Missing edge coverings of bipartite graphs and the geometry of the Hausdorff metric

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Abstract

In this paper, we examine the problem of finding the number k of elements at a given location on the line segment between two elements in the geometry the Hausdorff metric imposes on the set $\mathcal{H}(\mathbb{R}^n)$ of all nonempty compact sets in n-dimensional real space. We demonstrate that this problem is equivalent to counting the edge coverings of simple bipartite graphs. We prove the novel results that there exist no simple bipartite graphs with exactly 19 or 37 edge coverings, and hence there do not exist any two elements of $\mathcal{H}(\mathbb{R}^n)$ with exactly 19 or 37 elements at a given location lying between them – although there exist pairs of elements in $\mathcal{H}(\mathbb{R}^n)$ that have k elements at any given location between them for infinitely many values of k, including k from 1 to 18 and 20 to 36.

This paper extends results in the geometry of the Hausdorff metric given in [1]. In addition to our results about counting edge coverings, we give a brief introduction to this geometry.

1 Introduction

This paper presents results on counting the number k of elements at a given location between two elements in the geometry the Hausdorff metric imposes on the set $\mathcal{H}(\mathbb{R}^n)$ of all nonempty compact subsets of n-dimensional real space. In particular, we demonstrate a connection between this problem and counting the numbers of edge coverings of simple bipartite graphs.

In Section 2, we will give a brief introduction to the geometry of the Hausdorff metric, working through a discussion of the invariant k described above. In Section 3 we make explicit in Theorem 3.4 the correspondence between values of k and counting edge coverings of simple bipartite graphs. Then, we use this correspondence to provide a novel proof that k may never be 19 (Theorem 3.5); this proof, which comes from our graph theoretic results Theorems 5.4 and 5.5, is briefer than the proof given in [1], which is exhaustive in nature. Finally, the connection to graph theory is used to give the entirely new result Theorem 3.6 that k may never be 37.

The remainder of the paper is concerned with proving the novel results in graph theory Theorems 5.4 and 5.5, which are interesting in their own right: they state there exist no simple bipartite graphs with exactly 19 or 37 edge coverings. In Section 4, we prove some technical lemmas giving relationships between counting edge coverings of a graph and counting edge coverings of some of its subgraphs and then use these to prove our main graph theory results in Section 5.

Research into counting edge coverings of simple bipartite graphs was motivated by investigating a result in the geometry of the Hausdorff metric presented in [1], in which the authors give results on the number of sets k which exist at each location on a segment between sets A and B. It is shown in [1] that as the sets A and B vary, the invariant k may take on infinitely many distinct values, including 1-18, and 20-36, but k may never be 19; the next case left open by this work is the value 37.

These are fascinating results, and leave open the questions of what other numbers, if any, share this property; a sieve has shown that k may take on the values 38-40 as well; the next open case is 41. The question of what property of the numbers 19 and 37 causes this anomaly also remains open.

2 The geometry of the Hausdorff metric

This section gives an introduction to the geometry the Hausdorff metric imposes on the set $\mathcal{H}(\mathbb{R}^n)$ of all nonempty compact subsets of n-dimensional real space, working up to a discussion of the motivating problem for the work in this paper: finding the number k of elements at a given location lying between two elements. In the next section, we will make explicit the connection between the geometry of the Hausdorff metric and graph theory, and apply the graph theory results to this geometry.

Felix Hausdorff developed the Hausdorff metric h early in the 20^{th} century as a means of measuring the distance between compact sets in n-dimensional real space. The connection between graph theory and this geometry is related to the notion of betweenness. Unlike in Euclidean geometry, there can be more than one set at a given location between two sets in the geometry of the Hausdorff metric.

We set the convention that the sets A and B are arbitrary elements of $\mathcal{H}(\mathbb{R}^n)$ unless otherwise restricted and we let d_E denote the Euclidean metric on $\mathbb{R}^n$.

Definition 2.1. *Let* $A, B \in \mathcal{H}(\mathbb{R}^n)$*. The* Hausdorff metric *is defined as* $h(A, B) = \max\{d(A, B), d(B, A)\}$ *where* $d(A, B) = \max_{x \in A}\{d(x, B)\}$ *and* $d(x, B) = \min_{b \in B}\{d_E(x, b)\}$.

This h is a metric on $\mathcal{H}(\mathbb{R}^n)$ and $(\mathcal{H}(\mathbb{R}^n), h)$ is a complete metric space. If $A, B \in \mathcal{H}(\mathbb{R}^n)$ are singleton sets $A = \{a\}$ and $B = \{b\}$, then $h(A, B) = d_E(a, b)$.

