

Foundational Result

Theorem (Thue, 1909)

$|F(x, y)| \leqslant h$ (known as Thue's Inequality) has finitely many integer pair solutions

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Necessity of Hypotheses

- $\deg(F) \geqslant 3$ is necessary: if $d \in \mathbb{Z}_{>0}$ is not a square, then $F(x,y) = x^2 - dy^2$ is irreducible and homogeneous, and $|F(x,y)| \leqslant 1$ has infinitely many integer-pair solutions

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- $F(x,y)$ being irreducible is also necessary: if $F(x,y)$ has a linear factor, say $mx - ny$, then any integer multiple of (n,m) is a solution to $|F(x,y)| \leqslant h$
- The homogeneity condition is also necessary: if $F(x,y) = x^6 + y^3$, then any integer pair of the form $(n, -n^2)$ will be a solution to $|F(x,y)| \leqslant h$

Follow-Up Questions

Theorem (Thue, 1909)

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Approaches

1. Geometric
2. Algebraic

Changing Variables

Changing Variables

"Eliminating" h

Observe that $|x^5 + 3x^4y - y^5| \leqslant h$ if and only if

$$\left| \left(\frac{x}{h^{1/5}}\right)^5 + 3\left(\frac{x}{h^{1/5}}\right)^4 \left(\frac{y}{h^{1/5}}\right) - \left(\frac{y}{h^{1/5}}\right)^5 \right| \leqslant 1$$

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Fact

$|F(x,y)| \leqslant h$ *if and only if*

$$\left|F\left(\frac{x}{h^{1/n}}, \frac{y}{h^{1/n}}\right)\right| \leqslant 1$$

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Observe that $|x^5 + 3x^4y - y^5| \leqslant h$ with $y > 0$ if and only if

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Fact

$|F(x, y)| \leqslant h$ and $y > 0$ if and only if

$$\left|F\left(\frac{x}{y}, 1\right)\right| \leqslant \frac{h}{y^n}$$

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Suppose that $\kappa > n$*. Then any* $x, y \in \mathbb{Z}$ *with* $|F(x, y)| \leqslant h$ *has*

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Suppose that $\kappa > n$. Then any $x, y \in \mathbb{Z}$ with $|F(x,y)| \leqslant h$ has

$$\max(|x|, |y|) \leqslant C_{F,\kappa} e^{(\log h)^\kappa} = C_{F,\kappa} h^{(\log h)^{\kappa-1}}$$

where $C_{F,\kappa}$ is an effectively computable constant depending only on $F(x,y)$ and κ.

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Benefits

This gives an effective algorithm for solving Thue's inequality:

- Choose a $\kappa > n$
- Compute $C_{F,\kappa}$
- Test all pairs $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|, |y|) \leqslant C_{F,\kappa} e^{(\log h)^\kappa}$ to see if $|F(x,y)| \leqslant h$

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How Many Solutions?

- Define $N(F, h) := \#\{(x, y) \in \mathbb{Z}^2 : |F(x, y)| \leqslant h\}$

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- Baker's theorem immediately gives

$$N(F,h) \leqslant \left(2C_F h^{(\log h)^n} + 1\right)^2 \asymp_F h^{2(\log h)^n}$$

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Question

Is this what the growth rate of $N(F,h)$ actually looks like?

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$N(F, h)$	3	5	15	27	51	121	257	541

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Thue's Inequality
Introduction
Geometry
Algebra
Results

Counting Techniques
Different Solution Types
Large Solutions
Medium Solutions
Small Solutions
In Total

Binomials and Trinomials
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Note that $h^{0.32}$ grows much slower than $h^{2(\log h)^6}$

Visualizing Thue's Inequality

A Picture

$|F(x, y)| \leqslant h$ corresponds to a region of the xy-plane:

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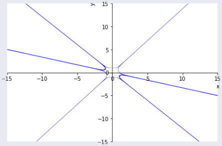
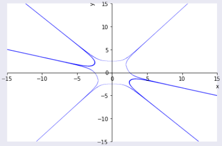


$|x^5 + 3x^4y - y^5| = 1$



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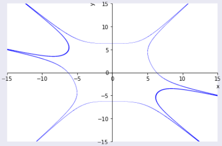

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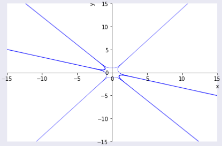

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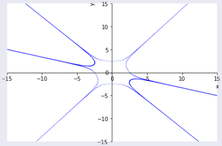

