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Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT,
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for any \( \kappa \in \text{Hom}(\Lambda(K(A)), K(A)) \), with \( |\kappa([1_A])| = J_1 \) and \( J_0 = \max \{ |\kappa(g_i)| : g_i = (i-1 \cdot e_0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r : 1 \leq i \leq r \} \), there exists a \( \delta \)-G-multiplicative contractive completely positive linear map \( \Phi : A \to M_{N_0 + \kappa([1_A])} \), such that \( \|\Phi\|_{\mathcal{P}} = (\kappa + [h_1] + [h_2] + \cdots + [h_k])|_{\mathcal{P}}. \)
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$$[\Phi]|_\mathcal{P} = (\kappa + [h_1] + [h_2] \cdots + [h_k])|_\mathcal{P}. \quad (e\,0.2)$$
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Let $\kappa \in \text{Hom}(\text{K}(A), \text{K}(K))$ and $S_i = \kappa(g_i)$, where $g_i = (i - 1) \otimes \ldots \otimes 0, \ldots, 0 \in \mathbb{Z}^r$, there exists a unital $G$-multiplicative contractive completely positive linear map $L: A \rightarrow M_{N_1}$ and a homomorphism $h: A \rightarrow M_{N_1}$ such that $[L]_{\mathcal{P}} = (\kappa + [h])_{\mathcal{P}}$,

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there exists a unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear map $L : A \to M_{N_1}$ and a homomorphism $h : A \to M_{N_1}$ such that
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N(\delta, G, \mathcal{P}, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

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$$ g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r, \quad \text{there exists a unital } \mathcal{G} \text{-} \delta \text{-multiplicative contractive completely positive linear map } L : A \to M_{N_1} \text{ and a homomorphism } h : A \to M_{N_1} \text{ such that} $$

$$ [L]|_{\mathcal{P}} = (\kappa + [h])|_{\mathcal{P}}, \quad (e0.4) $$
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}' \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N(\delta, G, \mathcal{P}, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$ and $S_i = \kappa(g_i)$, where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}'$, there exists a unital $G$-$\delta$-multiplicative contractive completely positive linear map $L : A \to M_{N_1}$ and a homomorphism $h : A \to M_{N_1}$ such that

$$[L]|_\mathcal{P} = (\kappa + [h])|_\mathcal{P}, \quad (e0.4)$$

where $N_1 = \sum_{i=1}^r (N(\delta, G, \mathcal{P}, i) \pm \Lambda_i) \cdot |S_i|$. 
Proof: Let $\psi^+_i : G \to \mathbb{Z}$ be a homomorphism defined by $\psi^+_i(g_i) = 1$, $\psi^+_i(g_j) = 0$, if $j \neq i$, and $\psi^+_i|_{\text{Tor}(G)} = 0$, and let $\psi^-_i(g_i) = -1$ and $\psi^-_i(g_j) = 0$, if $j \neq i$, and $\psi^-_i|_{\text{Tor}(G)} = 0$, $i = 1, 2, ..., r$. Note that $\psi^-_i = -\psi^+_i$, $i = 1, 2, ..., r$. Let $\Lambda_i = |\psi^+_i([1_A])|$, $i = 1, 2, ..., r$.

Let $\kappa^+_i, \kappa^-_i \in \text{Hom}_\Lambda(K(A), K(K))$ be such that $\kappa^+_i|_G = \psi^+_i$ and $\kappa^-_i = \psi^-_i$, $i = 1, 2, ..., r$. Let $N_0(i) \geq 1$ (in place of $N_0$) be required by ?? for $\delta, G, J_0 = 1$ and $J_1 = M_i$. Define $N(\delta, G, P, i) = N_0(i)$, $i = 1, 2, ..., r$.

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$. Then $\kappa|_G = \sum_{i=1}^r S_i \psi^+_i$, where $S_i = \kappa(g_i)$, $i = 1, 2, ..., r$.

By applying 4.1, one obtains $G$-$\delta$-multiplicative contractive completely positive linear maps $L^\pm_i : A \to M_{N_0(i) + \kappa^\pm_i([1_A])}$ and a homomorphism $h^\pm_i : A \to M_{N_0(i)}$ such that

$$[L^\pm_i]|_P = (\kappa^\pm_i + [h^\pm_i])|_P, \ i = 1, 2, ..., r. \quad (e0.5)$$
Define $L = \sum_{i=1}^{r} L_{i}^{\pm,|S_{i}|}$, where $L^{\pm,|S_{i}|} : A \rightarrow M_{|S_{i}|N_{0}(i)}$ defined by

$$L^{\pm,|S_{i}|}(a) = \text{diag}(L_{i}^{\pm}(a), ..., L_{i}^{\pm}(a))$$

for all $a \in A$. One checks that $L : A \rightarrow M_{N_{1}}$, where $N_{1} = \sum_{i=1}^{r} |S_{i}|(\Lambda_{i}^{'} + N(\delta, G, P, i))$ and $\Lambda_{i}^{'} = \psi_{i}^{+}([1_{A}])$, if $S_{i} > 0$, or $\Lambda_{i}^{'} = -\psi_{i}^{+}([1_{A}])$, if $S_{i} < 0$, is a unital $\delta$-$G$-multiplicative contractive completely positive linear map and

$$[L]_{P} = (\kappa + [h])_{P}$$

for some homomorphism $h : A \rightarrow M_{N_{1}}$. 
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras.
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Denote the mapping torus $M_{\phi_1, \phi_2}$ by $A = A(F_1, F_2, \phi_0, \phi_1) \subseteq C([0, 1], F_2) \oplus F_1$.

These $C^*$-algebras are called Elliott-Thomsen building block.

The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $C$. $A$ is said to be minimal, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$.

For $t \in (0, 1)$, define $\pi_t : A \to F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$.

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We also assume that, for any $0 < d < d_X/2$ and for any $d > \delta > 0$, there is a homeomorphism $r : X \setminus X^d - \delta \to X \setminus X^d$ such that $\text{dist}(r(x), x) < \delta$ for all $x \in X \setminus X^d - \delta$. 

(e 0.6)

(e 0.7)
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\[
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\]

\[
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where $X^d = \{x \in X : \text{dist}(x, Z) < d\}$. We also assume that, for any $0 < d < d_X/2$ and for any $d > \delta > 0$, there is a homeomorphism $r : X \setminus X^{d-\delta} \to X \setminus X^d$ such that

$$\text{dist}(r(x), x) < \delta \text{ for all } x \in X \setminus X^{d-\delta}. $$
Examples: $\mathcal{C} \subset \mathcal{D}_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$.)
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$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$
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Then $A \in D_d$. 

Note that $C(Y, F) \in D_1$. All theorems stated for $\text{PM}_r(C(X))$ so far works for $C^*$-algebras in $A_d$ for all $d \geq 1$. (Gong-L-Niu)
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Then $A \in D_d$.

Note that $C(Y, F) \in D_1$.

All theorems stated for $PM_r(C(X))P$ so far works for $C^*$-algebras in $A_d$ for all $d \geq 1$. (Gong-L-Niu)
We have a version of the following when $A$ has the form $PM_r(C(X))P$. 

Lemma 4.3. Let $A \in D_s$ be a unital $C^*$-algebra and let $\Delta : A^q_+ + \{0\} \to (0, 1)$ be a positive map. For any $\epsilon > 0$ and any finite subset $F$, there exist a finite subset $H \subset A^+ + \{0\}$ and an integer $L \geq 1$ satisfying the following: For any unital homomorphism $\phi : A \to M_k$ and any unital homomorphism $\psi : A \to M_R$ for some $R \geq Lk$ such that $\text{tr} \circ \psi(h) \geq \Delta(\hat{h})$ for all $h \in H$, 

there exist a unital homomorphism $\phi_0 : A \to M_R - k$ and a unitary $u \in M_R$ such that $\|\text{Ad}_{u} \circ \text{diag}(\phi(f), \phi_0(f)) - \psi(f)\| < \epsilon$ for all $f \in F$. 

Huaxin Lin

Lecture 4

June 9th, 2015, 9 / 20
We have a version of the following when $A$ has the form $PM_r(C(X))P$.

**Lemma 4.3.**

Let $A \in \mathcal{D}_s$ be a unital $C^*$-algebra and let $\Delta : \mathbb{A}^{q,1}_+ \setminus \{0\} \rightarrow (0, 1)$ be a positive map. For any $\epsilon > 0$ and any finite subset $\mathcal{F}$, there exist a finite subset $\mathcal{H} \subset \mathbb{A}^1_+ \setminus \{0\}$ and an integer $L \geq 1$ satisfying the following: For any unital homomorphism $\phi : A \rightarrow M_k$ and any unital homomorphism $\psi : A \rightarrow M_R$ for some $R \geq Lk$ such that

$$\text{tr} \circ \psi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}, \quad (e \text{0.9})$$

there exist a unital homomorphism $\phi_0 : A \rightarrow M_{R-k}$ and a unitary $u \in M_R$ such that

$$\|\text{Ad} u \circ \text{diag}(\phi(f), \phi_0(f)) - \psi(f)\| < \epsilon \quad (e \text{0.10})$$

for all $f \in \mathcal{F}$. 

Huaxin Lin

Lecture 4

June 9th, 2015, 9 / 20
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $P \subset K(A)$ be a finite subset.
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $\mathcal{D}_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$.
Lemma 4.4.

Let $A$ be a unital C*-algebra in $\mathcal{D}_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. 
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $\mathcal{D}_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map.
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A^+_q \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$,
Lemma 4.4.
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**Lemma 4.4.**

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $P \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $P$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $H \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(\Lambda \otimes C(\mathbb{T}), \mathbb{C})$
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$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\}, \quad (e \; 0.11)$$
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $P \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $P$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{ |\kappa(\beta(g_i))| : 1 \leq i \leq r \}, \quad (e\, 0.11)$$

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. 

$$i-1$$
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^{\times,1} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

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where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_{+1}^q \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{||\kappa(\beta(g_i))|| : 1 \leq i \leq r\},$$

(e 0.11)

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that $R \geq N(K + 1)$ and
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Let $A$ be a unital $C^*$-algebra in $\mathcal{D}_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \operatorname{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

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Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $\mathcal{P} \subset \mathcal{K}(A)$ be a finite subset. Suppose that $G \subset \mathcal{K}(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_{+}^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_{+} \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{ |\kappa(\beta(g_i))| : 1 \leq i \leq r \}, \quad (e \, 0.11)$$

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that $R \geq N(K + 1)$ and $\text{tr} \circ \phi(h) \geq \Delta(\hat{h})$ for all $h \in \mathcal{H}$, there exists a unitary $u \in M_R$ such that

$$\|[\phi(f), u]\| < \epsilon \text{ for all } f \in \mathcal{F} \quad \text{and} \quad (e \, 0.12)$$
Lemma 4.4.