Example 2.2. *Let A be the boundary of a square of side length 10 and B the disk of radius 5 both centered at the origin in $\mathbb{R}^2$ as shown in Figure 1. Then*

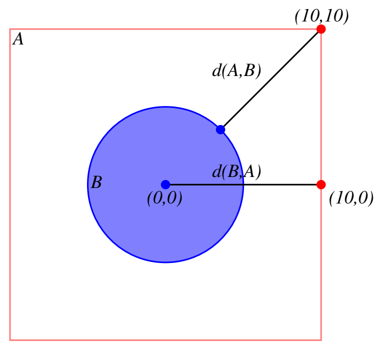

Figure 1: Distance from a compact set to a compact set.

$d(A,B) = d_E((0,10),(0,5)) = 5$ and $d(B,A) = d_E((0,0),(10,0)) = 10$. Notice that $d(A,B)$ is equal to $d_E(a,b)$ for several pairs of points (a,b) with $a \in A$ and $b \in B$.

In Euclidean geometry, a point $c \in \mathbb{R}^n$ is on the line segment between points a and b in $\mathbb{R}^n$ if and only if $d_E(a,b) = d_E(a,c) + d_E(c,b)$. We use the analog of this equality in $\mathcal{H}(\mathbb{R}^n)$ to describe the notion of betweenness that we will be using with the Hausdorff metric, which coincides with Blumenthal's definition of betweenness in metric geometries [3, Defintion 12.1].

Definition 2.3. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ be distinct. A set $C \in \mathcal{H}(\mathbb{R}^n)$ is* between *A and B if $h(A,B) = h(A,C) + h(C,B)$.*

We use the notation ACB if C is between A and B.

Definition 2.4. *In this paper we consider the* line segment *between A and B, denoted $[A,B]$, to consist of all elements in $\mathcal{H}(\mathbb{R}^n)$ between A and B.*

We say a set C satisfying ACB is *at the location* $s = h(A,C)$ away from A. Given two sets C, C' satisfying ACB and $AC'B$, we say they are at the same location between A and B if $h(A,C) = h(A,C')$ (or equivalently, $h(B,C) = h(B,C')$).

It should be noted that this definition of a line segment is distinct from other definitions. In [8, page x], for example, a closed segment in Euclidean space with endpoints a and b is taken to be $\{(1-t)a + tb \mid t \in [0,1]\}$; this is also called an affine segment. So, to generalize the notion of a line segment to $(\mathcal{H}(\mathbb{R}^n), h)$, one might think instead of the *affine segment* between two points $A, B \in \mathcal{H}(\mathbb{R}^n)$, $\{(1-t)A + tB \mid t \in [0,1]\}$.

Although in Euclidean space the affine segment between two points $a, b \in \mathbb{R}^n$ is the same as the set of all $c \in \mathbb{R}^n$ such that $d_E(a,b) = d_E(a,c) + d_E(c,b)$, an affine segment in $(\mathcal{H}(\mathbb{R}^n), h)$ is not in general the same as a line segment as defined for this paper. The affine segment with $A, B \in \mathcal{H}(\mathbb{R}^n)$ as endpoints has a unique point at each location (i.e. for each value of t) simply consisting of the weighted averages of all the choices of pairs points $a \in A$ and $b \in B$. For the example shown in Figure 2, the midpoint on the affine segment, $(A+B)/2$, consists of $C_1 \cup C_2 \cup C_3$. However for our notion of a location on a segment, more than one distinct set – in fact infinitely many, as in the example in Figure 2– can lie at the same location between A and B, as we discuss in the following example.

Example 2.5. *Consider the example in Figure 2. Here $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ are compact sets in $\mathbb{R}^2$. Let $C = C_1 \cup C_2 \cup C_3$ and $C' = C_1 \cup C_3$. Then C and C' satisfy ACB and $AC'B$, respectively, and are at the same location, i.e. $h(A, C) = h(A, C')$. Moreover, if we let C_2^* be any compact subset of C_2, then the set $C^* = C_1 \cup C_2^* \cup C_3$ satisfies AC^*B with $h(A, C^*) = h(A, C)$ as well.*

Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ and $0 < s < h(A, B)$. To help narrow down our search for sets between A and B located s away from A, we construct the minimal set C_s containing all the elements of the line segment $[A, B]$ (see Definition 2.4) that are located s away from A. We need the following definition:

Definition 2.6. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $s > 0$. The* s-neighborhood *of A is the set $(A)_s = \{x \in \mathbb{R}^n : d(x, A) \leq s\}$.*

Note that if $A \in \mathcal{H}(\mathbb{R}^n)$ and $s > 0$, then $(A)_s \in \mathcal{H}(\mathbb{R}^n)$, $h((A)_s, A) = s$, and any $C \in \mathcal{H}(\mathbb{R}^n)$ with $h(C, A) = s$ is a subset of $(A)_s$ [2, 4, 5]. So, given sets A and B, a set C satisfying ACB located s away from A, and thus $h(A, B) - s$ away from B, will be a subset of $(A)_s \cap (B)_{h(A,B)-s} =: C_s$. (In Example 2.5, $C = C_s$.)

