Definition 4.1 If an experiment can be repeated under the same condition, its outcome cannot be predicted with certainty, and the collection of its every possible outcome can be described prior to its performance, this kind of experiment is called random experiment. All the possible outcomes of a random experiment is denoted by Ω which is called sample space. A subset of Ω is called an event. An event cannot be decomposed is called a simple event. Two events are mutually exclusive if there are no intersections.

Example 4.1
Experiment: Roll a fair die on a hard, flat floor and observe the number appearing on the upper face. Is this a random experiment? If yes, what is the sample space? Define $E_i = \{$Observe a i$\}, i = 1, \cdots, 6$, $A = \{$Observe an odd number$\}$, and
$B = \{$Observe a number less than or equal to 4$\}$. Is A an event? Is A a simple event? Do A and B be mutually exclusive?

Classical Probability

Definition 4.2 Given a sample space $\Omega = \{E_1, E_2, \cdots, E_n\}$ with finite simple events, if we define a function $\mathbb{P}$ on Ω by:
(1)The empty set is denoted by ϕ. $\mathbb{P}(\phi) = 0$.
(2) $\mathbb{P}(E_i) = 1/n$ for each $1 \leq i \leq n$. This means that each simple event is equally likely.

(3) For each event A, $\mathbb{P}(A)$ is equal to the sum of the probabilities of simple events contained in A.
Then, $\mathbb{P}$ is called a classical probability on Ω.

Remark:
(1) $0 \leq \mathbb{P}(A) \leq 1$.
(2) $\mathbb{P}(\Omega) = 1$.

A more general definition of probability will be introduced later on.

Example 4.2 Consider the random experiment and events defined in Example 4.1, where
$A = \{$**Observe an odd number**$\}$, **and**
$B = \{$**Observe a number less than or equal to 4**$\}$. **Find the classical probability of event A and B.**
Solution:

$$\mathbb{P}(A) = \mathbb{P}(E_1) + \mathbb{P}(E_3) + \mathbb{P}(E_5) = 1/6 + 1/6 + 1/6 = 1/2$$

$$\mathbb{P}(B) = \mathbb{P}(E_1)+\mathbb{P}(E_2)+\mathbb{P}(E_3)+\mathbb{P}(E_4) = 1/6+1/6+1/6+1/6 = 2/3$$

Example 4.3 A jar contains four coins: a nickel, a dime, a quarter, and a half-dollar. Three coins are randomly selected from the jar.
a. List all the simple events in the sample space Ω.
b. This is a typical example of the classical probability. What is the probability that the selection will contain the half-dollar?
c. What is the probability that the total amount drawn

will equal 0.6 dollar or more?

Solution:
a. Denote:
N: nickel;
D: dime;
Q: quarter;
H: half-dollar.
and $E_1 = (NDQ)$,$E_2 = (NDH)$, $E_3 = (NQH)$, $E_4 = (DQH)$.
Then, $\Omega = \{E_1, E_2, E_3, E_4\}$.
b.

$\mathbb{P}$(**choose a half-dollar**) $= \mathbb{P}(E_2)+\mathbb{P}(E_3)+\mathbb{P}(E_4) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}$

c. The simple event along with their monetary values follow:

$$\begin{aligned} E_1 &= NDQ = \$0.4 \\ E_2 &= NDH = \$0.65 \\ E_3 &= NQH = \$0.80 \\ E_4 &= DQH = \$0.85 \end{aligned}$$

Hence,

$\mathbb{P}$(**total amount is \$0.6 or more**) $= \mathbb{P}(E_2)+\mathbb{P}(E_3)+\mathbb{P}(E_4) = \frac{3}{4}$.

The mn Rule
Consider an experiment that is performed in two stages. If the first stage can be accomplished in m different

ways and for each of these ways, the second stage can be accomplished in n different ways, then there are total mn different ways to accomplish the experiment.