Let $A$ be a unital $C^*$-algebra in $D_s$ and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_{+}^{q,1} \setminus \{0\} \rightarrow (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_{+} \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\},$$

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \rightarrow M_R$ such that $R \geq N(K + 1)$ and $\text{tr} \circ \phi(h) \geq \Delta(\hat{h})$ for all $h \in \mathcal{H}$, there exists a unitary $u \in M_R$ such that

$$\|[\phi(f), u]\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and}$$

$$\text{Bott}(\phi, u)|_{\mathcal{P}} = \kappa \circ \beta|_{\mathcal{P}}.$$
Proof: To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ is a subset of the unit ball. Let $\Delta_1 = (1/8)\Delta$ and $\Delta_2 = (1/16)\Delta$. Let $\epsilon_0 > 0$ and $\mathcal{G}_0 \subset A$ be a finite subset satisfy the following: If $\phi' : A \to B$ (for any unital $C^*$-algebra $B$) is a unital $\epsilon_0\mathcal{G}_0$-multiplicative contractive completely positive linear map and $u' \in B$ is a unitary such that

$$
\|\phi'(g)u' - u'\phi'(g)\| < 4\epsilon_0 \quad \text{for all} \quad g \in \mathcal{G}_0,
$$

then $\text{Bott}(\phi', u'|_{\mathcal{P}}$ is well defined. Moreover, if $\phi' : A \to B$ is another unital $\epsilon_0\mathcal{G}_0$-multiplicative contractive completely positive linear map then

$$
\text{Bott}(\phi', u'|_{\mathcal{P}} = \text{Bott}(\phi'', u''|_{\mathcal{P}},
$$

provided that

$$
\|\phi'(g) - \phi''(g)\| < 4\epsilon_0 \quad \text{and} \quad \|u' - u''\| < 4\epsilon_0 \quad \text{for all} \quad g \in \mathcal{G}_0.
$$

We may assume that $1_A \in \mathcal{G}_0$. Let
$G'_0 = \{ g \otimes f : g \in G_0 \text{ and } f = \{ 1_{C(T)}, z, z^* \} \}.$

where $z$ is the identity function on the unit circle $\mathbb{T}$. We also assume that if $\Psi' : A \otimes C(T) \to C$ (to some unital $C^*$-algebra $C$) is a $G'_0$-$\epsilon_0$-multiplicative contractive completely positive linear map, then there exist a unitary $u' \in C$ such that

$$\| \Psi'(1 \otimes z) - u' \| < 4\epsilon_0.$$ (e 0.17)

Without loss of generality, we may assume that $G_0$ is in the unital ball of $A$. Let $\epsilon_1 = \min\{\epsilon/64, \epsilon_0/512\}$ and $F_1 = F \cup G_0$. Let $H_0 \subset A_+ \setminus \{0\}$ (in place of $H$) be a finite subset and $L \geq 1$ be an integer required by 4.3 for $\epsilon_1$ (in place of $\epsilon$) and $F_1$ (in place of $F$) as well as $\Delta_2$ (in place of $\Delta$). Let $H_1 \subset A^1_+ \setminus \{0\}$ be finite subsets, $G_1 \subset A$ (in place of $G$) be a finite subset, $\delta_1 > 0$ (in place of $\delta$), $P_1 \subset K(A)$ (in place of $P$) be a finite subset, $H_2 \subset A_{s.a.}$ be a finite subset and $1 > \sigma > 0$ be required by ?? for $\epsilon_1$ (in place of $\epsilon$), $F_1$ (in place of $F$) and $\Delta_1$. We may assume that $[1_A] \in P_2$, $H_2$ is in the unit ball of $A$ and $H_0 \subset H_1$. 
Without loss of generality, we may assume that \( \delta_1, \sigma < \epsilon_1/16 \) and \( \mathcal{F}_1 \subset \mathcal{G}_1 \). Let \( \mathcal{P}_2 = \mathcal{P} \cup \mathcal{P}_1 \).

Suppose that \( A \) has irreducible representations of rank \( r_1, r_2, \ldots, r_k \). Fix one irreducible representation \( \pi_0 : A \to M_{r_1} \). Let \( N(p) \geq 1 \) (in place of \( N(\mathcal{P}_0) \)) and \( \mathcal{H}_0 \subset A_+^1 \setminus \{0\} \) (in place of \( \mathcal{H} \)) be a finite subset required by ?? for \( \{1_A\} \) (in place of \( \mathcal{P}_0 \)) and \( (1/3)\Delta \).

Let \( G_0 = G \cap K_0(A) \) and write \( G_0 = \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \text{Tor}(G_0) \), where

\[
\mathbb{Z}^{s_2} \oplus \text{Tor}(G_0) \subset \ker \rho_A.
\]

Let \( x_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \), \( j = 1, 2, \ldots, s_2 \). Note that \( A \otimes C(\mathbb{T}) \in \mathcal{A}_s \) and \( A \otimes C(\mathbb{T}) \) has irreducible representations of rank \( r_1, r_2, \ldots, r_k \). Let

\[
\bar{r} = \max\{|(\pi_0)_*0(x_j)| : 0 \leq j \leq s_1 + s_2\}.
\]

Let \( \mathcal{P}_3 \subset K(A \otimes C(\mathbb{T})) \) be a finite subset set containing \( \mathcal{P}_2 \), \( \{\beta(g_j) : 1 \leq j \leq r\} \) and a finite subset which generates \( \beta(\text{Tor}(G_1)) \).

Choose \( \delta_2 > 0 \) and finite subset
\[ \overline{G} = \{ g \otimes f : g \in G_2, \ f \in \{ 1, z, z^* \} \} \]

in \( A \otimes C(\mathbb{T}) \), where \( G_2 \subset A \) is a finite subset such that, for any unital \( \delta_2 \)-\( \overline{G} \)-multiplicative contractive completely positive linear map \( \Phi' : A \otimes C(\mathbb{T}) \to C \) (for any unital \( C^* \)-algebra \( C \) with \( T(C) \neq \emptyset \)), \( [\Phi']|_{\mathcal{P}_3} \) is well defined and

\[ [\Phi']|_{\text{Tor}(G_0) \oplus \beta(\text{Tor}(G_1))} = 0. \quad (e \ 0.18) \]

We may assume \( G_2 \supset G_1 \cup F_1 \).

Let \( \sigma_1 = \min \{ \Delta_2(\hat{h}) : h \in \mathcal{H}_1 \} \). Note \( K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \beta(K_1(A)) \) and \( \overline{K}(A \otimes C(\mathbb{T})) = \overline{K}(A) \oplus \beta(\overline{K}(A)) \). Consider the subgroup of \( K_0(A \otimes C(\mathbb{T})) \):

\[ \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \mathbb{Z}^{r} \oplus \text{Tor}(K_0(A) \oplus \beta(\text{Tor}(K_1(A))). \]

Let \( \delta_3 = \min \{ \delta_1, \delta_2 \} \). Let \( N(\delta_3, \overline{G}, \mathcal{P}_3, i) \) and \( \Lambda_i, \ i = 1, 2, \ldots, s_1 + s_2 + r \), be required by 4.2. (for \( A \otimes C(\mathbb{T}) \)). Choose an integer \( n_1 \geq N(p) \) such that

\[ \frac{(\sum_{i=1}^{s_1+s_2+r} N(\delta_3, \overline{G}, \mathcal{P}_3, i) + 1 + \Lambda_i)N(p)}{n_1 - 1} < \min \{ \sigma/16, \sigma_1/2 \}. \quad (e \ 0.19) \]
Choose $n > n_1$ such that

\[
\frac{n_1 + 2}{n} < \min\{\sigma/16, \sigma_1/2, 1/(L + 1)\}.
\] (e 0.20)

Let $\epsilon_2 > 0$ and let $\mathcal{F}_2 \subset A$ be a finite subset such that $[\Psi]|_{\mathcal{P}_2}$ is well defined.

Let $\epsilon_3 = \min\{\epsilon_2/2, \epsilon_1\}$ and $\mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2$.

Let $\delta_4 > 0$ (in place of $\delta$), $\mathcal{G}_3 \subset A$ (in place of $\mathcal{G}$) be a finite subset and let $\mathcal{H}_3 \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}_2$) required by Cor. 2.5 for $\epsilon_3$ (in place of $\epsilon$), $\mathcal{F}_3 \cup \mathcal{H}_1$ (in place of $\mathcal{F}$), $\delta_3/2$ (in place of $\epsilon_0$), $\mathcal{G}_2$ (in place of $\mathcal{G}_0$), $\Delta$, $\mathcal{H}_1$ (in place of $\mathcal{H}$), $\min\{\sigma/16, \sigma_1/2\}$ (in place of $\sigma$) and $n^2$ (in place of $K$) required by Cor. 2.5 (with $L_1 = L_2$).

Let $\mathcal{G} = \mathcal{F}_3 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ and let $\delta = \min\{\epsilon_3/16, \delta_4, \delta_3/16\}$. Let $\mathcal{G}_5 = \{g \otimes f : g \in \mathcal{G}_4, f \in \{1, z, z^*\}\}$.

Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_0$. Define $N_0 = (n + 1)N(p)\left(\sum_{i=1}^{s_1+s_2+r}N(\delta_3, \mathcal{G}_0, \mathcal{P}_3, i) + \Lambda_i + 1\right)$ and define $N = N_0 + N_0\bar{r}$. Fix any $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ with

\[
K = \max\{|\kappa(\beta(g_j))| : 1 \leq j \leq r\}.
\]
Let $R > N(K + 1)$. Suppose that $\phi : A \to M_R$ is a unital $G$-$\delta$-multiplicative contractive completely positive linear map such that
\[
\text{tr} \circ \phi(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}.
\tag{e 0.21}
\]

Then, by Cor. 2.5, there exists mutually orthogonal projections $e_0, e_1, e_2, \ldots, e_n \in M_R$ such that $e_1, e_2, \ldots, e_n$ are equivalent, $\text{tr}(e_0) < \min\{\sigma/64, \sigma_1/4\}$ and $e_0 + \sum_{i=1}^{n} e_i = 1_{M_R}$, and there exists a unital $\delta_3/2$-$G_2$-multiplicative contractive completely positive linear map $\psi_0 : A \to e_0 M_R e_0$ and a unital homomorphism $\psi : A \to e_1 M_R e_1$ such that
\[
\|\phi(f) - (\psi_0(f) \oplus \psi(f), \psi(f), \ldots, \psi(f))\| < \epsilon_3 \quad \text{for all } f \in F_3 \quad \text{and (e 0.22)}
\]
\[
\text{tr} \circ \psi(h) \geq \Delta(\hat{h})/3n \quad \text{for all } h \in \mathcal{H}_1 \quad \text{(e 0.23)}
\]

Let $\alpha \in \text{Hom}_{\Lambda}(K(A \otimes C(\mathbb{T})), K(M_r))$ be define as follows: $\alpha|_{K(A)} = [\pi_0]$ and $\alpha|_{\beta(K(A))} = \kappa|_{\beta(K(A))}$. Let
\[
\max\{|\kappa \circ \beta(g_i)| : i = 1, 2, \ldots, r, |\pi_0(x_j)| : 1 \leq j \leq s_1 + s_2\} \leq \max\{K, \bar{r}\}.
\]

Applying we obtain a unital $\delta_3$-$G$-multiplicative contractive completely positive linear map $\Psi : A \otimes C(\mathbb{T}) \to M_{N_1'}$, where
\[ N_1' \leq N_1 = \sum_{j=1}^{s_1+s_2+r} N(\delta_3, G_0, P_3, j) + \Lambda_i \) \max\{K, r\}, \] and a homomorphism \( H_0 : A \otimes C(\mathbb{T}) \to H_0(1_A)M_{N_1'}H_0(1_A) \) such that such that

\[ [\Psi]|_{P_3} = (\alpha + [H_0])|_{P_3}. \tag{e 0.24} \]

In particular, since \([1_A] \in P_2 \subset P_3\),

\[ \text{rank}\psi(1_A) = r_1 + \text{rank}(H_0). \]

Note that

\[ \frac{N_1' + N(p)}{R} \leq \frac{N_1 + N(p)}{N(K+1)} < 1/(n+1). \tag{e 0.25} \]

Let \( R_1 = \text{rank } e_1 \). Then \( R_1 \geq R/(n+1) \). So, from (e 0.25) \( R_1 \geq N_1 + N(p) \). In other words, \( R_1 - N_1' \geq N(p) \). Note that

\[ t \circ \psi(\hat{g}) \geq (1/3)\Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_0, \]

where \( t \) is the tracial state on \( M_{R_1} \). By applying to the case that \( \phi = \pi_0 \oplus H_0 \) and \( \mathcal{P}_0 = \{[1_A]\} \), we obtain a unital homomorphism
Define $\psi'_0 : A \otimes C(\mathbb{T}) \to \mathbf{e}_0 M_R e_0$ by $\psi'_0(a \otimes f) = \psi_0(a) \cdot f(1) \cdot e_0$ for all $a \in A$ and $f \in C(\mathbb{T})$, where $1 \in \mathbb{T}$. Define $\psi' : A \otimes C(\mathbb{T}) \to e_1 M_R e_1$ by $\psi'(a \otimes f) = \psi(a) \cdot f(1) \cdot e_0$ for all $a \in A$ and $f \in C(\mathbb{T})$. Let $E_1 = \text{diag}(e_1, e_2, \ldots, e_{n_{n_1}})$.

Define $L_1 : A \to E_1 M_R E_1$ by

$$L_1(a) = \pi_0(a) \oplus H_0 |_A(a) \oplus h_0(a \otimes 1) \oplus \left(\psi(f), \ldots, \psi(f)\right)$$

for $a \in A$ and define $L_2 : A \to E_1 M_R E_1$ by

$$L_2(a) = \psi(a \otimes 1) \oplus h_0(a \otimes 1) \oplus \left(\psi(f), \ldots, \psi(f)\right)$$

for $a \in A$. Note that

$$[L_1]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1} \quad (e\ 0.26)$$

$$\text{tr} \circ L_1(h) \geq \Delta_1(\hat{h}), \ \text{tr} \circ L_2(h) \geq \Delta_1(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \quad (e\ 0.27)$$

$$|\text{tr} \circ L_1(g) - \text{tr} \circ L_2(g)| < \sigma \text{ for all } g \in \mathcal{H}_2. \quad (e\ 0.28)$$

It follows from ?? that there exists a unitary $w_1 \in E_1 M_R E_1$ such that

$$\|\text{ad} \ w_1 \circ L_2(a) - L_1(a)\| < \epsilon_1 \text{ for all } a \in \mathcal{F}_1. \quad (e\ 0.29)$$
Define $E_2 = (e_1 + e_2 + \cdots + e_{n^2})$ and define $\Phi : A \rightarrow E_2 M_R E_2$ by

$$\Phi(f)(a) = \text{diag}(\psi(a), \psi(a), \ldots, \psi(a))$$

for all $a \in A$. \hspace{1cm} (e 0.30)

Then

$$\text{tr} \circ \Phi(h) \geq \Delta_2(\hat{h})$$

for all $h \in \mathcal{H}_0$. \hspace{1cm} (e 0.31)

By (e 0.20), $\frac{n}{n_{1+2}} > L + 1$. By applying 4.3, we obtain a unitary $w_2 \in E_2 M_R E_2$ and a unital homomorphism $H_1 : A \rightarrow (E_2 - E_1) M_R (E_2 - E_1)$ such that

$$\| \text{ad} w_2 \circ \text{diag}(L_1(a), H_1(a)) - \Phi(a) \| < \epsilon_1$$

for all $a \in \mathcal{F}_1$. \hspace{1cm} (e 0.32)

Put

$$w = (e_0 \oplus w_1 \oplus (E_2 - E_1))(e_0 \oplus w_2) \in M_R.$$

Define $H'_1 : A \otimes C(\mathbb{T}) \rightarrow (E_2 - E_1) M_R (E_2 - E_1)$ by

$$H'_1(a \otimes f) = H_1(a) \cdot f(1) \cdot (E_2 - E_1)$$

for all $a \in A$ and $f \in C(\mathbb{T})$. Define
$\Psi_1 : A \to M_R$ by

$$\Psi_1(f) = \psi_0(f) \oplus \Psi(f) \oplus h_0 \oplus \psi'(f), \ldots, \psi'(f) \oplus H'_1(f)$$

for all $f \in A \otimes C(\mathbb{T})$. It follows from (e 0.29), (e 0.32) and (e 0.22) that

$$\|\phi(a) - w^*\Psi_1(a \otimes 1)w\| < \epsilon_1 + \epsilon_1 + \epsilon_3 \text{ for all } a \in \mathcal{F}. \quad (e 0.34)$$

Now let $v \in M_R$ be a unitary such that

$$\|\Psi_1(1 \otimes z) - v\| < 4\epsilon_1. \quad (e 0.35)$$

Put $u = w^*vw$. Then, we estimate that

$$\|[\phi(a), u]\| < \min\{\epsilon, \epsilon_0\} \text{ for all } a \in \mathcal{F}_1. \quad (e 0.36)$$

Moreover, by (e 0.29), (e 0.24) and (e 0.15),

$$\text{Bott}(\phi, u)|_P = \kappa \circ \beta|_P. \quad (e 0.37)$$
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. 

Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_{1} + \{0\} \to (0, 1)$ be an order preserving map.

There exists a finite subset $H \subset A^q_{1} + \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$, a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A$, a finite subset $U \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ (k depends on A) for which $U \subset P$, and $N \in \mathbb{N}$ satisfying the following:

For any unital $G$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that $|\phi|_P = |\psi|_P$,

\[ \tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), a \in H_1, \]

\[ |\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in H_2, \]

\[ \text{dist}(\phi^\perp(u), \psi^\perp(u)) < \gamma_2, \quad \text{for all } u \in U, \]

\[ \|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon, \quad \text{for all } f \in F. \]
Theorem 4.5.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number.
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map.

There exists a finite subset $H_1 \subset A^q_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$, a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A$, a finite subset $U \subset U(M_k + 1)(A)/CU(M_k + 1)(A)$ (where $k$ depends on $A$) for which $U \subset P$, and $N \in \mathbb{N}$ satisfying the following:

For any unital $G$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that $\|\phi\|_P = \|\psi\|_P$, $(e 0.38)$

$\tau(\phi(a)) \geq \Delta(a), \tau(\psi(a)) \geq \Delta(a)$, for all $\tau \in T(C)$,

$\epsilon 0.39)$

$|\tau(\phi(a)) - \tau(\psi(a))| < \gamma_1$, for all $a \in H_2$,

$\epsilon 0.40)$

and $\|\tau(\phi^\wedge(u)) - \tau(\psi^\wedge(u))\| < \gamma_2$, for all $u \in U$,

$\epsilon 0.41)$

there exists a unitary $W \in C \otimes M_N$ such that $\|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon$, for all $f \in F$. $(e 0.42)$
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^{q,1}_{+} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^1_{+} \setminus \{0\}$,
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, $\epsilon > 0$..
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_{+}^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $H_1 \subset A_{+}^1 \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$
Theorem 4.5.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$,
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_+^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$,
Theorem 4.5. Let $A \in D_d$ for some integer $d \geq 1$. Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_{+1}^q \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a finite subset $H_1 \subset A_{+1}^1 \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$ and a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A$, a finite subset $U \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$)
Theorem 4.5.