To show that C_s is the minimal such set, it suffices to prove that $h(A, C_s) = s$. To see this, note that since $h(A, B) = r$, by definition of the Hausdorff metric there exist $a \in A$, $b \in B$ such that $h(a, b) = r$. Then, similarly, there exists c on the line segment $\overline{ab}$ such that $h(a, c) = s$, and $c \in C_s$ by definition. Then it is a simple exercise in using definition of the Hausdorff metric to show that $h(A, C_s) = s$.

The counting problem we are interested in is concerned with sets A and B with a finite number of sets at each location between them. If C_s is finite, then we are in such a situation, as in the following example:

Example 2.7. *Consider the sets $A = \{(0, 0), (1, 1)\}$ and $B = \{(0, 1), (1, 0)\}$ in $\mathbb{R}^2$ as shown in Figure 3. In this case, $h(A, B) = 1$. Fix some $0 < s <$*

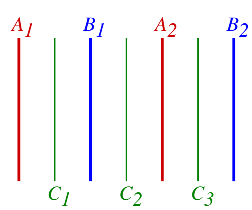

Figure 2: Sets A, B, and C in $\mathcal{H}(\mathbb{R}^2)$. The lines are parallel, and equally spaced.

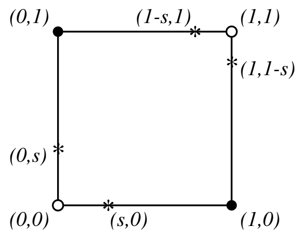

Figure 3: Seven sets at a distance s from A

1. *Then* $C_s = \{(s,0),(0,s),(1,1-s),(1-s,1)\}$. *The complete list of sets between* A *and* B *at the location* s *away from* A *is:* C_s, $\{(s,0),(0,s),(1,1-s)\}$, $\{(s,0),(0,s),(1-s,1)\}$, $\{(s,0),(1,1-s),(1-s,1)\}$, $\{(0,s),(1,1-s),(1-s,1)\}$, $\{(s,0),(1-s,1)\}$, *and* $\{(0,s),(1,1-s)\}$.

In Example 2.7, there are exactly 7 sets between A and B located s away from A for any choice of s [1]. In the article [1], the authors show that if there are finitely many sets at one location between sets A and B, then there are exactly the same finite number of elements at each location between A and B.

Definition 2.8. *Let* $A, B \in \mathcal{H}(\mathbb{R}^n)$. *If there are finitely many sets at each location between* A *and* B, *then we denote this number by* $\#([A,B])$ *(see Definition 2.4).*

This paper is concerned with the problem of characterizing possible values of $\#([A,B])$ for any $A, B \in \mathcal{H}(\mathbb{R}^n)$. In [1], the authors prove that if there exist only finitely many sets at every location between A and B, then:

$$h(A,B) = d(a,B) \text{ for all } a \in A \text{ and } h(A,B) = d(b,A) \text{ for all } b \in B \tag{2.1}$$

Remark 2.9. *For any* $A, B \in \mathcal{H}(\mathbb{R}^n)$ *such that* $[A,B]$ *satisfies* (2.1), $d_E(a,b) \geq h(A,B)$ *for any* $a \in A, b \in B$.

The converse to (2.1) is not true in general, but it is true if we restrict ourselves to *finite* sets A and B: any pair of finite sets A and B satisfying (2.1) has a finite number of sets at each location between them. By Theorem 9.1 of [1], if A and B are not both finite, but there is a finite number of sets m at each location between them, finite A' and B' such that $\#([A',B']) = m$ (and hence, satisfying (2.1)) can be constructed using the Finite Conversion algorithm of [1]. So, in characterizing possible values $\#([A,B])$, we may narrow our focus to pairs of finite sets A and B satisfying (2.1), and we refer to such a pair as a *finite configuration*, as in [1].

Example 2.7 shows a finite configuration, and another such example is shown in Figure 4, where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and $C_s = \{c_1, c_2, c_3\}$. In this case, C_s itself and $\{c_1, c_3\}$ are between A and B located s away from A. However, not every subset of C_s satisfies these conditions. In this example, any subset of C_s that does not contain c_1 or c_3 does not lie between A and B, so $\#([A,B]) = 2$.

