The Basic Homotopy Lemma, III

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June 9th, 2015,

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$$\mathrm{length}(\{u_t\}) \leq 2\pi + \epsilon. \tag{e 0.2}$$

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$$\underline{K}(A) = \bigoplus_{i=0,1} K_i(A) \bigoplus \bigoplus_{i=0,1} \bigoplus_{n>1} K_i(A, \mathbb{Z}/n\mathbb{Z}).$$

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$$\|L(f \otimes 1)u_t - u_t L(f \otimes 1)\| < \epsilon \text{ for all } f \in \mathcal{F} \qquad \text{(e 0.15)}$$

and $t \in [0,1]$*. Moreover,* $\mathrm{length}(\{u_t\}) \leq \pi + \epsilon$*.*

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Proof : Let $\Delta_1 = (1/2)\Delta$, $\mathcal{F}_0 = \{f \otimes 1 : 1 \otimes z : f \in \mathcal{F}\}$ and let $B = A \otimes C(\mathbb{T})$. Then B has the form $QM_r(C(X \times T)Q$. Let $\mathcal{H}' \subset B_+ \setminus \{0\}$ (in place of $\mathcal{H}$) be a finite subset, $\mathcal{G}_1 \subset A \otimes C(\mathbb{T})$ (in place of $\mathcal{G}$) be a finite subset, $\delta_1 > 0$ (in place of δ), $\mathcal{P}' \subset \underline{K}(B)$ (in place of $\mathcal{P}$) be a finite subset required by Theorem 2. 1(for B instead of A) for $\epsilon/16$ (in place of ϵ), $\mathcal{F}_0$ (in place of $\mathcal{F}$) and Δ. Without loss of generality, we may assume that there are finite subsets $\mathcal{H}_1' \subset A_+ \setminus \{0\}$ and $\mathcal{H}_2' \subset C(\mathbb{T})_+ \setminus \{0\}$ such that

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$$\mathcal{P}' = \mathcal{P}_0 \sqcup \mathcal{P}_1, \tag{e 0.16}$$

where $\mathcal{P}_0 \subset \underline{K}(A)$ and $\mathcal{P}_1 \subset \beta(\underline{K}(A))$ are finite subsets.

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Now suppose that $L : A \otimes C(\mathbb{T}) \to M_k$ and a unitary $u \in M_k$ satisfy the assumption with the above $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{G}$, $\mathcal{P}$, δ and σ.

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It follows that

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tr \circ \psi(h) \ge tr \circ L(h) - \sigma \ge \Delta_1(\hat{h}) \qquad \text{(e 0.21)}
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Let $w_1 = \mathrm{Ad}\, U \circ \phi(1 \otimes z)$.

Then

$$\begin{aligned}
\|u - w\| &\leq \|u - L(1 \otimes z)\| + \|L(1 \otimes z) - \mathrm{Ad}\, U \circ \psi(1 \otimes z)\| & \text{(e 0.23)}\\
&< \delta + \epsilon/16 < \epsilon/8. & \text{(e 0.24)}
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$$\|u_t-u\|<\epsilon/8,\ \ \|u_t-w\|<\epsilon/8,\ \ u_0=u,\ \ u_{1/2}=w \qquad \text{(e 0.25)}$$

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It follows (from Theorem 1.1) that there exists a continuous path of unitaries $\{u_t : t \in [1/2, 1]\} \subset M_k$ such that

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for all $t \in [1/2,1]$ and $f \in A\otimes 1$.

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Furthermore,

$$\|u_t L(f \otimes 1) - L(f \otimes 1)u_t\| < \epsilon \text{ for all } f \in \mathcal{F} \tag{e 0.30}$$

and $t \in [0, 1]$.