Let \( A \in \mathcal{D}_d \) for some integer \( d \geq 1 \). Let \( F \subset A \), let \( \epsilon > 0 \) be a positive number and let \( \Delta : A^q_{+} \setminus \{0\} \to (0, 1) \) be an order preserving map. There exists a finite subset \( \mathcal{H}_1 \subset A^1_{+} \setminus \{0\} \), \( \gamma_1 > 0 \), \( \gamma_2 > 0 \), \( \delta > 0 \), a finite subset \( \mathcal{G} \subset A \) and a finite subset \( \mathcal{P} \subset K(A) \), a finite subset \( \mathcal{H}_2 \subset A \), a finite subset \( \mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A)) \) (\( k \) depends on \( A \)) for which \([\mathcal{U}] \subset \mathcal{P}\).
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following:
Theorem 4.5.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/\overline{CU(M_{k+1}(A))}$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$. 

\[(e \ 0.38) \quad \tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \quad a \in \mathcal{H}_1, \]

\[(e \ 0.39) \quad |\tau(\phi(a)) - \tau(\psi(a))| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2, \]

\[(e \ 0.40) \quad \text{dist}(\phi^\sharp(u), \psi^\sharp(u)) < \gamma_2, \quad \text{for all } u \in \mathcal{U}, \]

\[(e \ 0.41) \quad \text{there exists a unitary } W \in C \otimes M_N \text{ such that } \|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon, \quad \text{for all } f \in \mathcal{F}.\]
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_+^1 \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that

\[
\|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon,
\]

for all $f \in \mathcal{F}$. (e 0.42)
Theorem 4.5.
Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\varepsilon > 0$ be a positive number and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_+^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that

$$[\phi]|_\mathcal{P} = [\psi]|_\mathcal{P},$$
Theorem 4.5.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that

\begin{equation}
[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},
\end{equation}

\begin{equation}
\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \ a \in \mathcal{H}_1,
\end{equation}
Theorem 4.5.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\varepsilon > 0$ be a positive number and let $\Delta : A_{+}^{1,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_+^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A)) / CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in \mathcal{C}$ such that

$$[\phi]|_\mathcal{P} = [\psi]|_\mathcal{P},$$  \hspace{1cm} (e 0.38)

$$\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \quad a \in \mathcal{H}_1,$$  \hspace{1cm} (e 0.39)

$$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2,$$
Theorem 4.5.

Let \( A \in D_d \) for some integer \( d \geq 1 \). Let \( \mathcal{F} \subset A \), let \( \epsilon > 0 \) be a positive number and let \( \Delta : A_+^{q,1} \setminus \{0\} \to (0, 1) \) be an order preserving map. There exists a finite subset \( \mathcal{H}_1 \subset A_+^1 \setminus \{0\} \), \( \gamma_1 > 0 \), \( \gamma_2 > 0 \), \( \delta > 0 \), a finite subset \( \mathcal{G} \subset A \) and a finite subset \( \mathcal{P} \subset K(A) \), a finite subset \( \mathcal{H}_2 \subset A \), a finite subset \( \mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A)) \) (\( k \) depends on \( A \)) for which \([\mathcal{U}] \subset \mathcal{P}\), and \( N \in \mathbb{N} \) satisfying the following: For any unital \( \mathcal{G}\)-\( \delta \)-multiplicative contractive completely positive linear maps \( \phi, \psi : A \to C \) for some \( C \in \mathcal{C} \) such that

\[
[\phi]|_\mathcal{P} = [\psi]|_\mathcal{P}, \tag{e 0.38}
\]

\[
\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \quad a \in \mathcal{H}_1, \tag{e 0.39}
\]

\[
|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2, \tag{e 0.40}
\]

and \( \text{dist}(\phi^\dagger(u), \psi^\dagger(u)) < \gamma_2, \quad \text{for all } u \in \mathcal{U}, \tag{e 0.41} \)
Theorem 4.5.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset \mathcal{K}(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in \mathcal{C}$ such that

$$[\phi]|_\mathcal{P} = [\psi]|_\mathcal{P}, \quad (e\ 0.38)$$

$$\tau(\phi(a)) \geq \Delta(a), \ \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \ a \in \mathcal{H}_1, \quad (e\ 0.39)$$

$$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2, \quad (e\ 0.40)$$

and dist($\phi^\dagger(u), \psi^\dagger(u)$) < $\gamma_2$, for all $u \in \mathcal{U}$, \quad (e\ 0.41)

there exists a unitary $W \in C \otimes M_N$ such that
Theorem 4.5. Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^1_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in \mathcal{C}$ such that

\[ [\phi]|_\mathcal{P} = [\psi]|_\mathcal{P}, \tag{e 0.38} \]

\[ \tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \quad a \in \mathcal{H}_1, \tag{e 0.39} \]

\[ |\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2, \tag{e 0.40} \]

and \[ \text{dist}(\phi^\dagger(u), \psi^\dagger(u)) < \gamma_2, \quad \text{for all } u \in \mathcal{U}, \tag{e 0.41} \]

there exists a unitary $W \in C \otimes M_N$ such that

\[ \|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon, \quad \text{for all } f \in \mathcal{F}. \tag{e 0.42} \]
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$.

Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition so that

$\pi_{t_i} \circ \phi(g) \approx \pi_{t'_i} \circ \phi(g)$

and

$\pi_{t_i} \circ \psi(g) \approx \pi_{t'_i} \circ \psi(g)$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i], i = 1, 2, \ldots, n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$, such that

$w_i \pi_{t_i} \circ \phi(g) w_i^* \approx \pi_{t'_i} \circ \psi(g) w_i^* \approx \phi_{i+1} \circ \phi(g) \approx \phi_i \circ \phi(g)$.

We may also assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$.

Note that

$(w_i^* w_i+1) \pi_{t_i} \circ \phi(g) (w_i^* w_i+1) \approx w_i^* w_i+1 \pi_{t'_i} \circ \psi(g) w_i \approx \phi_{i+1} \circ \phi(g) \approx \phi_i \circ \phi(g)$.
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$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g)$$

and

We need to apply the Homotopy Lemma.
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

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be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \text{ and } \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g)$$  \hspace{1cm} (e 0.43)
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

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$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \text{ and } \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (\text{e} \ 0.43)$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$. 
Idea of the proof:

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We may also assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$.

Note that

$$\left( w_i^{*} + w_{i+1} \right) \pi_t \circ \phi(g) \left( w_{i+1} + w_i \right) \approx w_i^{*} \pi_{t+1} \circ \psi(g) w_i \approx \phi_{i+1} \circ \phi(g) \approx \phi_i \circ \phi(g) \quad (e \ 0.45)$$

We need to apply the Homotopy Lemma.
Idea of the proof:

Let \( C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1 \). Let

\[
0 = t_0 < t_1 < \cdots < t_n = 1
\]

be a partition so that

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\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e0.43)
\]

for all \( g \in G \), provided \( t, t' \in [t_{i-1}, t_i], \ i = 1, 2, \ldots, n \).

By applying Theorem 2.1, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \),
\( w_0 \in h_0(F_1) \), if \( i = 0 \),
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

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We may also assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$.

Note that

$$w_i \pi_{t_i} \circ \phi(g) (w_i w_{i+1}) \approx w_{i+1} \pi_{t_{i+1}} \circ \psi(g) w_i \approx \phi_{i+1} \circ \phi(g) \quad (e\ 0.45)$$

We need to apply the Homotopy Lemma.
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

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for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i], \ i = 1, 2, \ldots, n$.

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$$w_i \pi_{t_i} \circ \phi(g) w_i^* \approx \pi_{t_i} \circ \psi(g). \quad (e 0.44)$$
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

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for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, ..., n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$, such that

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We may also assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$. 


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By applying Theorem 2.1, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \),
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\[
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Note that

\[
(w_{i+1}^* w_i) \pi_{t_i} \circ \phi(g) (w_i^* w_{i+1}) \approx w_{i+1}^* \pi_{t_{i+1}} \circ \psi(g) w_{i+1}
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Idea of the proof:

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By applying Theorem 2.1, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \), \( w_0 \in h_0(F_1) \), if \( i = 0 \), and \( w_1 \in h_1(F_1) \), if \( i = 1 \), such that

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(w_{i+1}^* w_i) \pi_{t_i} \circ \phi(g)(w_i^* w_{i+1}) \approx w_{i+1}^* \pi_{t_{i+1}} \circ \psi(g) w_{i+1}
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for all \( g \in G \), provided \( t, t' \in [t_{i-1}, t_i] \), \( i = 1, 2, \ldots, n \).

By applying Theorem 2.1, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \), \( w_0 \in h_0(F_1) \), if \( i = 0 \), and \( w_1 \in h_1(F_1) \), if \( i = 1 \), such that

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We may also assume that there is a unitary \( w_e \in F_1 \) such that \( h_0(w_e) = w_0 \) and \( h_1(w_e) = w_n \).

Note that

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\]

\[
\approx \phi_{i+1} \circ \phi(g) \approx \phi_i \circ \phi(g).
\]  

(e 0.45)

We need to apply the Homotopy Lemma.
Need to change $w_i$ to something $z_i w_i$ to make “bott” element trivial,
Need to change $w_i$ to something $z_i w_i$ to make “bott” element trivial, which is quite demanding.
Need to change $w_i$ to something $z_i w_i$ to make “bott” element trivial, which is quite demanding. In order not to accumulate errors, the condition ($e 0.41$) is used.
Need to change $w_i$ to something $z_i w_i$ to make “bott” element trivial, which is quite demanding. In order not to accumulate errors, the condition $(e^{0.41})$ is used. We also need to take care of “end points”. 
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. 
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. 

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Huaxin Lin  
Lecture 4  
June 9th, 2015, 14 / 20
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. 

There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence $0 \rightarrow \text{Aff}(T(A))/\rho_A(K_0(A)) \rightarrow U(M_k(A))/CU(M_k(A)) \rightarrow K_1(A) \rightarrow 0$.

Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. If $\phi: A \rightarrow B$ is a unital homomorphism then $\phi^\# : U(M_k(A))/CU(M_k(A)) \rightarrow U(M_k(B))/CU(M_k(B))$. Slightly modification, if $\phi$ is almost multiplicative, $\phi^\#$ can also be defined.
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. 

Slightly modification, if $\phi$ is almost multiplicative, $\phi^\dagger$ can also be defined.
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Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $igcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. 