Next, we introduce the idea of adjacency, which will help us calculate $\#([A,B])$.

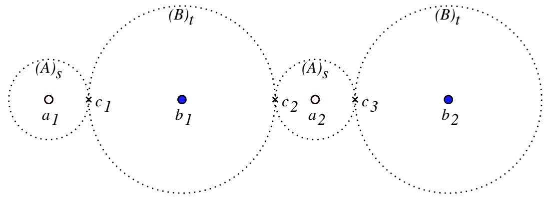

Figure 4: Seven sets at a distance s from A

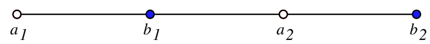

Figure 5: A finite configuration, with adjacencies

Definition 2.10. *Let $[A, B]$ be a finite configuration. A point $a \in A$ is* adjacent *to a point $b \in B$ if $d_E(a, b) = h(A, B)$.*

In Figure 5, we have added line segments connecting adjacent points to the finite configuration $[A, B]$ of Figure 4. The following result formalizes correspondence between elements of C_s and pairs of adjacent points.

Proposition 2.11. *Given a finite configuration $[A, B]$, for any fixed s such that $0 < s < h(A, B)$, there is a bijective correspondence between pairs of adjacent points in the configuration and elements of C_s.*

Proof. Let $[A, B]$ be a finite configuration and $h(A, B) = r$. Fix a value s such that $0 < s < r$ and let $t := r - s$.

By Remark 2.9, we may write

$$C_s = (A)_s \cap (B)_t = \bigcup_{a \in A, b \in B} (\{a\})_s \cap (\{b\})_t = \bigcup \{(\{a\})_s \cap (\{b\})_t \mid d_E(a, b) = r\}.$$

For any pair of points $a \in A$ and $b \in B$ such that $d_E(a, b) = r$, the set $(\{a\})_s \cap (\{b\})_t$ consists of a single point. To prove the desired correspondence, it remains only to show that the dilations of distinct pairs of adjacent points in our configuration will have distinct intersections. Suppose we have pairs of points (a, b) and (a', b') such that $(\{a\})_s \cap (\{b\})_t = (\{a'\})_s \cap (\{b'\})_t = \{c\}$. Then by the triangle inequality, $d_E(a, c) = s, d_E(c, b') = t \Rightarrow d_E(a, b') \leq s + t = r$, so $d_E(a, b') = r$. The line segments $\overline{ab}$ and $\overline{ab'}$ are collinear since they are both collinear with the segment $\overline{ac}$. Since c lies between a and b as well as between a and b', we have $b = b'$. Similarly, $a = a'$. □

Proposition 2.12. *Let $[A, B]$ be a finite configuration. Fix an arbitrary s such that $0 < s < h(A, B)$ and let $C' \subseteq C_s$ be a (compact) set. The set C' is between A and B at the location s away from A if and only if each point in $A \cup B$ belongs to some pair of adjacent points corresponding to an element of C' via the construction given in the proof of Proposition 2.11.*

Proof. Let $[A, B]$ be a finite configuration with $h(A, B) = r$. Let $0 < s < r$ and $t = r - s$.

First we show the following:

$$\text{For any } a \in A, b \in B, \text{ and } c \in C_s, d_E(a, c) \geq s \text{ and } d_E(b, c) \geq t \tag{2.2}$$

Given any $c \in C_s$, by Proposition 2.11 there exist some $a' \in A$ and $b' \in B$ such that $\{c\} = (\{a'\})_s \cap (\{b'\})_t$. Suppose there exists $a \in A$ such that $d_E(a, c) < s$. Then, by triangle inequality, $d_E(a, b') < s + t = r$, which contradicts Remark 2.9.

Now, let C' be an arbitrary set satisfying $AC'B$, $h(A, C') = s$ (and hence $h(C', B) = t$). Let $a \in A$. As discussed previously, $C' \subseteq C_s$. We see that $d(a, C') = s$, and so by (2.2) there is a point $c_a \in C'$ such that $d_E(a, c_a) = s$. A similar argument shows there is a point $b \in B$ with $d_E(c_a, b) = t$. By triangle inequality, $d_E(a, b) \leq s + t = h(A, B)$, and so by Remark 2.9, $d_E(a, b) = r$. Since $\{c_a\} = (\{a\})_s \cap (\{b\})_t$, a is a member of the adjacent pair of points (a, b) corresponding to c_a. If we instead start with a point $b \in B$, then we can produce an adjacent pair of points (a, b) corresponding to a point in C' by a similar argument. This proves the forward implication of this proposition.

