Math 5320, 3/28/18 Name:

Worksheet 26: Ruler and compass constructions

1. Use your ruler and compass to construct a line perpendicular to the line below:

2. Suppose the following two points are spaced one unit apart in our coordinate system. Construct a line with length $\sqrt{2}$. Construct a line with length $\sqrt{3}$.

The rules of ruler and compass construction in general:

We always start with two given points (and we take the distance between them to be "1"). These points are *constructed*. Any other constructed points must be obtained as intersection points of constructed lines and/or constructed circles that we can draw as follows:

The ruler lets us draw a constructed line passing through any two constructed points. Note that we're not allowed to use the hash marks on the side of the ruler. For ruler and compass constructions, we're really only using the ruler as a straight edge, not as a measurement tool. The compass lets us draw a constructed circle centered on any constructed point that passes through any other constructed point.

A point, or a line, or a circle is called *constructible* if it can be constructed in finitely many steps using these rules.

However, we'll be interested in whether a lot of other things are constructible, that is, whether they can be obtained via a ruler and compass construction: numbers, angles, and regular polygons. So that we can talk about these things, we introduce coordinates to our constructions so that the first two points we're given are always at $(0, 0)$ and $(1, 0)$.

- A number r is said to be constructible if the point at $(r, 0)$ is constructible.
- An angle is constructible if it is possible to construct two lines meeting at that angle.
- A regular polygon is constructible if it's possible to make one out of constructed lines (any size).

3. Can you adapt your work from problem 2 show that $\sqrt{2}$ and $\sqrt{3}$ are constructible?

OK, now it's time to bring some field theory into this. Suppose we've started a ruler and compass construction and all the coordinates of all the constructed points are in some field $F \subset \mathbb{R}$. Any new constructed points will come from the intersection of two lines in our construction, the intersection of a line and a circle, or the intersection of two circles. The purpose of the next few questions is to help us get a feel for when these new constructed points have coordinates in F and, if they don't have coordinates in F, what sort of field extension of F they'll have coordinates in.

4. Suppose we have two points (a_0, b_0) and (a_1, b_1) that are already constructed. What is the equation for a line that passes through them both? What is the equation for a circle centered at (a_0, b_0) and passing through (a_1, b_1)?

5. What is the point of intersection of $y = 2x$ and $y = 5x - 1$?

6. In a ruler and compass construction whose constructed points all have coordinates in F, the intersection of any two lines will have have coordinates in F. Does this seem plausible from the problems you've solved?

7. Where do the circle and line $y^2 + x^2 = 1$ and $y = 2x - 2$ intersect?

Finding the intersection of a circle and a line amounts to solving a quadratic equation. In a ruler and compass construction whose constructed points all have coordinates in F, a circle and line will intersect if the discriminant D of this quadratic equation is non-negative and the coordinates of the intersection will be in $F(\sqrt{D})$.

8. Where do the circles $y^2 + x^2 = 1$ and $(x - 2)^2 + y^2 = 4$ intersect?

As you solved the previous problem, you probably noticed that if you take the difference between two equations for circles, you get a linear equation. The points where the circles intersect are exactly the same as the points where this line and either of the circles intersect. The line will have coordinates in F (how do we know that?), so from our previous discussion, we know that the intersection points of two circles in our ruler and compass construction will have coordinates in F or a degree 2 field extension of F.

(See Theorem 15.5.6) In summary, we've shown that any constructible point will have coordinates in a field $K \subseteq \mathbb{R}$ where $[K : \mathbb{Q}]$ is a power of 2. That is, for some integer n, there is a chain of field extensions

$$\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n = K$$

where $[F_{i+1} : F_i] = 2$ for all i.

Now we're ready to show that it's impossible in general to trisect an angle with ruler and compass.

9. Why is an angle θ constructible if and only if $\cos(\theta)$ is?

10. If it were possible to trisect an angle with a ruler and compass, then since $60°$ is constructible (why?), then $\frac{1}{3} \times 60° = 20° = \frac{\pi}{9}$ would be constructible. It can be verified, if you wish, that $\alpha = 2\cos(\frac{\pi}{9}) = e^{i\theta} + e^{-i\theta}$ satisfies the equation $x^3 - 3x - 1$, which has no rational roots. Why does this information allow us to conclude that $\pi/9$ can't be constructed, and therefore that there is no universal method for trisecting angles with a ruler and compass?

It's also impossible to construct certain regular polygons.

11. Why is a regular n-gon constructible if and only if the angle $\frac{2\pi}{n}$ is?

12. Suppose we instead think of ruler and compass constructions as sitting on the complex plane. How should we adapt our understanding of the coordinates of the constructed points?

13. In light of the last two questions, what is the degree of ζ_7 and how can it be used to prove that a regular 7-gon cannot be constructed using ruler and compass? What further conclusions can you draw about which regular n-gons aren't constructible or could be constructible?

14. If you'd like to work ahead, a couple of problems in the problem set due April 4 are related to what you've done with ruler and compass: Ch. 5.2(b) Construct a pentagon with ruler and compass (Hint: $\cos(\frac{2\pi}{5}) = \frac{\sqrt{5}-1}{4}$) and Ch. 5.3: Is a regular 9-gon constructible?