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Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even

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$$

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$$
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Huaxin Lin
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June 9th, 2015, 14 / 20
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

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Let \( A \) be a unital \( C^* \)-algebra and let \( U(A) \) be the unitary group of \( A \). Denote by \( CU(A) \) the closure of the commutator subgroup of \( U(A) \). When \( A \) has stable rank one \( CU(A) \subset U_0(A) \). We will consider the group \( U(A)/CU(A) \). Or \( U(M_k(A))/CU(M_k(A)) \). Or even \( \bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A))) \). There is a metric on \( U(M_k(A))/CU(M_k(A)) \). Let us assume that \( A \) has stable rank \( \leq k \). C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

\[
0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(M_k(A))/CU(M_k(A)) \to K_1(A) \to 0.
\]

Let \( B \) is another unital \( C^* \)-algebra of stable rank at most \( k \). If \( \phi: A \to B \) is a unital homomorphism then \( \phi^\dagger : U(M_k(A))/CU(M_k(A)) \to U(M_k(B))/CU(M_k(B)) \).
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^\infty (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

$$0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(M_k(A))/CU(M_k(A)) \to K_1(A) \to 0.$$ 

Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. If $\phi : A \to B$ is a unital homomorphism then $\phi^\dagger : U(M_k(A))/CU(M_k(A)) \to U(M_k(B))/CU(M_k(B))$. Slightly modification, if $\phi$ is almost multiplicative, $\phi^\dagger$ can also be defined.
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in C$,
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in C$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW.
Definition

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Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-G-multiplicative, then $L_e$ is also $\delta$-G-multiplicative.
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Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$,
Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in C$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a 1-dim NCCW as defined.
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Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a 1-dim NCCW as defined. Let $L_1, L_2 : A \rightarrow C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $F \subset A$ be a subset. Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$
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$$\|w_0^* \pi_0 \circ L_1(a)w_0 - \pi_0 \circ L_2(a)\| < \epsilon \quad \text{and}$$

$$\|w_1^* \pi_1 \circ L_1(a)w_1 - \pi_1 \circ L_2(a)\| < \epsilon \quad \text{for all } a \in \mathcal{F}.$$
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$$\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| < \epsilon \quad \text{and} \quad (e 0.47)$$

$$\|w_1^* \pi_1 \circ L_1(a) w_1 - \pi_1 \circ L_2(a)\| < \epsilon \quad \text{for all } a \in \mathcal{F}. \quad (e 0.48)$$
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\[
\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| < \epsilon \quad \text{and} \quad (e \, 0.47)
\]

\[
\|w_1^* \pi_1 \circ L_1(a) w_1 - \pi_1 \circ L_2(a)\| < \epsilon \quad \text{for all } a \in F. \quad (e \, 0.48)
\]

Then there exists a unitary $u \in F_1$ such that
Lemma

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Then there exists a unitary $u \in F_1$ such that

$$\|\phi_0(u)^* \pi_0 \circ L_1(a) \phi_0(u) - \pi_0 \circ L_2(a)\| < \epsilon \text{ and} \quad (e0.49)$$
Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a 1-dim NCCW as defined. Let $L_1, L_2 : A \to C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$ and $w_1 \in \pi_1(C) \subset F_2$ such that

$$\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| \ < \ \epsilon \ \text{and} \quad (e \ 0.47)$$

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$$\|\phi_1(u)^* \pi_1 \circ L_1(a) \phi_1(u) - \pi_1 \circ L_2(a)\| \ < \ \epsilon \ \text{for all } a \in \mathcal{F}. \quad (e \ 0.50)$$
Proof:

Write \( F_1 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k} \) and \( F_2 = M_{r_1} \oplus M_{r_2} \oplus \cdots \oplus M_{r_l} \). We may assume that, \( \ker \phi_0 \cap \ker \phi_1 = \{0\} \).

We may assume that \( \phi_0|_{M_{n_i}} \) is injective, \( i = 1, 2, \ldots, k(0) \) with \( k(0) \leq k \), \( \phi_0|_{M_{n_i}} = 0 \) if \( i > k(0) \), and \( \phi_1|_{M_{n_i}} \) is injective, \( i = k(1), k(1) + 1, \ldots, k \) with \( k(1) \leq k \), \( \phi_1|_{M_{n_i}} = 0 \), if \( i < k(1) \). Write \( F_{1,0} = \bigoplus_{i=1}^{k(0)} M_{n_i} \) and \( F_{1,1} = \bigoplus_{j=k(1)}^{k} M_{n_j} \). Note that \( k(1) \leq k(0) + 1 \), \( \phi_0|_{F_{1,0}} \) and \( \phi_1|_{F_{1,1}} \) are injective. Note \( \phi_0(F_{1,0}) = \phi_0(F_1) = \pi_0(C) \) and \( \phi_1(F_{1,1}) = \phi_1(F_1) = \pi_1(C) \). Let \( \psi_0 = (\phi_0|_{F_{1,0}})^{-1} \) and \( \psi_1 = (\phi_1|_{F_{1,1}})^{-1} \).

For each fixed \( a \in A \), since \( L_i(a) \in C \) (\( i = 0, 1 \)), there are elements

\[
g_{a,i} = g_{a,i,1} \oplus g_{a,i,2} \oplus \cdots \oplus g_{a,i,k(0)} \oplus \cdots \oplus g_{a,i,k} \in F_1,
\]

such that \( \phi_0(g_{a,i}) = \pi_0 \circ L_i(a) \) and \( \phi_1(g_{a,i}) = \pi_1 \circ L_i(a) \), \( i = 1, 2, \ldots, k \), where \( g_{a,i,j} \in M_{n_j}, j = 1, 2, \ldots, k \), and \( i = 1, 2 \). Note that such \( g_{a,i} \) is unique since \( \ker \phi_0 \cap \ker \phi_1 = \{0\} \). Since \( w_0 \in \pi_0(C) = \phi_0(F_1) \), there is a unitary

\[
u_0 = \nu_{0,1} \oplus \nu_{0,2} \oplus \cdots \oplus \nu_{0,k(0)} \oplus \cdots \oplus \nu_{0,k}
\]

such that \( \phi_0(\nu_0) = w_0 \).
Note that the first \( k(0) \) components of \( u_0 \) is uniquely determined by \( w_0 \) (since \( \phi_0 \) is injective on this part) and the components after \( k(0) \)'s components can be chosen arbitrarily (since \( \phi_0 = 0 \) on this part). Similarly there exist

\[
u_1 = u_{1,1} \oplus u_{1,2} \oplus \cdots \oplus u_{1,k(1)} \oplus \cdots \oplus u_{1,k}\]

such that \( \phi_1(u_1) = w_1 \)

Now by e 0.47 and e 0.48, we have

\[
\| \phi_0(u_0)^* \phi_0(g_{a,1}) \phi_0(u_0) - \phi_0(g_{a,2}) \| < \epsilon \quad \text{and} \quad \quad (e \ 0.51)
\]

\[
\| \phi_1(u_1)^* \phi_1(g_{a,1}) \phi_1(u_1) - \phi_1(g_{a,2}) \| < \epsilon \quad \text{for all} \quad a \in \mathcal{F}. \quad (e \ 0.52)
\]

Since \( \phi_0 \) is injective on \( F_1^i \) for \( i \leq k(0) \) and \( \phi_1 \) is injective on \( F_1^i \) for \( i > k(0) \) (note that we use \( k(1) \leq k(0) + 1 \)), we have

\[
\| (u_{0,i})^* (g_{a,1,i}) u_{0,i} - (g_{a,2,i}) \| < \epsilon \quad \forall \quad i \leq k(0) \quad \text{and} \quad \quad (e \ 0.53)
\]

\[
\| (u_{1,i})^* (g_{a,1,i}) u_{1,i} - (g_{a,2,i}) \| < \epsilon \quad \forall \quad i > k(0) \quad \quad (e \ 0.54)
\]

for all \( a \in \mathcal{F} \).
Let \( u = u_{0,1} \oplus \cdots \oplus u_{0,k(0)} \oplus u_{1,k(0)+1} \oplus \cdots \oplus u_{1,k} \in F_1 \)—that is for the first \( k(0) \)'s components of \( u \), we use \( u_0 \)'s corresponding components, and for the last \( k - k(0) \) components of \( u \), we use \( u_1 \)'s. From e0.53 and e0.53. we have

\[
\|u^*g_{a,1}u - g_{a,2}\| < \epsilon \quad \text{for all } a \in \mathcal{F}.
\]

Apply \( \phi_0 \) and \( \phi_1 \) to the above inequality, we get e0.49 and e0.50 as desired.
Proof of Theorem 4.5. There is $n_0$ such that $n_0x = 0$ for all $x \in K_i(A \otimes C(T))$, $i = 0, 1$. Set $N = n_0!$. Put $\Delta_1$ be defined above for the given $\Delta$.

Let $\mathcal{H}_1' \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) for $\epsilon/32$ (in place of $\epsilon$) and $\mathcal{F}$ required by 3.5.

Let $\delta_1 > 0$ (in place of $\delta$), $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset and let $\mathcal{P}_0 \subset K(A)$ (in place of $\mathcal{P}$) be a finite subset required by 3.5 for $\epsilon/32$ (in place of $\epsilon$), $\mathcal{F}$ and $\Delta_1$. We may assume that $\delta_1 < \epsilon/32$ and $(2\delta_1, \mathcal{G}_1)$ is a KK-pair.