Conversely, let each point in $A \cup B$ be a member of some pair of adjacent points corresponding (via the construction given in Proposition 2.11) to an element of C'. Then, for any $a \in A$ and $b \in B$, there is a point $c_a \in C'$ such that $d_E(a, c_a) = s$ and a point $c_b \in C'$ such that $d_E(b, c_b) = t$, hence $d(a, C') = s$ and $d(b, C') = t$. Then $d(A, C') = s$ and $d(B, C') = t$. Similarly, for any given $c \in C'$, there is a point $a \in A$ and a point $b \in B$ such that $d_E(c, a) = s$ and $d_E(c, b) = t$, hence $d(c, A) \leq s$ and $d(c, B) \leq t$. Then $d(C, A) \leq s$ and $d(C, B) \leq t$. Thus $h(A, C') = s$ and $h(B, C') = t$ and so C' satisfies $AC'B$, as desired. □

We have proved that finding $\#([A, B])$ for a finite configuration amounts to finding pairs of adjacent points, which leads us to express this problem in terms of graph theory, as shown in the next section.

3 Application of graph theory to the geometry of the Hausdorff metric

In this section, we make the connection between the number of edge coverings bipartite graphs have and $\#([A, B])$ for finite configurations $[A, B]$.

In this paper we are only concerned with counting edge coverings of simple graphs so we use the term "graph" in place of "simple graph." The reader may refer to [7] for an introduction to graph theory.

Definition 3.1 ([7, p. 120])**.** *An edge covering of a graph $G = \{\mathcal{V}, \mathcal{E}\}$ is a subset of edges $E \subseteq \mathcal{E}$ that covers all vertices of the graph, that is, each vertex of G is incident to at least one edge from E.*

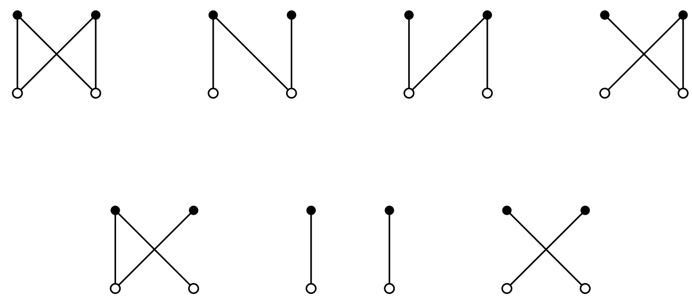

Figure 6: The edge coverings of $K_{2,2}$

We consider the graph $\{\emptyset, \emptyset\}$ (i.e. with vertex set and edge set both the empty set) to vacuously have 1 edge covering. It is necessary to make this definition because such graphs may come up when using our formulas for counting edge coverings, for instance when applying (4.1) to a graph G' consisting of two vertices and one edge between them.

See Figure 6 for an example of a graph with all its edge coverings.

We will denote the number of edge coverings of a graph G as $\#(G)$.

Definition 3.2. *For any finite configuration* $[A, B]$*, we define the* graph associated with the configuration*,* $G_{A,B}$*, to be the graph with one vertex for each point in* $A \cup B$ *where two vertices share an edge only if they correspond to two adjacent points in* A *and* B*.*

Example 3.3. *Drawing lines between adjacent points in a configuration shows what the associated graph should look like; Figure 5 shows the graph associated to the configuration in Figure 4.*

By construction, $G_{A,B}$ is bipartite. In any given configuration $[A, B]$, every point is adjacent to at least one point in the other set by the condition (2.1), so $G_{A,B}$ is also linked.

Theorem 3.4. *For any finite configuration* $[A, B]$*,* $\#([A, B]) = \#(G_{A,B})$*.*

Proof. By the construction of the bipartite graph associated to a finite configuration, the set of edges in an edge covering of $G_{A,B}$ corresponds to a set of adjacent points in $[A, B]$ so that each point in $A \cup B$ belongs to some pair of adjacent points. Proposition 2.12 concludes the proof of this theorem. □

Figure 6 shows the edge coverings of the graph associated to the configuration in Figure 3. The reader may wish to compare these edge coverings with the list of sets at a given location between A and B listed in Example 2.7 for a nice visual illustration of the correspondence discussed in Proposition 2.12 and Theorem 3.4.