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Define $L : A \otimes C(\mathbb{T}) \to M_k$ by

$$L(a \otimes f) = (1-p)L_1(a \otimes f)(1-p) \oplus \mathrm{diag}(\psi_0(a), \overbrace{\psi_1(a), ..., \psi_1(a)}^{n})f(w_1).$$

One obtains a continuous path of unitaries $\{w_t \in [0,1]\} \subset M_k$ such that

$$w_0 = u, \;\; w_1 = w_0'' \oplus \psi_0(1 \otimes z) \oplus \mathrm{diag}(w_{1,1}', w_{1,2}', ..., w_{1,n}') \quad \text{(e 0.45)}$$

$$\|w_t\phi(f) - \phi(f)w_t\| < \epsilon \;\; \text{for all} \;\; f \in \mathcal{F}, \quad \text{(e 0.46)}$$

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for all $a \in A$ and $f \in C(\mathbb{T})$. It follows that

$$L(f \otimes 1) \approx \phi(f) \;\text{ and }\; L(1 \otimes z \approx w_1 \tag{e 0.48}$$

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$$L(f \otimes 1) \approx \phi(f) \ \text{ and } \ L(1 \otimes z \approx w_1 \qquad \text{(e 0.48)}$$

One also has that

$$\mathrm{tr} \circ L(h_1 \otimes h_2) \ge \Delta(\hat{h_1}) \cdot \tau_m(h_2)/4 \qquad \text{(e 0.49)}$$

Lemma 2.12.

Let A be a unital subhomogeneous C^*-algebra. Let $\epsilon > 0$, let $\mathcal{F} \subset A$ be a finite subset and let $\sigma_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi : A \to M_n$ (for some integer $n \geq 1$) is a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map. Then, there exists a projection $p \in M_n$ and a unital homomorphism $\phi_0 : A \to pM_np$ such that

$$\begin{aligned} \|p\phi(a) - \phi(a)p\| &< \epsilon \text{ for all } a \in \mathcal{F}, \\ \|\phi(a) - [(1-p)\phi(a)(1-p) + \phi_0(a)]\| &< \epsilon \text{ for all } a \in \mathcal{F} \text{ and} \\ tr(1-p) &< \sigma_0, \end{aligned} \tag{e 0.50}$$

where tr is the normalized trace on M_n.

Proof of 3.4 There exists an integer $n \geq 1$ such that

$$(1/n)\sum_{j=1}^{n} f(e^{\theta + j2\pi i/n}) \geq (63/64)\tau_m(f) \tag{e 0.51}$$

for all $f \in \mathcal{H}_2$ and for any $\theta \in [-\pi, \pi]$. We may also assume that $16\pi/n < \epsilon$.
Let

$$\sigma_1 = (1/2^{10}) \inf\{t(h) : h \in \mathcal{H}_1\} \inf\{\tau_m(g) : g \in \mathcal{H}_2\}\}.$$

Let $\mathcal{F}' = \{f \otimes 1, f \otimes z : f \in \mathcal{F} \cup \mathcal{H}_1\}$. Let $\delta_1 > 0$ (in place of δ) and $\mathcal{G}_1 \subset A \otimes C(\mathbb{T})$ (in place of $\mathcal{G}$) be a finite subset required by **3.5** for $\epsilon/32$ (in place of ϵ), $\mathcal{F}'$ (in place of $\mathcal{F}$) and $\sigma_1/16$ (in place of σ_0). Without loss of generality, one may assume that

$$\mathcal{G}_1 = \{g \otimes 1, 1 \otimes z : g \in \mathcal{G}_2\},$$

where $\mathcal{G}_2 \subset A$ is a finite subset.