Moreover, we may assume that $\delta_1$ is so small that if $\|uv - vu\| < 3\delta_1$, then the Exel formula

$$\tau(bott_1(u, v)) = \frac{1}{2\pi \sqrt{-1}}(\tau(\log(u^*vuv^*)))$$

holds for any pair of unitaries $u$ and $v$ in any unital $C^*$-algebra $C$ with tracial rank zero and any $\tau \in T(C)$ (see Theorem 3.6 of [?]). Moreover if $\|v_1 - v_2\| < 3\delta_1$, then

$$bott_1(u, v_1) = bott_1(u, v_2).$$
Let \( g_1, g_2, \ldots, g_{k(A)} \in U(M_m(A)(A)) \) \((m(A) \geq 1\) is an integer\) be a finite subset such that \( \{\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{k(A)}\} \subset J_c(K_1(A)) \) and such that \( \{[g_1], [g_2], \ldots, [g_{k(A)}]\} \) forms a set of generators for \( K_1(A) \). Let \( U = \{\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{k(A)}\} \subset J_c(K_1(A)) \) be a finite subset. Let \( U_0 \subset A \) be a finite subset such that

\[
\{g_1, g_2, \ldots, g_{k(A)}\} = \{(a_{i,j}) : a_{i,j} \in U_0\}.
\]

Let \( \delta_u = \min\{1/256m(A)^2, \delta_1/16m(A)^2\} \), \( G_u = F \cup G_1 \cup U_0 \) and let \( P_u = P_0 \).

Let \( \delta_2 > 0 \) (in place of \( \delta \)), let \( G_2 \subset A \) (in place of \( G \)) and let \( H_2' \subset A_+ \setminus \{0\} \) (in place of \( H \)) and let \( N_1 \geq 1 \) (in place of \( N \)) be an integer required by 4.4 for \( \delta_u \) (in place of \( \epsilon \)), \( G_u \) (in place of \( F \)), \( P_u \) (in place of \( P \)) and \( \Delta \) and with \( \bar{g}_j \) (in place of \( g_j \)), \( j = 1, 2, \ldots, k(A) \) (with \( k(A) = r \)).

Let \( d = \min\{\Delta(\hat{h}) : h \in H_2'\} \). Let \( \delta_3 > 0 \) and let \( G_3 \subset A \otimes C(\mathbb{T}) \) be finite subset satisfying the following: For any \( \delta_3\)-\( G_3 \)-multiplicative contractive completely positive linear map \( L' : A \otimes C(\mathbb{T}) \rightarrow C' \) (for any unital \( C^* \)-algebra \( C' \) with \( T(C') \neq \emptyset \)),

\[
|\tau([L](\beta(\bar{g}_j)))| < d/8, \quad j = 1, 2, \ldots, k(A).
\]

(e 0.55)
Without loss of generality, we may assume that

\[ G_3 = \{ g \otimes z : g \in G'_3 \text{ and } z \in \{1, z, z^*\} \}, \]

where \( G'_3 \subset A \) is a finite subset (by choosing a smaller \( \delta_3 \) and large \( G'_3 \)). Let \( \epsilon''_1 = \min\{d/27m(A)^2, \delta_u/2, \delta_2/2m(A)^2, \delta_3/2m(A)^2\} \) and let \( \bar{\epsilon}_1 > 0 \) (in place of \( \delta \)) and \( G_4 \subset A \) (in place of \( G \)) be a finite subset required by ?? for \( \epsilon'' \) (in place of \( \epsilon \)) and \( G_u \cup G'_3 \). Put

\[ \epsilon_1 = \min\{\epsilon'_1, \epsilon''_1, \bar{\epsilon}_1\}. \]

Let \( G_5 = G_u \cup G'_3 \cup G_4 \).

Let \( \mathcal{H}'_3 \subset A^+ \) (in place of \( \mathcal{H}_1 \)), \( \delta_4 > 0 \) (in place of \( \delta \)), \( G_6 \subset A \) (in place of \( G \)), \( \mathcal{H}'_4 \subset A_{s.a.} \) (in place of \( \mathcal{H}_2 \)), \( \mathcal{P}_1 \subset K(A) \) (in place of \( \mathcal{P} \)) and \( \sigma_4 > 0 \) (in place of \( \sigma_2 \)) be the finite subset and constants required by Theorem 2.1 \( \epsilon_1/4 \) (in place of \( \epsilon \)) and \( G_5 \) (in place of \( \mathcal{F} \)) and \( \Delta \).

Let \( N_2 \geq N_1 \) such that \( (k(A) + 1)/N_2 < d/8 \). Choose \( \mathcal{H}'_5 \subset A_+ \setminus \{0\} \) and \( \delta_5 > 0 \) and a finite subset \( G_7 \subset A \) such that, for any \( M_m \) and unital \( \delta_5\)-\(G_7\)-multiplicative contractive completely positive linear map \( L' : A \to M_m \), if \( \text{tr} \circ L'(h) > 0 \) for all \( h \in \mathcal{H}'_5 \), then \( m \geq N_2((8/d) + 1) \).
Let $\delta = \min\{\epsilon_1/16, \delta_4/4m(A)^2, \delta_5/4m(A)^2\}$, let $G = G_5 \cup G_6 \cup G_7$ and let $P = P_u \cup P_1$. Let

$$\mathcal{H}_1 = \mathcal{H}'_1 \cup \mathcal{H}'_2 \cup \mathcal{H}'_3 \cup \mathcal{H}'_4 \cup \mathcal{H}'_6$$

and let $\mathcal{H}_2 = \mathcal{H}'_4$. Let $\gamma_1 = \sigma_4$ and let

$$0 < \gamma_2 < \min\{d/16m(A)^2, \delta_u/9m(A)^2, 1/256m(A)^2\}.$$  

Now suppose that $C \in \mathcal{C}$ and $\phi, \psi : A \to C$ be two unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear maps satisfying the assumption for the above given $\Delta$, $\mathcal{H}_1$, $\delta$, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}_2$, $\gamma_1$, $\gamma_2$ and $\mathcal{U}$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\|\pi_t \circ \phi(g) - \pi_{t'} \circ \phi(g)\| < \epsilon_1/16$$  and  

$$\|\pi_t \circ \psi(g) - \pi_{t'} \circ \psi(g)\| < \epsilon_1/16$$

for all $g \in \mathcal{G}$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

We write $C = A(F_1, F_2, h_0, h_1)$, $F_1 = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_{F(1)}}$ and $F_2 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_{F(2)}}$. By the choice of $\mathcal{H}'_5$,

$$n_j \geq N_2(8/d + 1) \text{ and } m_s \geq N_2(8/d + 1),$$  

(e 0.58)
\[ 1 \leq j \leq F(2), \ 1 \leq s \leq F(1). \] By applying Theorem 2.1, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \), \( w_0 \in h_0(F_1) \), if \( i = 0 \), and \( w_1 \in h_1(F_1) \), if \( i = 1 \), such that

\[ \|w_i \pi_t \circ \phi(g)w_i^* - \pi_t \circ \psi(g)\| < \epsilon_1/16 \text{ for all } g \in G_5. \] (e 0.59)

It follows from 0.8 that we may assume that there is a unitary \( w_e \in F_1 \) such that \( h_0(w_e) = w_0 \) and \( h_1(w_e) = w_n \).

By (e 0.41), let \( \omega_j \in M_{m(A)}(C) \) be a unitary such that \( \omega_j \in CU(M_{m(A)}(C)) \) and

\[ \|\langle (\phi \otimes \text{id}_{M_{m(A)}}(g_j^*)\rangle \langle (\psi \otimes \text{id}_{M_{m(A)}})(g_j)\rangle - \omega_j\| < \gamma_2, \ j = 1, 2, ..., k(A). \]

Write

\[ \omega_j = \prod_{l=1}^{e(j)} \exp(\sqrt{-1}a_j^{(l)}) \]

for some selfadjoint element \( a_j^{(l)} \in M_{m(A)}(C), \ l = 1, 2, ..., e(j), \ j = 1, 2, ..., k(A). \) Write

\[ a_j^{(l)} = (a_j^{(l,1)}, a_j^{(l,2)}, ..., a_j^{(l,n_{F(2)})}) \text{ and } \omega_j = (\omega_j,1, \omega_j,2, ..., \omega_j,F(2)) \]
\[ C([0, 1], F_2) = C([0, 1], M_{n_1}) \oplus \cdots \oplus C([0, 1], M_{n_{F(2)}}), \]
where \( \omega_{j,s} = \exp(\sqrt{-1}a_j^{(l,s)}) \), \( s = 1, 2, \ldots, F(2) \). Then
\[ e(j) \sum_{l=1}^{n_s} n_s(t \otimes \text{Tr}_{m(A)})(a_j^{(l,s)}(t)) \frac{e(j)}{2\pi} \in \mathbb{Z}, \quad t \in [0, 1], \]
where \( t_s \) is the normalized trace on \( M_{n_s}, s = 1, 2, \ldots, F(2) \). In particular,
\[ e(j) \sum_{l=1}^{n_s} n_s(t \otimes \text{Tr}_{m(A)})(a_j^{(l,s)}(t)) = e(j) \sum_{l=1}^{n_s} n_s(t \otimes \text{Tr}_{m(A)})(a_j^{(l,s)}(t')) \quad (e.0.60) \]
for all \( t, t'' \in [0, 1] \).
Let \( W_i = w_i \otimes \text{id}_{M_m(A)} \), \( i = 0, 1, \ldots, n \) and \( W_e = w_e \otimes \text{id}_{M_m(F_1)} \). Then
\[ \| \pi_i(\langle \phi \otimes \text{id}_{M_m(A)}(g_j^*) \rangle) W_i(\pi_i(\langle \phi \otimes \text{id}_{M_m(A)}(g_j) \rangle) W_i^* - \omega_j(t_i) \| < 3m(A)^2\varepsilon_1 + 2\gamma_2 < 1/32. \]  
\[ \text{We also have} \]
\[ \| \langle \phi_e \otimes \text{id}_{M_m(A)}(g_j^*) \rangle W_e(\langle \phi_e \otimes \text{id}_{M_m(A)}(g_j) \rangle) W_e^* \]
\[- \pi_e(\omega_j) \| < 3m(A)^2\varepsilon_1 + 2\gamma_2 < 1/32. \]
\[ (e.0.63) \]
\[ (e.0.64) \]
It follows from (e 0.61) that there exists selfadjoint elements 
\( b_{i,j} \in M_{m(A)}(F_2) \) such that

\[
\exp(\sqrt{-1} b_{i,j}) = \omega_j(t_i)^*(\pi_i(\phi \otimes \text{id}_{M_{m(A)}})(g_j^*)) W_i(\pi_i(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j) \rangle) W_i^*, \tag{e 0.65}
\]

\[
\omega_j(t_i)^*(\pi_i(\phi \otimes \text{id}_{M_{m(A)}})(g_j^*)) W_i(\pi_i(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j) \rangle) W_i^*, \tag{e 0.66}
\]

and \( b_{e,j} \in M_{m(A)}(F_1) \) such that

\[
\exp(\sqrt{-1} b_{e,j}) = \pi_e(\omega_j)^*(\pi_e(\phi \otimes \text{id}_{M_{m(A)}})(g_j^*)) W_e(\pi_e(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j) \rangle) W_e^*, \tag{e 0.67}
\]

\[
\pi_e(\omega_j)^*(\pi_e(\phi \otimes \text{id}_{M_{m(A)}})(g_j^*)) W_e(\pi_e(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j) \rangle) W_e^*, \tag{e 0.68}
\]

and

\[
\| b_{i,j} \| < 2 \arcsin(3m(A)^2/4 + 2\gamma_2), \ j = 1, 2, \ldots, k(A), \tag{e 0.69}
\]