We have proved that for every finite configuration $[A, B]$, there is an associated bipartite graph. Conversely, given any bipartite graph G, there exists a finite configuration so that its associated bipartite graph is G. The Configuration Construction Theorem ([1], Theorem 6.4) states that given finite sets A and B and a collection of unordered pairs (a, b), $a \in A$, $b \in B$, there exists n large enough so that a configuration $[A', B']$ with bijective correspondences between the elements of A, B and the points of A', B', respectively, where the unordered pairs (a, b) correspond to the adjacent points of the configuration, can be realized in $\mathbb{R}^n$.The two sets and unordered pairs specified by the vertices and edges of a bipartite graph G fulfill the hypothesis of the Configuration Construction Theorem. We can now prove the following theorem:

Theorem 3.5. *There do not exist any* $A, B \in \mathcal{H}(\mathbb{R}^n)$ *such that* $\#([A, B]) = 19$.

Proof. By Theorem 3.4 and the above discussion of the Finite Configuration Theorem [1], the set of all values of $\#([A, B])$ for finite configurations $[A, B]$ is equal to the set of all values of $\#(G)$ where G is a linked bipartite graph. The result then follows from Theorem 5.4. □

By a similar argument, the following theorem is a consequence of Theorem 5.5.

Theorem 3.6. *There do not exist any* $A, B \in \mathcal{H}(\mathbb{R}^n)$ *such that* $\#([A, B]) = 37$.

4 Counting edge coverings

We now start to present the technical lemmas needed to prove the main graph theory results, Theorems 5.4 and 5.5, of the paper. The central results of the section are the formulas in Lemmas 4.3 and 4.8 relating the number of edge coverings of any graph to the numbers of edge coverings of some of its subgraphs. Then we also present several corollaries applying these lemmas to produce results about the divisibility and bounds on the number of edge coverings of a graph. We also use these observations in two examples in this section to verify that two nonbipartite graphs we have produced have 19 and 37 edge coverings, respectively.

Many results in this paper will refer to the degrees of vertices, and we find it convenient to make the following definiton.

Definition 4.1. *A vertex which is neither a pendant nor a pendant neighbor is called* amicable.

Given a graph $G = \{\mathcal{V}, \mathcal{E}\}$, the degree of a vertex $v \in \mathcal{V}$ in G is denoted by $\deg_G(v)$.

The following result lets us count the edge coverings of graphs with a certain property.

Lemma 4.2. *Let* $G = \{\mathcal{V}, \mathcal{E}\}$ *be a linked graph where every vertex is either a pendant or a pendant neighbor. Let* E *be the set of all edges* $(u, v) \in \mathcal{E}$ *where* u *and* v *are both pendant neighbors, but neither is a pendant. Then* $\#(G) = 2^{|E|}$.

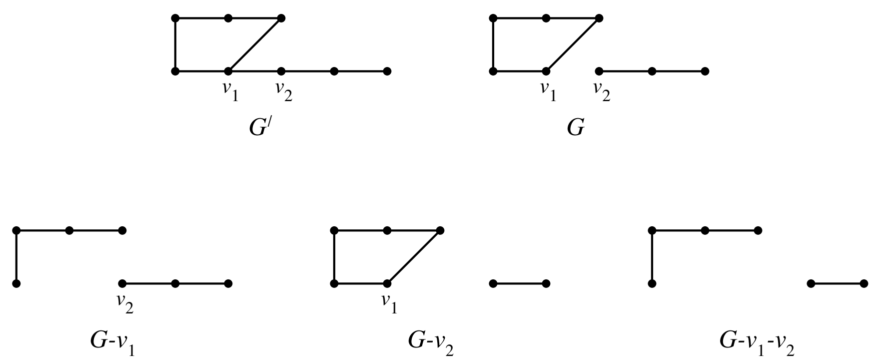

Figure 7: Illustrating Theorem 4.3

Proof. Every covering consists of all pendant edges plus some set of non-pendant edges. □

The following lemma shows how adding an edge to a graph affects the number of edge coverings.

Lemma 4.3. *Let $G = \{\mathcal{V}, \mathcal{E}\}$ be a graph. Let $v_1, v_2 \in \mathcal{V}$ and suppose $(v_1, v_2) \notin \mathcal{E}$. Let $G' = \{\mathcal{V}, \mathcal{E}'\}$ where $\mathcal{E}' = \mathcal{E} \cup \{(v_1, v_2)\}$. Then,*

$$\#(G') = 2 \cdot \#(G) + \#(G - v_1) + \#(G - v_2) + \#(G - v_1 - v_2). \tag{4.1}$$

Proof. The collection of all edge coverings of G' can be partitioned into those edge coverings that have (v_1, v_2) as a member and those where (v_1, v_2) is not a member. The set of all edge coverings of G' that have (v_1, v_2) as a member can be further partitioned into four disjoint sets: those where v_1 and v_2 each either have or don't have incident edges other than (v_1, v_2). □

The following corollary is a consequence of applying Lemma 4.3 inductively. See [6] for a different proof as well as some results on counting the numbers of edge coverings of more complicated graphs from the point of view of configurations (see Section 2) in the geometry of the Hausdorff metric.