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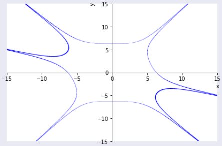

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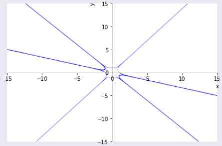

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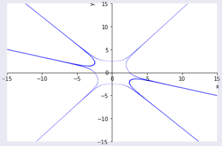

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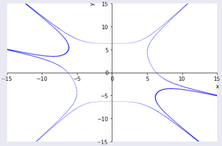

$$|x^5 + 3x^4y - y^5| = 10^4$$

$$N(F,h) = \text{number of lattice points “inside” } |F(x,y)| \leqslant h$$
$$\approx \mathrm{vol}\{(x,y) \in \mathbb{R}^2 : |F(x,y)| \leqslant h\}$$

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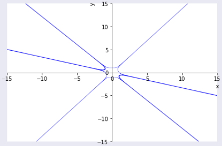

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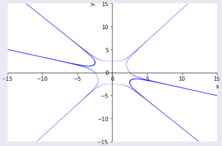

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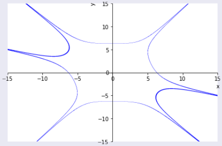

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$$
\begin{aligned}
N(F,h) &= \text{number of lattice points "inside" } |F(x,y)| \leqslant h \\
&\approx \mathrm{vol}\{(x,y) \in \mathbb{R}^2 : |F(x,y)| \leqslant h\} \\
&=: V(F,h)
\end{aligned}
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Exploring dependence on h

Volume

In general

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Implication

Since $N(F,h) \approx V(F,h) = h^{2/n}V(F,1)$, it would make sense if $N(F,h) \approx h^{2/n} \cdot C_F$ where C_F is a constant depending only on F

Exploring dependence on h

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Theorem (Mahler, 1934)

For any $F \in \mathbb{Z}[x, y]$ irreducible and homogeneous of degree $n \geqslant 3$, there exists a constant $C(F)$ so that

$$|N(F, h) - V(F, h)| \leqslant C(F) \cdot h^{\frac{1}{n-1}}$$

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Corollary

$$h^{-2/n} N(F, h) = V(F, 1) + O_F(h^{-\frac{1}{n+3}})$$

i.e. $N(F, h) = O_F(h^{2/n})$

Confirming Our Hypothesis

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From Mahler's Theorem

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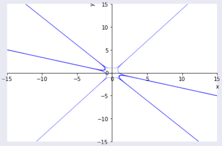
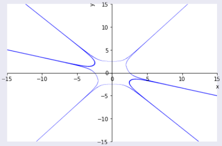
The "Long Tendrils"

Some Pictures

Recall that we had the previous pictures:



$$|x^5 + 3x^4y - y^5| = 1$$



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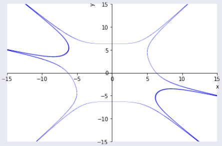

$$|x^5 + 3x^4y - y^5| = 10^4$$

Question

What's the deal with the linear parts?

A Connection to $\mathbb{Q}$

Translating to One Variable

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  - Immediate: if $\prod_{i=1}^{n}\left|\frac{p}{q} - \alpha_i\right| = \frac{1}{q^n}$, then there exists i so that $\left|\frac{p}{q} - \alpha_i\right| \leqslant \frac{1}{q}$

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- By symmetry, we could also count rational approximations to roots of $g(Y) = F(1, Y)$

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- We can connect primitive solution counts to total solution counts using partial summation methods.

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- But $f(\frac{x}{y}) = f(\alpha) = 0$, so $F(x, y) = 0$.
- Hence, the line $Y = \frac{X}{\alpha}$ is a subset of $\{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leqslant h\}$
- Therefore, real roots α correspond with tendrils of slope $\frac{1}{\alpha}$

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Let $F(x, y) = x^5 + 3x^4y - y^5$

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$f(x) = F(x, 1) = x^5 + 3x^4 - 1$
$f(x)$ has real roots $\alpha_1 \approx -2.99$, $\alpha_2 \approx -0.82$, and $\alpha_3 \approx 0.72$.

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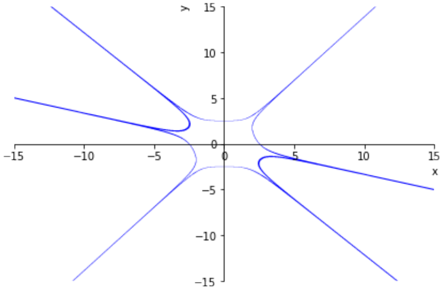

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If $g(x) \in \mathbb{R}[x]$ has degree n, then g has no more than n real roots.