\( i = 0, 1, \ldots, n, e. \)

We write

\[
b_{i,j} = (b_{i,j}^{(1)}, b_{i,j}^{(2)}, \ldots, b_{i,j}^{F(2)}) \in F_2 \quad \text{and}
\]

\[
b_{e,j} = (b_{e,j}^{(1)}, b_{e,j}^{(2)}, \ldots, b_{e,j}^{F(1)}) \in F_1. \tag{e 0.70}
\]
We also have that
\[ h_0(b_{e,j}) = b_{0,j} \quad \text{and} \quad h_1(b_{e,j}) = b_{n,j}. \] (e 0.71)

Note that
\[
(\pi_i(\langle \phi \otimes \text{id}_{M_{m(A)}} (g_j^*) \rangle)) W_i(\pi_i(\langle \phi \otimes \text{id}_{M_{m(A)}} (g_j) \rangle)) W_i^* = \pi_i(\omega_j) \exp(\sqrt{-1}b_{i,j}),
\] (e 0.72)
\[ j = 1, 2, \ldots, k(A) \] and \[ i = 0, 1, \ldots, n, \ e. \]

Then,
\[
\frac{n_s}{2\pi} (t_s \otimes \text{Tr}_{M_{m(A)}}) (b^{(s)}_{i,j}) \in \mathbb{Z},
\] (e 0.74)
where \( t_s \) is the normalized trace on \( M_{n_s}, s = 1, 2, \ldots, F(2), \)
\[ j = 1, 2, \ldots, k(A), \] and \[ i = 0, 1, \ldots, n. \] We also have
\[
\frac{m_s}{2\pi} (t_s \otimes \text{Tr}_{M_{m(A)}}) (b^{(s)}_{e,j}) \in \mathbb{Z}
\] (e 0.75)
where \( t_s \) is the normalized trace on \( M_{m_s}, s = 1, 2, \ldots, F(1), \)
\[ j = 1, 2, \ldots, k(A). \] Let
\[
\lambda^{(s)}_{i,j} = \frac{n_s}{2\pi} (t_s \otimes \text{Tr}_{M_{m(A)}}) (b^{(s)}_{i,j}) \in \mathbb{Z},
\]
where \( t_s \) is the normalized trace on \( M_{n_s}, s = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, k(A) \) and \( i = 0, 1, 2, \ldots, n \).

Let

\[
\lambda^{(s)}_{e,j} = \frac{m_s}{2\pi} (t_s \otimes \text{Tr}_{M_{m(A)}})(b^{(s)}_{e,j}) \in \mathbb{Z}
\]

where \( t_s \) is the normalized trace on \( M_{m_s}, s = 1, 2, \ldots, F(1) \) and \( j = 1, 2, \ldots, k(A) \). Let

\[
\lambda_{i,j} = (\lambda^{(1)}_{i,j}, \lambda^{(2)}_{i,j}, \ldots, \lambda^{(F(2))}_{i,j}) \in \mathbb{Z}^{F(2)} \quad \text{and} \quad \lambda_{e,j} = (\lambda^{(1)}_{e,j}, \lambda^{(2)}_{e,j}, \ldots, \lambda^{(F(1))}_{e,j}) \in \mathbb{Z}^{F(1)}.
\] (e 0.76)

We have

\[
\left| \frac{\lambda^{(s)}_{i,j}}{n_s} \right| < \frac{d}{4}, \ s = 1, 2, \ldots, F(2), \quad \text{and} \quad (e 0.77)
\]

\[
\left| \frac{\lambda^{(s)}_{e,j}}{m_s} \right| < \frac{d}{4}, \ s = 1, 2, \ldots, F(1), \quad (e 0.78)
\]

\( j = 1, 2, \ldots, k(A), \ i = 0, 1, 2, \ldots, n. \)
Define \( \alpha_i^{(0,1)} : K_1(A) \to \mathbb{Z}^F(2) \) by mapping \([g_j]\) to \(\lambda_{i,j}, j = 1, 2, \ldots, k(A)\) and \(i = 0, 1, 2, \ldots, n\), and define \( \alpha_e^{(0,1)} : K_1(A) \to \mathbb{Z}^F(1) \) by mapping \([g_j]\) to \(\lambda_{e,j}, j = 1, 2, \ldots, k(A)\). We write \( K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \beta(K_1(A)) \) (see ?? for the definition of \( \beta \)). Define \( \alpha_i : K_*(A \otimes C(\mathbb{T})) \to K_*(F_2) \) as follows: On \( K_0(A \otimes C(\mathbb{T})) \), define

\[
\alpha_i|_{K_0(A)} = [\pi_i \circ \phi]|_{K_0(A)}, \quad \alpha_i|_{\beta(K_1(A))} = \alpha_i \circ \beta|_{K_1(A)} = \alpha_i^{(0,1)} \quad (e \text{ 0.79})
\]

and on \( K_1(A \otimes C(\mathbb{T})) \),

\[
\alpha_i|_{K_1(A \otimes C(\mathbb{T}))} = 0, \quad (e \text{ 0.80})
\]

\(i = 0, 1, 2, \ldots, n\), and define \( \alpha_e \in \text{Hom}(K_*(A \otimes C(\mathbb{T})), K_*(F_1))\), by

\[
\alpha_e|_{K_0(A)} = [\pi_e \circ \phi]|_{K_0(A)}, \quad \alpha_e|_{\beta(K_1(A))} = \alpha_i \circ \beta|_{K_1(A)} = \alpha_e^{(0,1)} \quad (e \text{ 0.81})
\]

on \( K_0(A \otimes C(\mathbb{T})) \) and \((\alpha_e)|_{K_1(A \otimes C(\mathbb{T}))} = 0\). Note that

\[
(h_0)_* \circ \alpha_e = \alpha_0 \quad \text{and} \quad (h_1)_* \circ \alpha_e = \alpha_n. \quad (e \text{ 0.82})
\]
Since \( A \otimes C(\mathbb{T}) \) satisfies the UCT, the map \( \alpha_e \) can be lifted to an element of \( KK(A \otimes C(\mathbb{T}), F_1) \) which is still denoted by \( \alpha_e \). Then define

\[
\alpha_0 = \alpha_e \times [h_0] \quad \text{and} \quad \alpha_n = \alpha_e \times [h_1]
\]

in \( KK(A \otimes C(\mathbb{T}), F_2) \). For \( i = 1, \ldots, n - 1 \), also pick a lifting of \( \alpha_i \) in \( KK(A \otimes C(\mathbb{T}), F_2) \), and still denote it by \( \alpha_i \). We estimate that

\[
\| (w_i^* w_{i+1}) \pi_t \circ \phi(g) - \pi_t \circ \phi(g)(w_i^* w_{i+1}) \| < \epsilon_1/4 \quad \text{for all} \quad g \in G_5,
\]

\( i = 0, 1, \ldots, n - 1 \). Let \( \Lambda_{i,i+1} : C(\mathbb{T}) \otimes A \to F_2 \) be a unital contractive completely positive linear map given by the pair \( w_i^* w_{i+1} \) and \( \pi_t \circ \phi \) (by ??, see 2.8 of [?]). Denote \( V_{i,j} = \langle \pi_t \circ \phi \otimes \text{id}_{M_m(A)}(g_j) \rangle, \quad j = 1, 2, \ldots, k(A) \) and \( i = 0, 1, 2, \ldots, n - 1 \).

Write

\[
V_{i,j} = (V_{i,j,1}, V_{i,j,2}, \ldots, V_{i,j,F(2)}) \in F_2, \quad j = 1, 2, \ldots, k(A), \quad i = 0, 1, 2, \ldots, n.
\]

Similarly, write

\[
W_i = (W_{i,1}, W_{i,2}, \ldots, W_{i,F(2)}) \in F_2, \quad i = 0, 1, 2, \ldots, n.
\]
We have

\[ \| W_i V_{i,j} W_i^* V_{i,j} W_{i+1} V_{i,j} W_{i+1}^* - 1 \| < \frac{1}{16} \]  
\[ \| W_i V_{i,j} W_i^* V_{i,j} V_{i+1,j} W_{i+1} V_{i+1,j} W_{i+1}^* - 1 \| < \frac{1}{16} \]

and there is a continuous path \( Z(t) \) of unitaries such that \( Z(0) = V_{i,j} \) and \( Z(1) = V_{i+1,j} \). Since

\[ \| V_{i,j} - V_{i+1,j} \| < \frac{\delta_1}{12}, \quad j = 1, 2, \ldots, k(A), \]

we may assume that \( \| Z(t) - Z(1) \| < \frac{\delta_1}{6} \) for all \( t \in [0, 1] \). We also write

\[ Z(t) = (Z_1(t), Z_2(t), \ldots, Z_{F(2)}(t)) \in F_2 \quad \text{and} \quad t \in [0, 1]. \]