Let F_k be the kth Fibonacci number, where $F_0 = 0$ and $F_1 = F_2 = 1$.

Corollary 4.4.

(a) A graph consisting of a trail with n vertices has F_{n-1} edge coverings.

(b) A graph consisting of an n-circuit has $F_{n-1} + F_{n+1}$ (i.e. the nth Lucas number) edge coverings.

Next, we use Lemma 4.3 and Corollary 4.4 to show that the graph G' in Figure 7 has 37 edge coverings.

Example 4.5. *Consider the graph G' in Figure 7. Each proper subgraph of G' shown consists of two connected components. Observe that G consists of a 5-circuit and a trail with 3 vertices, so by Corollary 4.4 (a) and (b), $\#(G) = (F_4 + F_6) \cdot F_2 = 11$. Similarly, $\#(G - v_2) = (F_4 + F_6) \cdot F_1 = 11$, $\#(G - v_1) = F_3 \cdot F_2 = 2$, and $\#(G - v_1 - v_2) = F_3 \cdot F_1 = 2$. Substituting into* (4.1)*, we get $\#(G') = 2 \cdot 11 + 11 + 2 + 2 = 37$.*

The following statement is a generalization of Lemma 4.3. We prove it using Lemma 4.3 and Corollary 4.4, and so list as a further corollary to Lemma 4.3. In this corollary (which will be used in the proof of Corollary 4.7), we show how the number of edge coverings of a graph is affected by adding (a) a trail $w_1, \dots w_k$ between two vertices v_1 and v_2 or (b) a cycle $v_1, w_1, \dots, w_k, v_1$ containing only one vertex v_1 of the original graph.

Corollary 4.6. *Let $G = \{\mathcal{V}, \mathcal{E}\}$ be a graph.*

(a) Let $v_1, v_2 \in \mathcal{V}$ and let $G' = \{\mathcal{V}', \mathcal{E}'\}$ where $\mathcal{V}' = \mathcal{V} \cup \{w_1, \dots, w_k\}$ for some $k \geq 1$ and where $\mathcal{E}' \supset \mathcal{E}$ and $\mathcal{E}' \setminus \mathcal{E} = \{(v_1, w_1), (v_2, w_k)\} \cup \{(w_{l-1}, w_l) \mid 2 \leq l \leq k\}$. Then,

$$\#(G') = F_{k+3} \cdot \#(G) + F_{k+2} \cdot \#(G - v_1) + F_{k+2} \cdot \#(G - v_2) + F_{k+1} \cdot \#(G - v_1 - v_2).$$

(b) Let $v_1 \in \mathcal{V}$ and let $G' = \{\mathcal{V}', \mathcal{E}'\}$ where $\mathcal{V}' = \mathcal{V} \cup \{w_1, \dots, w_k\}$ for some $k \geq 1$ and where $\mathcal{E}' \supset \mathcal{E}$ and $\mathcal{E}' \setminus \mathcal{E} = \{(v_1, w_1), (v_1, w_k)\} \cup \{(w_{l-1}, w_l) | 2 \leq l \leq k\}$. Then,

$$\#(G') = F_{k+3} \cdot \#(G) + (F_{k+2} + F_k) \cdot \#(G - v_1).$$

Proof. Let G' be as in the hypothesis for (a). Let the graph $G_0 = \{\mathcal{V}', \mathcal{E}' - e_1 - e_2\}$ where $e_1 = (v_1, w_1)$ and $e_2 = (v_2, w_k)$. Observe G_0 consists of two linked components; one component is G and the other is the trail of vertices $w_1, w_2, \dots, w_k$. By Corollary 4.4 the component consisiting of the trail of vertices $w_1, \dots, w_k$ has F_{k-1} edge coverings. Thus,

$$\#(G_0) = F_{k-1} \cdot \#(G) \tag{4.2}$$

To complete the proof of part (a), we apply (4.1) to G_0 and e_1 and then to $G_0 \cup \{e_1\}$ and e_2.

Now let G' be as in the hypothesis for part (b). Let $G_0 = \{\mathcal{V}', \mathcal{E}' - e_1 - e_2\}$ where e_1 is the edge between v_1 and w_1 and e_2 is the edge between v_1 and w_k. The graph G_0 is identical to the graph G_0 in the proof of part (a), so the following equation still holds:

$$\#(G_0) = F_{k-1} \cdot \#(G) \tag{4.3}$$

To complete the proof of part (b), we apply (4.1) to G_0 and e_1 and then to $G_0 \cup \{e_1\}$ and e_2. □

In the next two corollaries to Lemma 4.3, we present results about $\#(G)$ and divisors of $\#(G)$ for a graph G with certain properties.