Example 4.4 A driver wants to go from city A to B, then to c. If from A to B there 5 different ways and B to C there 6 different ways. Totally are there how many different ways from A to C?

Solution: By mn rule, there are 30 different ways.
The Extended mn Rule
Consider an experiment that is performed in k stages. If the first stage can be accomplished in n_1 different ways and for each of these ways, the second stage can be accomplished in n_2 different ways,$\cdots$, and the k stage can be accomplished in n_k different ways, then there are total $n_1 n_2 \cdots n_k$ different ways to accomplish the experiment.

Example 4.5
There are 10 people stand in a line to take a photo. Two photos are said different if at least two people's positions are changed. How many different photos can we take?
Solution:
There are 10 different ways to choose a person to stand on the first position; After taking out the first person,

we only have 9 remaining people. Then, there are 9 different ways to choose a person to stand on the second position; and so on. By the extended mn rule totally we can take $10 \times 9 \cdots \times 1 = 3628800$ different photos.

Example 4.6

Suppose that a family will certainly have 5 children, but each child being a boy or girl is totally uncertain with equal probability. How many simple events in the sample space? What are the probabilities of the following events:

a. $A = \{$Last baby is a boy$\}$.

b. $B = \{$The first and the second are girls$\}$.

Solution:

In this question, order is important. Consider that there are five positions in a line. Each position has two choices:boy or girl. By extended mn rule, the total number of simple events in the sample space is equal to $2^5 = 32$.

a. The last baby is a boy, then the last position is occupied and the first four positions have options. A has 2^4 different simple events. Therefore, the $\mathbb{P}(A) = 2^4/2^5 = 1/2$.

b. The first and second are girls, then the first and second positions are occupied and the remaining three positions have options. B has 2^3 different simple events. Therefore, the $\mathbb{P}(B) = 2^3/2^5 = 1/4$.

Example 4.7

There are 10 people and 50 big rooms. Suppose that each person has equal probability 1/50 to go to any one of the 50 rooms. Let A={for 10 specified rooms, each of these 10 rooms has exactly one person }. Find out the probability of event A.

Solution:

By extended mn rule, the number of simple events in the sample space is 50^{10}. If we denote $n! = n \times (n-1) \times \cdots 1$, by the extended mn rule, the number of simple events in A is equal to 10!. Therefore,

$$\mathbb{P}(A) = 10!/(50^{10})$$

Definition 4.3 A permutation of n different objects is an ordering arrangement of this n objects.

Counting Rule for Permutations I The number of ways we can arrange n distinct objects is

$$P_n^n = n!.$$

Proof:

This is equivalent to the photo problem that there are n positions in a line and n different people. Two photos are counted as different if at least two people's positions are different in the photos. How many different photos can we take? The first position can be occupied by one of n people. The second position can be occu-

pied by remaining $n-1$ people, $\cdots$, the last position only can be occupied by the last person. Therefore, totally it is $n \times (n-1) \times \cdots, 1 = n!$.

Counting Rule for Permutations II **The number of ways we can arrange n distinct objects, taking them r at a time, is**

$$P_r^n = n!/(n-r)!.$$

Proof:

This is equivalent to the photo problem that there are r positions in a line and n different people. Two photos are counted as different if at least two people's positions or two people are different in the photos. How many different photos can we take? The first position can be occupied by one of n people. The second position can be occupied by remaining $n-1$ people, $\cdots$, the last position, r^{th} position, can be occupied by one of remaining $n-(r-1)$ people. Therefore, by extended mn rule, totally it is

$$\begin{aligned}
& n \times (n-1) \times \cdots, n-(r-1) = n \times (n-1) \times \cdots \times (n-r+1) \\
& = \frac{n \times (n-1) \times \cdots \times (n-r) \times (n-r-1) \times \cdots \times 1}{(n-r) \times (n-r-1) \times \cdots \times 1} \\
& = n!/(n-r)!
\end{aligned}$$

Example 4.8

There are three letters A, B, C. How many different ordered, two letter alphabets can we get?

