

# Defining $\mathbb{G}_m$ and Yoneda and group objects

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When I started learning algebraic geometry, I often saw the notation  $\mathbb{G}_m$ , which refers to an algebraic group called the multiplicative group. But when I looked up definitions, they seemed to disagree.

This write-up has two portions. The first is a discussion of the definition of  $\mathbb{G}_m$ , and the second is a discussion of group objects.

## 1 $\mathbb{G}_m$ : A definition

**Definition 1.** Sometimes  $\mathbb{G}_m$  is defined as  $\text{Spec}(\mathbb{Z}[x^\pm] := [x, x^{-1}]$ , or  $\text{Spec}(k[x^\pm])$  for a field  $k$ , depending on what base we are working over. These schemes can also be denoted by  $\mathbb{A}^1 \setminus \{0\}$  they consist of the affine line without the point at 0 corresponding to the ideal  $(x)$ .

**Definition 2.** However, in some places, given a scheme  $X$ ,  $\mathbb{G}_{mX}$  is defined as

- (a) the group of units of the ring of global sections of  $X$ :  $\mathcal{O}_X(X)^*$  (or, depending on the notation you prefer,  $\Gamma(X, \mathcal{O}_X)^*$ )
- (b) or it is sometimes defined as the sheaf that assigns to an open set  $U$  in  $X$ , that is, the group of units of the ring of sections of the structure sheaf  $\mathcal{O}_X$  over  $U$ .

These definitions are clearly quite different: they are a scheme, a group, and a sheaf, respectively.

I was also confused by Definition 2 given that  $\mathbb{G}_m$  is an algebraic group, which is a scheme.

For a moment I thought perhaps the sheaf given in Definition 2(b) would be the structure sheaf of the scheme from Definition 1, if I could just pick the base field properly, but that's of course not right for a lot of reasons, including, for instance that a structure sheaf consists of rings, not groups, or the fact that  $k[x^\pm]$  contains elements that aren't invertible.

But the ideas really are closely related if we find the right way of looking at them. Algebraic geometers often view the same structures in different ways. A big part of studying for my qualifying exam was learning the “dictionary” for how to go back and forth between these points of view. A number of these correspondences are so ingrained, or considered so standard in the field, that authors don’t make them explicit, even when using both methods at the same time, which can cause confusion for novices.

The correspondence at work in this case is that between a representable functor that maps from schemes to abelian groups and its representing object: Definition 1 gives the representing object, Definition 2(a) tells us where the functor maps an arbitrary scheme  $X$ , and Definition 2(b) is the sheaf we get from restricting the domain of the functor to the open sets contained in  $X$ . The second definition is the functor’s representing object and the first definition is the functor, after restricting the domain to the open subschemes of  $X$ . There’s actually quite a bit going on here.

Consider the representable functor

$$F := \text{Hom}_{\text{Scheme}}(-, \text{Spec}(\mathbb{Z}[x^{\pm}])) : \text{Scheme} \rightarrow \text{Ab}$$

$$X \mapsto \text{Hom}_{\text{Scheme}}(X, \text{Spec}(\mathbb{Z}[x^{\pm}]))$$

Note that

$$\text{Hom}_{\text{Scheme}}(X, \text{Spec}(\mathbb{Z}[x^{\pm}])) = \text{Hom}_{\text{Ring}}(\mathbb{Z}[x^{\pm}], \mathcal{O}_X(X)).$$

Any ring map (keep in mind these are maps between commutative, unital rings so that unity maps to unity) from  $\mathbb{Z}[x, x^{-1}]$  to another ring is completely determined by where  $x$  maps to, which must be to an invertible element since  $x$  is invertible. So the maps in  $\text{Hom}_{\text{Ring}}(\mathbb{Z}[x^{\pm}], \mathcal{O}_X(X))$  correspond to elements of  $\mathcal{O}_X(X)^*$ .

Hence,  $F$  is the functor that maps a scheme  $X$  to the group of units of its global sections.

## 2 Group objects

In the previous section it was mentioned that algebraic geometers often go back and forth between a scheme (or variety if you like)  $X$  and the contravariant representable functor  $\text{Hom}_{\text{Scheme}}(-, X) : \text{Scheme}^{op} \rightarrow \text{Set}$  mapping from schemes to sets. Such a functor is sometimes called the “functor of points” of  $X$ . This is a reasonable name because if we apply it to  $\text{Spec}(k)$  for  $k$  a field, we get  $\text{Hom}(\text{Spec}(k), X) = X(k)$ , the points in  $X$  with coordinates in  $k$ , ie, the “ $k$ -points” of  $X$ .

It’s also reasonable to think of a scheme and its functor of points as in some ways interchangeable because the Yoneda Lemma tells us that the functor mapping from schemes to functors from schemes to sets sending a scheme to the functor it represents is fully faithful. More concretely, given schemes  $X$  and  $Y$ ,

$$\text{Hom}_{\text{Scheme}}(X, Y) \cong \{\text{Natural transformations } \text{Hom}(-, X) \Rightarrow \text{Hom}(-, Y)\}.$$

That is, any natural transformation between those two representable functors comes from postcomposing with a morphism from  $X$  to  $Y$ : if we have a map between the  $Z$ -points of  $X$  and  $Y$  for each scheme  $Z$  and these maps satisfy certain naturality conditions, then they arise from a map from  $X$  to  $Y$ .

This correspondence also helps us make sense of a group object in a category, which is an object  $G$  and morphisms  $m : G \times G \rightarrow G$ ,  $i : G \rightarrow G$  and  $e : \star \rightarrow G$ , which give multiplication, taking the inverse, and picking out the identity ( $\star$  is meant to represent a final object in the category). These morphisms should furthermore satisfy some identities. I won't include those diagrams here, but the idea is that if  $G$  is a set, then these data and axioms are equivalent to putting a group structure on  $G$ .

For any scheme  $Z$ , the morphisms  $m$ ,  $i$  and  $e$  make the set of  $Z$ -points of  $G$  a group. And, naturality tells us that given a map between schemes  $Z \rightarrow Z'$ , there is a map between groups  $G(Z') \rightarrow G(Z)$ .

Algebraic groups like  $\mathbb{G}_m$  are group objects in the category of algebraic varieties.