We obtain a continuous path

\[ W_i V_{i,j} W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^* \]

which is in \( CU(M_{nm(A)}) \) for all \( t \in [0, 1] \) and

\[ \| W_i V_{i,j} W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^* - 1 \| < \frac{1}{8} \quad \text{for all} \quad t \in [0, 1]. \]
It follows that

$$(1/2\pi \sqrt{-1})(t_s \otimes \text{Tr}_{M_m(A)})[\log(W_{i,s} V_{i,j,s} W_{i,s}^* V_{i,j,s}^* Z_s(t)^* W_{i+1,s} Z_s(t) W_{i+1,s}^*)]$$

is a constant, where $t_s$ is the normalized trace on $M_{n_s}$. In particular,

$$(1/2\pi \sqrt{-1})(t_s \otimes \text{Tr}_{M_m(A)})(\log(W_{i,s} V_{i,j,s} W_{i,s}^* W_{i+1,s} V_{i,j,s} W_{i+1}^*)) = (1/2\pi \sqrt{-1})(t_s \otimes \text{Tr}_{M_m(A)})(\log(W_{i,s} V_{i,j,s} W_{i,s}^* V_{i,j} V_{i+1,j,s} W_{i+1} V_{i,j,s} W_{i+1}^*)).$$

Also

$$W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* = (\omega_j(t_i) \exp(\sqrt{-1}b_{i,j}))^* \omega_j(t_i) \exp(\sqrt{-1}b_{i+1,j})$$

$$= \exp(-\sqrt{-1}b_{i,j}) \omega_j(t_i)^* \omega_j(t_{i+1}) \exp(\sqrt{-1}b_{i+1,j}).$$

Note that, by (???) and (e 0.56), for $t \in [t_i, t_{i+1}]$,

$$\|\omega_j(t_i)^* \omega_j(t) - 1\| < 3(3\epsilon_1 + 2\gamma_2) < 3/32,$$

$$j = 1, 2, ..., k(A), i = 0, 1, ..., n - 1.$$
By Lemma 3.5 of [?],

\[(t_s \otimes \text{Tr}_{m(A)})(\log(\omega_{j,s}(t_i)^*\omega_{j,s}(t_{i+1}))) = 0. \quad (e \ 0.90)\]

It follows that (by the Exel formula, using (??), (e \ 0.88) and (e \ 0.90))

\[t \otimes \text{Tr}_{m(A)})(\text{bott}_1(V_{i,j}, W_i^* W_{i+1})) \quad (e \ 0.91)\]

\[= \left(\frac{1}{2\pi\sqrt{-1}}\right)(t \otimes \text{Tr}_{m(A)})(\log(V_{i,j}^* W_i^* W_{i+1} V_{i,j} W_{i+1}^* W_i)) \quad (e \ 0.92)\]

\[= \left(\frac{1}{2\pi\sqrt{-1}}\right)(t \otimes \text{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* W_i)) \quad (e \ 0.92)\]

\[= \left(\frac{1}{2\pi\sqrt{-1}}\right)(t \otimes \text{Tr}_{m(A)})(\log(\exp(-\sqrt{-1}b_{i,j})\omega_j(t_i)^*\omega_j(t_{i+1}) \exp(\sqrt{-1}b_{i+1,j})) \quad (e \ 0.93)\]

\[= \left(\frac{1}{2\pi\sqrt{-1}}\right)[(t \otimes \text{Tr}_{k(n)})(-\sqrt{-1}b_{i,j}) + (t \otimes \text{Tr}_{k(n)})(\log(\omega_j(t_i)^*\omega_j(t_{i+1}))) \quad (e \ 0.93)\]

\[+ (t \otimes \text{Tr}_{k(n)})(\sqrt{-1}b_{i,j})] \quad (e \ 0.93)\]

\[= \frac{1}{2\pi}(t \otimes \text{Tr}_{k(n)})(-b_{i,j} + b_{i+1,j}) \quad (e \ 0.94)\]
for all \( t \in T(F_2) \). In other words,

\[
bott_1(V_{i,j}, W_i^* W_{i+1}) = -\lambda_{i,j} + \lambda_{i+1,j}
\]  
\text{(e 0.95)}

\( j = 1, 2, ..., m(A), \ i = 0, 1, ..., n - 1 \).

Consider \( \alpha_0, ..., \alpha_n \in KK(A \otimes C(\mathbb{T}), F_2) \) and \( \alpha_e \in KK(A \otimes C(\mathbb{T}), F_1) \).

Note that

\[
|\alpha_i(g_j)| = |\lambda_{i,j}|
\]

and by (e 0.77), one has

\[
m_s, n_j \geq N_2(8/d + 1).
\]

By applying 4.4 (using (e 0.78), among other items), there are unitaries \( z_i \in F_2, i = 1, 2, ..., n - 1 \), and \( z_e \in F_1 \) such that

\[
\| [z_i, \pi_t \circ \phi(g)] \| < \delta_u \text{ for all } g \in G_u
\]

\[
\text{Bott}(z_i, \pi_t \circ \phi) = \alpha_i \text{ and Bott}(z_e, \pi_e \circ \phi) = \alpha_e.
\]  
\text{(e 0.96)}

Put

\[
z_0 = h_0(z_e) \text{ and } z_n = h_1(z_e).
\]
One verifies (by (e 0.83)) that

$$Bott(z_0, \pi_{t_0} \circ \phi) = \alpha_0 \text{ and } Bott(z_n, \pi_{t_n} \circ \phi) = \alpha_n.$$  \hfill (e 0.98)

Let \( U_{i,i+1} = z_i(w_i)^*w_{i+1}(z_{i+1})^*, \ i = 0, 1, 2, \ldots, n - 1. \) Then

$$\| [U_{i,i+1}, \pi_{t_i} \circ \phi(g)]\| < \min\{\delta_1, \delta_2\}, \quad g \in G_u, \ i = 0, 1, 2, \ldots, n - 1.$$  \hfill (e 0.99)

Moreover, for \( i = 0, 1, 2, \ldots, n - 1, \)

$$\text{bott}_1(U_{i,i+1}, \pi_{t_i} \circ \phi) = \text{bott}_1(z_i, \pi_{t_i} \circ \phi) + \text{bott}_1((w_i^*w_{i+1}, \pi_{t_i} \circ \phi))$$

$$+ \text{bott}_1((z_{i+1}^*, \pi_{t_i} \circ \phi)$$

$$= (\lambda_{i,j}) + (-\lambda_{i,j} + \lambda_{i+1,j}) + (-\lambda_{i+1,j})$$

$$= 0.$$

Note that for any \( x \in \bigoplus_{*=0,1} \bigoplus_{k=1}^{\infty} K_*(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}), \) one has \( N_x = 0. \) Therefore

$$\text{Bott}(\left\{ U_{i,i+1}, \ldots, U_{i,i+1} \right\}) \left( \left\{ \pi_{t_i} \circ \phi, \ldots, \pi_{t_i} \circ \phi \right\}\right) |_P = N\text{Bott}(U_{i,i+1}, \pi_{t_i} \circ \phi)|_P = 0.$$  \hfill (e 0.100)
\( i = 0, 1, 2, \ldots, n - 1. \)  

Note that, by the assumption (e 0.39),

\[
    t_s \circ \pi_t \circ \phi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1',
\]

where \( t_s \) is the normalized trace on \( M_{ns}, 1 \leq s \leq F(2) \).

By applying \( ?? \), using (e 0.101), (e 0.99) and (e 0.100), there exists a continuous path of unitaries, \( \{ \tilde{U}_{i,i+1}(t) : t \in \left[ t_i, t_{i+1} \right] \} \subset F_2 \otimes M_N(\mathbb{C}) \) such that

\[
    \tilde{U}_{i,i+1}(t_i) = \text{id}_{F_2 \otimes M_N(\mathbb{C})}, \quad \tilde{U}_{i,i+1}(t_{i+1}) = (z_i w_i^* w_{i+1} z_{i+1}^*) \otimes 1_{M_N(\mathbb{C})},
\]

and

\[
    \| \tilde{U}_{i,i+1}(t) (\pi_{t_i} \circ \phi(f), \ldots, \pi_{t_i} \circ \phi(f)) \tilde{U}_{i,i+1}(t)^* - (\pi_{t_i} \circ \phi(f), \ldots, \pi_{t_i} \circ \phi(f)) \| < \epsilon
\]

for all \( f \in \mathcal{F} \) and for all \( t \in \left[ t_i, t_{i+1} \right] \). Define \( W \in C \otimes M_N \) by

\[
    W(t) = (w_i z_i^* \otimes 1_{M_N}) \tilde{U}_{i,i+1}(t) \text{ for all } t \in \left[ t_i, t_{i+1} \right],
\]
$i = 0, 1, \ldots, n - 1$. Note that $W(t_i) = w_i z_i^* \otimes 1_{M_N}$, $i = 0, 1, \ldots, n$. Note also that

$$W(0) = w_0 z_0^* \otimes 1_{M_N} = h_0(w_e z_e^*) \otimes 1_{M_N}$$

and

$$W(1) = w_n z_n^* \otimes 1_{M_N} = h_1(w_e z_e^*) \otimes 1_{M_N}.$$ 

So $W \in C \otimes M_N$. One then checks that, by (e 0.56), (e 0.103), (e 0.96) and (e 0.59), for $t \in [t_i, t_{i+1}]$,

$$\| W(t)((\pi_t \circ \phi)(f) \otimes 1_{M_N}) W(t)^* - (\pi_t \circ \psi)(f) \otimes 1_{M_N} \| \leq \epsilon$$

for all $f \in \mathcal{F}$. 

$$\| W(t)((\pi_t \circ \phi)(f) \otimes 1_{M_N}) W(t)^* - W(t)((\pi_{t_i} \circ \phi)(f) \otimes 1_{M_N}) W^*(t) \|$$

$$+ \| W(t)(\pi_{t_i} \circ \phi)(f) W(t)^* - W(t_i)(\pi_{t_i} \circ \phi)(f) W(t_i)^* \|$$

$$+ \| W(t_i)((\pi_{t_i} \circ \phi)(f) \otimes 1_{M_N}) W(t_i)^* - (w_i(\pi_{t_i} \circ \phi)(f) w_i^*) \otimes 1_{M_N} \|$$

$$+ \| w_i(\pi_{t_i} \circ \phi)(f) w_i^* - \pi_{t_i} \circ \psi(f) \|$$

$$+ \| \pi_{t_i} \circ \psi(f) - \pi_t \circ \phi(f) \|$$

$$< \epsilon_1/16 + \epsilon/32 + \delta_u + \epsilon_1/16 + \epsilon_1/16 < \epsilon$$