Solution:

$$AB \quad AC \quad BC$$
$$BA \quad CA \quad CB$$

According to the permutation counting formula

$$P_2^3 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 6$$

Example 4.9

There are 26 letters $A, B, C, \cdots, X, Y, Z$. How many different ordered, five letter alphabets can we get?

Solution:

According to the permutation counting formula

$$P_5^{26} = \frac{26!}{(26-5)!} = 7893600$$

A more general definition of probability is as follows.

Definition 4.4 Given a sample space $\Omega = \{E_1, E_2, \cdots, E_n\}$ with n simple events, where n is finite or infinite. if we define a function $\mathbb{P}$ on Ω by:
(1)The empty set is denoted by ϕ. $\mathbb{P}(\phi) = 0$.
(2) For each simple event E_i, its probability $\mathbb{P}(E_i)$ is defined and $0 \leq \mathbb{P}(E_i) \leq 1$. Thus, "equally likely" is a special case of this definition.
(3) For each event A, $\mathbb{P}(A)$ is equal to the sum of the probabilities of simple events contained in A.
Then, $\mathbb{P}$ is called a probability on Ω.

Example 4.10
In a pocket there are 3 black and 2 white balls. Balls are identical except their colors. Randomly drawing a ball and observing its color. How many simple events in the sample space? Can you define a reasonable probability on the sample space? Is this equally likely?
Solution:

$$\Omega = \{W, B\}$$
$$\mathbb{P}(W) = \frac{2}{5}, \quad \mathbb{P}(B) = \frac{3}{5}.$$

Definition 5.1 A combination is an order ignored selection of objects from a larger group of objects.

We know that the number of different permutations of r different objects is

$$P_r^r = r!.$$

The P_r^n can be thought as the multiplication of two numbers:
(1) number of ways to select r different objects from n different objects. We denote this number by C_r^n
(2) number of ways to get different arrangements for each selected r objects. This is just P_r^r.
Therefore,

$$P_r^n = C_r^n \times P_r^r$$

or

$$C_r^n = P_r^n / P_r^r = \frac{n!}{r!(n-r)!}$$

Counting Rule for Combination The number of different combinations of n different objects that can be formed, taking them r at a time, is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Example 5.1 A student prepares for a quiz by studying a list of ten problems. She only can solve six of them. For the quiz, the instructor selects five questions at random from the list of ten. What is the probability of event A that the student can solve all five problems on the exam?

Solution: The number of simple events in the sample space is

$$C_5^{10} = \frac{10!}{5!(10-5)!}.$$

The event A that the student can solve all five problems can be described in the way that the five problems are selected from the six problems she learned. Thus, the number of simple events in the event A is

$$C_5^6 = \frac{6!}{5!(6-5)!}$$

and

$$\mathbb{P}(A) = \frac{C_5^6}{C_5^{10}}$$

Example 5.2 We want to choose 5 people from 20 people to organize a traveling group. Are there how many

different ways to choose a group?
Solution: In this example order is not important, therefore, there are C_5^{20}.

Example 5.3 Suppose that 10 defective computers are included in a shipment of 1000 computers.If you test 20 computers in this 1000 computers, what is the probability of event A that you found two defective computers?
Solution:

$$\mathbb{P}(A) = \frac{C_2^{10} C_{18}^{990}}{C_{20}^{1000}}.$$

Definition 5.2 The intersection of events A and B, denoted by $A \cap B$, is all the simple events belonging to both A and B. The union of events A and B, denoted by $A \cup B$, is all the simple events belonging to either A or B. The complement of event A, denoted by A^c, is all the simple events belonging to sample space Ω but A.