Let $\mathcal{H}_1' \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}_2$) be a finite subset required by **??** for $\min\{\epsilon/32, \sigma_1/16\}$ (in place of ϵ), $\mathcal{F} \cup \mathcal{H}_1$ (in place of $\mathcal{F}$), $\mathcal{H}_1$ (in place of $\mathcal{H}$), $(190/258)\Delta$ (in place of Δ) and $\sigma_1/16$ (in place of σ) and integer n. Put

$$\mathcal{H}' = \{h_1 \otimes h_2, h_1 \otimes 1, 1 \otimes h_2 : h_1 \in \mathcal{H}_1 \text{ and } h_2 \in \mathcal{H}_2\}.$$

Let $\mathcal{G}_3 = \mathcal{G}_2 \cup \mathcal{H}_1 \cup \mathcal{H}_1'$. To simplify the notation, without loss of generality, one may assume that $\mathcal{G}_3$ and $\mathcal{F}'$ are all in the unit ball of $A \otimes C(\mathbb{T})$. Let $\delta_2 = \min\{\epsilon/64, \delta_1/2, \sigma_1/16\}$.
Let $\mathcal{G}_4 \subset A$ be a finite subset (in place of $\mathcal{G}$) and let δ_3 (in place of δ) be positive as required by Lemma **3.3** for $\mathcal{G}_3$ (in place of $\mathcal{F}_0$), $\mathcal{F}'$ (in place of $\mathcal{F}$), and δ_2 (in place of ϵ).
Let $\mathcal{G} = \mathcal{G}_4 \cup \mathcal{G}_3$ and $\delta = \min\{\delta_1/4, \delta_2/2, \delta_3/2\}$. Now let $\phi : A \to M_k$ be a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and $u \in M_k$ be a unitary such that (e 0.33) and (e 0.34) hold for the above δ, σ, $\mathcal{G}$ and $\mathcal{H}_1'$.
It follows from Lemma A that there exists a δ_2-$\mathcal{G}_3$-multiplicative

contractive completely positive linear map $L_1 : A \otimes C(\mathbb{T}) \to M_k$ such that

$$\|L_1(g \otimes 1) - \phi(g)\| < \delta_2 \text{ for all } g \in \mathcal{G}_2 \text{ and} \tag{e 0.52}$$

$$\|L_1(1 \otimes z) - u\| < \delta_2. \tag{e 0.53}$$

We then have that

$$\mathrm{tr} \circ L_1(h \otimes 1) \geq \mathrm{tr} \circ \phi(h) - \delta_2 \tag{e 0.54}$$

$$\geq \Delta(\hat{h}) - \sigma_1/16 \geq (191/256)\Delta(\hat{h}) \tag{e 0.55}$$

for all $h \in \mathcal{H}_1$.

It follows **3.5** that there exists a projection $p \in M_k$ and a unital homomorphism $\psi : A \otimes C(\mathbb{T}) \to pM_kp$ such that

$$\|pL_1(f) - L_1(f)p\| < \min\{\epsilon/32, \sigma_1/16\} \text{ for all } f \in \mathcal{F}', \quad (e\,0.56)$$
$$\|L_1(f) - (1-p)L_1(f)(1-p) + \psi(f)\| < \min\{\epsilon/32, \sigma_1/16\} \quad (e\,0.57)$$
$$\text{for all } f \in \mathcal{F}' \text{ and } \mathrm{tr}(1-p) < \sigma_1/16. \quad (e\,0.58)$$

Note that $pM_kp \cong M_m$ for some $m \le k$. It follows from (e 0.55), (e 0.34), (e 0.57) and (e 0.58) that

$$\mathrm{tr} \circ \psi(h) \ge (191/256)\Delta(\hat{h}) - \sigma_1/16 - \sigma_1/16 \ge (190/256)\Delta(\hat{h}) \quad (e\,0.59)$$

for all $h \in \mathcal{H}_1$.

