1. Let $\varphi : R \to S$ be a ring homomorphism. Show that the image of $\varphi$, denoted by $\varphi(R)$, is a subring of $S$.

(See also section A.3 of the textbook for a discussion of Zorn’s lemma)

A partial ordering of a set $S$ is a relation $s \leq s'$ which may hold between some elements of $S$ and satisfies the following axioms for all $s, s', s'' \in S$:

(i) $s \leq s$
(ii) if $s \leq s'$ and $s' \leq s''$, then $s \leq s''$
(iii) if $s \leq s'$ and $s' \leq s$, then $s = s'$.

A partial ordering is called a total ordering if, in addition

(iv) for all $s, s' \in S$, either $s \leq s'$ or $s' \leq s$.

An element $m \in S$ is maximal if there is no $s$ such that $m \leq s$ except for $m$ itself.

If $A$ is a subset of a partially ordered set $S$, then an upper bound for $A$ is an element $b \in S$ such that for all $a \in A$, $a \leq b$.

A partially ordered set $S$ is inductive if every totally ordered subset $T$ of $S$ has an upper bound.

**Lemma** (Zorn’s Lemma). An inductive partially ordered set $S$ has at least one maximal element.

2. Let $R$ be a ring and $I$ an ideal in $R$. Use Zorn’s Lemma to prove that $I$ is contained in a maximal ideal.