Example 5.4 Consider the random experiment of rolling a fair die. Define $A = \{E_1, E_2, E_3, E_4\}$, $B = \{E_1, E_3, E_5\}$ What are $A \cap B$, $A \cup B$, and A^c?
A more general definition of probability is defined as follows:
Definition 5.3 Given a sample space Ω, if we define a function $\mathbb{P}$ on Ω by:
(1)The empty set is denoted by ϕ. $\mathbb{P}(\phi) = 0$.
(2) For a sequence of mutually exclusive events A_i, let

$A = \cup_{i=1}^{\infty} A_i$, **then**

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

(3) For each event B, $0 \le \mathbb{P}(B) \le 1$ and $\mathbb{P}(\Omega) = 1$.
Then, $\mathbb{P}$ is called a probability on Ω.

Example 5.5 Roll a fair die. Define $E_i = \{$ Observe a i $\}$, $A = \{$ Observe an odd number$\}$, and $B = \{$ Observe a number less or equal to 4$\}$. Find the probabilities $\mathbb{P}(A \cup B)$, $\mathbb{P}(A \cap B)$, and $\mathbb{P}(A^c)$.

$$\mathbb{P}(A \cup B) = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) + \mathbb{P}(E_4) + \mathbb{P}(E_5) = \frac{5}{6}$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(E_1) + \mathbb{P}(E_3) = \frac{2}{6}$$

$$\mathbb{P}(A^c) = \mathbb{P}(E_2) + \mathbb{P}(E_4) + \mathbb{P}(E_6) = \frac{3}{6}$$

Example 5.6 Consider an experiment of flipping a coin as many times as necessary until a head turns up. Define a probability on the sample space. Define $A = \{$ The time to first time observe a head is bigger than one$\}$. Find $\mathbb{P}(A)$.
Solution: Let E_i be the event of first time observing a head at i^{th} flipping. Then, $\Omega = \{E_1, E_2, \cdots\}$. Define

$$\mathbb{P}(E_i) = \frac{1}{2^i}$$

Then, $\mathbb{P}(\Omega) = \sum_{i=1}^{\infty} \mathbb{P}(E_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$**. If we define that for any given event** A**,** $\mathbb{P}(A)$ **is equal to the sum of the probabilities of the simple events in** A**. Then,** $\mathbb{P}$ **is a probability on** Ω**.** $\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \mathbb{P}(E_1) = 1 - \frac{1}{2} = \frac{1}{2}$**. Probability of an event** A **is a number to indicate how big the occurring possibility of event** A**. Sometimes given an event, say** B**, occurs, it can affect the occurring probability of another event** A**.**

Example 5.7 Consider an experiment of rolling a fair die. Define the typical classical probability on the sample space. Define $A = \{$ **Observe a number less or equal to 3**$\}$**. If we already knew that** A **occurred, then what is** $\mathbb{P}(E_1)$**.**
Solution: If we denote that the "probability" of E_1**, given that** A **has occurred by** $\mathbb{P}(E_1|A)$**. Then**

$$\mathbb{P}(E_1|A) = \frac{1}{3}$$

Definition 5.4 The conditional probability of an event A**, given that event** B **has occurred, is**

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B))}{\mathbb{P}(B)} \quad \textbf{if} \quad \mathbb{P}(B) \neq 0$$

The conditional probability of an event B**, given that event** A **has occurred, is**

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B))}{\mathbb{P}(A)} \quad \textbf{if} \quad \mathbb{P}(A) \neq 0$$

Definition 5.5 Two events A and B are said to be independent if and only if

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

or

$$\mathbb{P}(B|A) = \mathbb{P}(B).$$

Otherwise, the two events are said to be dependent.