By Cor A (Lecture 2) there are mutually orthogonal projections $e_0, e_1, e_2, ..., e_n \in pM_kp$ such that $e_1, e_2, ..., e_n$ are equivalent, there are unital homomorphisms $\psi_0 : A \otimes C(\mathbb{T}) \to e_0M_ke_0$ and $\psi_1 : A \otimes C(\mathbb{T}) \to e_1M_ke_1$ such that

$$\|\psi(f) - \mathrm{diag}(\psi_0(f), \overbrace{\psi_1(f), ..., \psi_1(f)}^{n})\| < \min\{\epsilon/32, \sigma_1/6\} \quad (e\,0.60)$$
$$\text{for all } f \in \mathcal{F}_1 \text{ and } \mathrm{tr}(e_0) < \sigma_1/16 \quad (e\,0.61)$$

Let $w_0' = \psi_1(1 \otimes z)$. One may write

$$w_0' = \mathrm{diag}(\exp(ia_1), \exp(ia_2), ..., \exp(ia_n)),$$

where $a_j \in e_j M_k e_j$ is a selfadjoint element with $\|a_j\| \le \pi$. By linear algebra, it is easy to find a continuous path of unitaries $\{w_{t,j}' : t \in [0,1]\} \subset e_j M_k e_j$ such that

$$w_{0,j}' = \exp(ia_j), \;\; w_{1,j}' = \exp(i(2\pi j/n)), \tag{e 0.62}$$

$$\text{and } \; \mathrm{length}(\{w_{t,j}'\}) \le \pi + \epsilon/4. \tag{e 0.63}$$

Moreover, one can choose such $w_{t,j}'$ that it commutes with every element in $\psi_1(f)$, $f \in A$. There is a unitary $w_0'' \in (1-p)M_k(1-p)$ such that

$$\|w_0'' - (1-p)L_1(1 \otimes z)(1-p)\| < \epsilon/16. \tag{e 0.64}$$

Put

$$u_0' = w_0'' \oplus \psi_0(1 \otimes z) \oplus w_0'. \tag{e 0.65}$$

Then u_0 is a unitary and

$$\|u - u_0'\| \;\; \le \;\; \|u - L_1(1 \otimes z)\| + \|L_1(1 \otimes z) - u_0'\| \tag{e 0.66}$$

$$\le \;\; \delta_2 + \epsilon/16 < \epsilon/8. \tag{e 0.67}$$

One obtains a continuous path of unitaries $\{w_t \in [0,1]\} \subset M_k$ such that

$$
\begin{aligned}
w_0 = u, \;\; w_1 = w_0'' \oplus \psi_0(1\otimes z) \oplus \mathrm{diag}(w_{1,1}', w_{1,2}', ..., w_{1,n}') \quad & (e\,0.68)\\
\|w_t\phi(f) - \phi(f)w_t\| < \epsilon \;\text{ for all }\; f \in \mathcal{F}, \quad & (e\,0.69)\\
\text{and } \;\mathrm{length}(\{w_t\}) \le \pi + \epsilon. \quad & (e\,0.70)
\end{aligned}
$$

Define $L : A \otimes C(\mathbb{T}) \to M_k$ by

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for all $a \in A$ and $f \in C(\mathbb{T})$. It follows that

$$\|L(f\otimes 1) - \phi(f)\| < \epsilon \;\text{ for all }\; f \in \mathcal{F} \;\text{ and }\; \|L(1\otimes z) - w_1\| < \epsilon.$$

One also has that

$$
\begin{aligned}
L(h_1 \otimes h_2) &\geq \mathrm{tr}((\psi_0(h_1) + n\mathrm{tr}(\psi_1(h_1 \otimes 1)))\mathrm{tr}(h_2(w_1)) \\
&\geq \mathrm{tr} \circ \psi(h_1)(\frac{1-\sigma_1/16}{n})\sum_{j=1}^{n} h_2(e^{i2\pi j/n}) - \sigma_1/6 && \text{(e 0.71)} \\
&\geq (190/256)\Delta(\hat{h_1})(\frac{1-\sigma_1/16}{n})\sum_{j=1}^{n} h_2(e^{i2\pi j/n}) - \sigma_1/6 && \text{(e 0.72)} \\
&\geq (190/256)\Delta(\hat{h_1})(63/64)(1-\sigma_1/16)\tau_m(h_2) - \sigma_1/6 \\
&\geq (190/256)\Delta(\hat{h_1})((63/64)(1-1/2^{14})\tau_m(h_2) - (1/2^{12})t(h_1)\tau_m(h_2) \\
&\geq \Delta(\hat{h_1}) \cdot \tau_m(h_2)/4
\end{aligned}
$$

for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$.