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Lemma (Schmidt, 1987)

Suppose $g(x) \in \mathbb{R}[x]$ has $s+1$ nonzero terms and $g(0) \neq 0$. Then g has no more than $2s$ real roots.

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Answer

Maybe we need to consider some complex roots, but we only need to consider good approximations to $\ll s$ roots

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Conjecture (Mueller and Schmidt, 1987)

s^2 can be replaced by s and $(1 + \log h^{1/n})$ is unnecessary.

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Theorem (Bennett, 2001)

$ax^n - by^n = 1$ has at most one solution in positive integers x and y.

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Theorem (Thomas, 2000)

For $n \geqslant 39$ and F a trinomial, the number of solutions to $|F(x, y)| = 1$ is less than or equal to 48.

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Theorem (Akhtari and Bengoechea, 2020)

The number of positive, primitive solutions of $|F(x, y)| \leqslant h$ when h is small relative to the discriminant of F is $\ll s \log s$. If $n \geqslant s^2$, then the number of positive, primitive solutions is $\ll s$.

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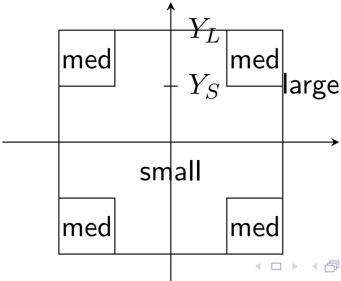

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- Technique: archimedean Newton polygons

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$$\left|\alpha - \frac{x}{y}\right| \leqslant \frac{K}{y^{n/s}} \quad \text{or} \quad \left|\alpha^* - \frac{y}{x}\right| < \frac{K}{x^{n/s}}$$

Medium Solution Setup

Lemma (Mueller and Schmidt, 1987)

There is a set S of roots of $f(x) = F(x, 1)$ and a set S^* of roots of $g(y) = F(1, y)$ both with size $\ll s$ so that for any solution to $|F(x, y)| \leqslant h$ with $|x|, |y| > Y_S$, there exists $\alpha \in S$ or $\alpha^* \in S^*$ so that

$$\left|\alpha - \frac{x}{y}\right| \leqslant \frac{K}{y^{n/s}} \quad \text{or} \quad \left|\alpha^* - \frac{y}{x}\right| < \frac{K}{x^{n/s}}$$

where K depends on F and h

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Moral

There's a set of $\ll s$ algebraic numbers so that any primitive solution to $|F(x,y)| \leqslant h$ with $x, y > Y_S$ gives a rational number $\frac{x}{y}$ or $\frac{y}{x}$ which is close to one of those algebraic numbers.

Counting

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Goal

Fix $\alpha \in S$ and count the number of rationals which satisfy

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Setup

- Recall that a (positive) medium solution has $Y_S < x, y < Y_L$
- Fix α, enumerate the medium solutions which satisfy the above inequality, and order them so that

$$Y_S < y_0 \leqslant y_1 \leqslant \cdots \leqslant y_t < Y_L$$

The Gap Principle

- Use the fact that if $\frac{x_i}{y_i}$ and $\frac{x_{i+1}}{y_{i+1}}$ are close to α, they are close to each other:

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- This is known as The Gap Principle

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Counting with Gaps

Using $y_{i+1} > \dfrac{y_i^{\frac{n}{s}-1}}{K}$ together with $Y_S < y_0 \leqslant y_1 \leqslant \cdots \leqslant y_t < Y_L$, we can find bounds on t.

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$$Y_L \geqslant y_t \geqslant \frac{y_{t-1}^{\frac{n}{s}-1}}{K}$$

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$$Y_L \geqslant y_t \geqslant \frac{y_{t-1}^{\frac{n}{s}-1}}{K} \geqslant \frac{\left(\frac{y_{t-2}^{\frac{n}{s}-1}}{K}\right)^{\frac{n}{s}-1}}{K}$$

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$$Y_L K^{\frac{-1}{\frac{n}{s}-2}} \geqslant \left(Y_S K^{\frac{-1}{\frac{n}{s}-2}}\right)^{(\frac{n}{s}-1)^t}$$

and solve the inequality for t to find...