Example 5.8 Consider an experiment of rolling two fair dice . Define the typical classical probability on the sample space.

$$\{(E_1, E_1), (E_1, E_2), \cdots, (E_6, E_6)\}$$

Define $A = \{$ Observe 1 on the first die$\}$. Define $B = \{$ Observe 2 on the second die$\}$. Events A and B are independent?
Solution: First,

$$\mathbb{P}(A) = \mathbb{P}((E_1, E_1)) + \mathbb{P}((E_1, E_2)) + \mathbb{P}((E_1, E_3)) +$$
$$\mathbb{P}((E_1, E_4)) + \mathbb{P}((E_1, E_5)) + \mathbb{P}((E_1, E_6)) = \tfrac{6}{36}.$$

and

$$\mathbb{P}(B) = \mathbb{P}((E_1, E_2)) + \mathbb{P}((E_2, E_2)) + \mathbb{P}((E_3, E_2)) +$$
$$\mathbb{P}((E_4, E_2)) + \mathbb{P}((E_5, E_2)) + \mathbb{P}((E_6, E_2)) = \tfrac{6}{36}.$$

Second,

$$\mathbb{P}(A \cap B) = \mathbb{P}((E_1, E_2)) = \frac{1}{36}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(A|B) = & \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/36}{6/36} = \frac{1}{6} \\ & = \mathbb{P}(A)\end{aligned}$$

<u>**Additive Rule of Probability**</u> **Given two events A and B,**

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

If A and B are mutually exclusive, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

<u>**Multiplicative Rule of Probability**</u> **Given two events A and B, then**

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A) = \mathbb{P}(B)\mathbb{P}(A|B).$$

If A and B are independent,then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Similarly, if A, B, and C are independent (?), then

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

<u>**Remark:**</u> **Events A, B, and C are independent if and**

only if

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A)\mathbb{P}(B),\\ \mathbb{P}(A \cap C) &= \mathbb{P}(A)\mathbb{P}(C),\\ \mathbb{P}(B \cap C) &= \mathbb{P}(B)\mathbb{P}(C),\\ \mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)\end{aligned}$$

Example 5.9 Suppose that in a box there are 2 distinct, white balls and 3 distinct, black balls. We successively, randomly draw two balls with replacement. Define the typical classical probability on the sample space and define $A = \{$ first drawing is a white ball $\}$ and $B = \{$ Second drawing is a black ball $\}$. Find $\mathbb{P}(A \cup B)$ and $\mathbb{P}(A \cap B)$. Does $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ hold? How about changing the experiment into without replacement?

Solution: The sample space is

$$\begin{aligned}\Omega = \{ & (W_1, W_1), (W_1, W_2), (W_1, B_1),\\ & \cdots, (B_3, B_3)\}\end{aligned}$$

There are $5 \times 5 = 25$ **simple events. Since**

$$\mathbb{P}(A) = \frac{C_1^2 C_1^5}{25},$$

$$\mathbb{P}(B) = \frac{C_1^5 C_1^3}{25},$$

and

$$\mathbb{P}(A \cap B) = \frac{C_1^2 C_1^3}{25},$$

we have

$$\begin{aligned}\mathbb{P}(A\cup B) = &\ \mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B)\\ &= \frac{C_1^2C_1^5}{25}+\frac{C_1^5C_1^3}{25}-\frac{C_1^2C_1^3}{25}=\frac{19}{25}.\end{aligned}$$

With replacement,

$$\begin{aligned}\mathbb{P}(A\cap B) = &\ \frac{6}{25}=\frac{10}{25}\times\frac{15}{25}\\ &= \mathbb{P}(A)\mathbb{P}(B).\end{aligned}$$

If without replacement,

$$\Omega = \ \{(W_1, W_2), \cdots, \}$$

There are $5\times 4 = 20$ **simple events. Since**

$$\mathbb{P}(A)=\frac{C_1^2C_1^4}{20},$$

$$\mathbb{P}(B)=\frac{C_1^2C_1^3+C_1^3C_1^2}{20},$$

and

$$\mathbb{P}(A\cap B)=\frac{C_1^2C_1^3}{20},$$

therefore,

$$\begin{aligned}\mathbb{P}(A\cap B) = &\ \frac{6}{20}\neq\frac{6}{25}=\frac{8}{20}\times\frac{12}{20}\\ &= \mathbb{P}(A)\mathbb{P}(B).\end{aligned}$$