Lemma 3.5.

Let $A = PM_r(C(X))P$ *and let* $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ *be a non-decreasing map.*

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$$u_0 = v,\ u_1 = 1,\ \text{and}\ \|\phi(f)u_t - u_t\phi(f)\| < \epsilon \tag{e 0.73}$$

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Let $A = PM_r(C(X))P$ and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset \underline{K}(A)$ satisfying the following: For any unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\phi : A \to M_k$ (for some integer $k \geq 1$) and any unitary $v \in M_k$ such that

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for all $t \in [0,1]$ and $f \in \mathcal{F}$. Moreover,

$$\mathrm{length}(\{u_t\}) \leq 2\pi + \epsilon. \tag{e 0.74}$$

Definition

Let A be a unital C^*-algebra with $T(A) \neq \emptyset$ and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be a non-decreasing map. Suppose that $\tau_m : C(\mathbb{T}) \to \mathbb{C}$ is the tracial state given by the normalized Lesbegue measure. Define $\Delta_1 : (A \otimes C(\mathbb{T}))_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ by

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Let Δ_1 be as above.

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Let Δ_1 be as above. We will apply 3.2 and 3.4. Let $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_2 \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ be finite subsets, $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset, $\delta_1 > 0$ (in place of δ) and $\mathcal{P} \subset \underline{K}(A)$ be a finite subset required by **3.2**

Proof:

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By applying **3.4** one obtains a continuous path of unitaries $\{u_t : t \in [0, 1/2]\} \subset M_k$ such that

$$u_0 = u, \;\; u_1 = w, \;\; \|u_t\phi(g) - \phi(g)u_t\| < \min\{\delta_1, \epsilon/4\} \qquad (\mathrm{e}\,0.76)$$

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By (e 0.77), (e 0.78), (e 0.81) and (e 0.79), applying **3.2**, Moreover, there is a unital contractive completely positive linear map $L: A \otimes C(\mathbb{T}) \to M_k$ such that

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Moreover, there is a unital contractive completely positive linear map $L : A \otimes C(\mathbb{T}) \to M_k$ such that

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By (e 0.77), (e 0.78), (e 0.81) and (e 0.79), applying **3.2**, there is a continuous path of unitaries $\{u_t \in [1/2, 1]\} \subset M_k$ such that

$$u_{1/2} = w, \;\; u_1 = 1,$$

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By (e 0.77), (e 0.78), (e 0.81) and (e 0.79), applying **3.2**, there is a continuous path of unitaries $\{u_t \in [1/2, 1]\} \subset M_k$ such that

$$u_{1/2} = w, \;\; u_1 = 1, \;\; \|u_t\phi(f) - \phi(f)u_t\| < \epsilon/4 \text{ for all } f \in \mathcal{F} \qquad (e\,0.82)$$

Moreover, there is a unital contractive completely positive linear map $L : A \otimes C(\mathbb{T}) \to M_k$ such that

$$\|L(g \otimes 1) - \phi(g)\| < \min\{\delta_1, \epsilon/4\} \text{ for all } g \in \mathcal{G}_1 \cup \mathcal{F}, \tag{e 0.77}$$
$$\|L(1 \otimes z) - w\| < \min\{\delta_1, \epsilon/4\} \tag{e 0.78}$$
$$\text{and } \mathrm{tr} \circ L(h_1 \otimes h_2) \ge \Delta(h_1)\tau_m(h_2)/4 \tag{e 0.79}$$

for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$. Furthermore,

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