Counting

Lemma (K., 2021)

If $n \geqslant 3s$ and there are $t+1$ medium solutions associated to α, then

$$t \leqslant \frac{\log\left[\frac{\log Y_L K^{-1/(\frac{n}{s}-2)}}{\log Y_S K^{-1/(\frac{n}{s}-2)}}\right]}{\log\left(\frac{n}{s}-1\right)}$$

Moreover, this bound is sharp.

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Moreover, this bound is sharp.

Something more useful

Reducing the above constants into terms of n, s, h, H, using $n \geqslant 3s$, and applying the fact that there are $\ll s$ roots α that we need to care about, we find...

Counting Medium Solutions

Theorem (K., 2021)

The number of primitive medium solutions to $|F(x,y)| \leqslant h$ *when* $n \geqslant 3s$ *is*

$$\ll s\left(1+\log\left(s+\frac{\log h}{\max(1,\log H)}\right)\right)$$

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Recall:

Conjecture

If $h \leqslant H^{1-\frac{s}{n}-\rho}$, *then the number of primitive solutions to* $|F(x,y)| \leqslant h$ *is bounded by a function only of* s *and* ρ

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Challenges

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When $n > 4se^{2\Phi}$, the number of primitive small solutions to $|F(x,y)| \leqslant h$ is

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"Definition"

Here, Φ measures the "sparsity" of F and satisfies $\log^3 s \leqslant e^{\Phi} \ll s$

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Recap

Bounds for Different Types of Solutions

Recall:

- The number of large primitive solutions is $\ll s$
- The number of medium primitive solutions is $\ll s\left(1 + \log s + \log^+\left(\frac{\log h}{\max(1, \log H)}\right)\right)$ when $n \geqslant 3s$.
- The number of small primitive solutions is $\ll s e^{\Phi} h^{2/n}$ when $n > 4se^{2\Phi}$.

Summing Up

As a consequence:

Theorem (K., 2022)

When $n > 4se^{2\Phi}$, the number of primitive solutions to $|F(x,y)| \leqslant h$ is

$$\ll se^{\Phi}\left(1 + \log^{+}\left(\frac{\log h^{1/\log^3 s}}{\max(1, \log H)}\right)\right) h^{2/n}$$

Compare to:

Theorem (Mueller and Schmidt, 1987)

The number of integer pair solutions to $|F(x,y)| \leqslant h$ is

$$\ll s^2 h^{2/n}(1 + \log h^{1/n})$$

Binomials

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Theorem (Bennett, 2001)

$ax^n - by^n = 1$ *has at most one solution in positive integers* x *and* y

Trinomials

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Recall that we showed the Gap Principle previously: when $\frac{x}{y}$ and $\frac{x'}{y'}$ both approximate the same root of f and $y' \geqslant y > Y_S$, we had

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In general, K is very large.

Theorem (Thomas, 2000)

When $s = 2$, K can be improved substantially and Y_S can be taken to be less than 1 (eliminating any small solutions). As a consequence, there are explicit bounds on the number of solutions to $|F(x,y)| = 1$. If $n \geqslant 39$, then there are no more than 48 solutions to $|F(x,y)| = 1$.

Trinomials

Theorem (K., 2021)

When $s = 2$, there are no more than $C(n)$ solutions to $|F(x,y)| = 1$ where $C(n)$ is defined by

n	6 - 7	8	9 - 11	12 - 16	17 - 38	39 - 218	$\geqslant 219$
$C(n)$	128	96	72	64	56	48	40

See `https://arxiv.org/abs/2210.09631` for more details.

Trinomial Computations

Computations indicate that for the following degrees (vertical axis) and heights (horizontal axis), the maximum number of solutions to $|F(x,y)| = 1$ is given in the following table:

H	1	2	3	4	5	6	7	8	9	10	11	12	13
$n = 6$	8	6	8	8	6	6	6	6	8	6	6	6	6
$n = 7$	8	6	8	8	6	6	6	6	8	6	6	6	6
$n = 8$	8	6	8	8	6	6	6	6	8	6	6	6	6
$n = 9$	8	6	8	8	6	6	6	6	8	6	6	-	-
$n = 10$	8	6	8	8	6	6	6	6	8	-	-	-	-
$n = 11$	8	6	8	8	6	6	6	6	8	-	-	-	-
$n = 12$	8	6	8	8	6	6	6	-	-	-	-	-	-
$n = 13$	8	6	8	8	6	6	-	-	-	-	-	-	-
$n = 14$	8	6	8	8	6	6	-	-	-	-	-	-	-
$n = 15$	8	6	8	-	-	-	-	-	-	-	-	-	-

Thank you!

Questions?