Corollary 4.7. *Let $G = \{\mathcal{V}, \mathcal{E}\}$ be a connected graph such that every amicable vertex in G has degree 2. If the collection of amicable vertices is nonempty, then either $\#(G)$ is divisible by a Fibonacci number greater than 1 or $\#(G)$ is a Lucas number.*

Proof. Let G be as in the hypothesis and fix an amicable vertex $v \in V$. We will construct a trail of amicable vertices containing v as follows.

Since v has degree 2, it must share edges with two other vertices, v' and v'', which are either pendant neighbors or amicable.

If one of the vertices sharing an edge with v is amicable, without loss of generality we call it v' and we include it in the trail. Since v' is amicable, by hypothesis it has degree 2, and so has one other edge, which it shares either a pendant neighbor or an amicable vertex. If this vertex is amicable, we include it in our trail, and so on.

We continue to construct this trail of amicable vertices until we come to an amicable vertex w_1 which shares an edge with a pendant neighbor or shares an edge with an amicable vertex which has already appeared in the trail we are constructing. Then if v'' is amicable, we include it in our trail and continue again until we come to an amicable vertex w_k which shares an edge with a pendant neighbor or shares an edge with an amicable vertex which has already appeared in the trail.

We have constructed a trail of distinct amicable vertices $w_1, \ldots, w_k$ which includes (and possibly consists of only) $v = w_i$ for some i. The vertices w_1 and w_k are situated in one of the following ways:

1. w_1 and w_k each share an edge with distinct pendant neighbor vertices v_1 and v_2. (Note this includes the case where $v = w_1 = w_k$.)
2. w_1 and w_k each share an edge with the same pendant neighbor vertex, v_1.
3. v_1 and v_2 share an edge.

In case 1, since v_1 and v_2 are pendant neighbors, removing either of them isolates a vertex. So, applying Corollary 4.6(a) to the graph $G - \{w_1, \ldots, w_k\}$ shows $\#(G) = F_{k+3} \cdot \#(G \setminus \{w_1, \ldots, w_k\})$ (the names of the vertices we have chosen coincide with the notation used in Corollary 4.6(a)). For case 2, applying Corollary 4.6(b) similarly shows that $\#(G)$ must be divisible by a Fibonacci number. In case 3, $\#(G)$ must consist entirely of this cycle of distinct amicable vertices. By Corollary 4.4(b), it follows that $\#(G)$ is a Lucas number. $\square$

Now we present a formula that shows how removing a vertex from a graph affects its number of edge coverings.

Lemma 4.8. *Let $G = \{\mathcal{V}, \mathcal{E}\}$ be a graph and let v be any vertex in $\mathcal{V}$. Let $\deg_G(v) = m$ where v shares edges with the distinct vertices $v_1, \ldots, v_m$. Then*

$$\#(G) = (2^m - 1) \cdot \#(G - v) + \sum_{k=1}^{m} 2^{m-k} \sum_{1 \le i_1 < \cdots < i_k \le m} \#(G - v - v_{i_1} - \cdots - v_{i_k}) \quad (4.4)$$

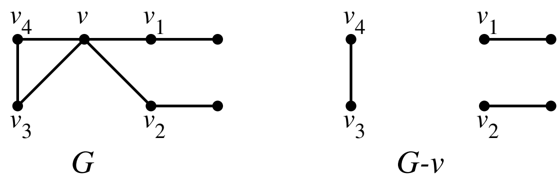

Figure 8: An example of using Theorem 4.8

Proof. To count the edge coverings, group them according to which subset I of $\{v_1, \ldots, v_m\}$ is covered only by edges incident to v. Note that if I is empty, we must use at least one edge incident to v. □

We apply the formula (4.4) to demonstrate that the graph G in Figure 8 has 19 edge coverings.

Example 4.9. *Consider the graph in Figure 8. Note that* $\#(G - v) = 1$ *and* $\#(G - v - v_3 - v_4) = 1$*. For any* $V \subseteq \{v_1, v_2, v_3, v_4\}$ *except* $V = \{v_3, v_4\}$*,* $G - v - V$ *has at least one isolated vertex. So, applying* (4.4) *yields* $\#(G) = (2^4 - 1) \cdot 1 + 2^2 \cdot 1 = 19$.

We use Lemma 4.8 to determine when removing a vertex from a graph does not change its number of edge coverings.

Corollary 4.10. *Let* $G = \{\mathcal{V}, \mathcal{E}\}$ *be a bipartite graph such that* $\#(G) \neq 0$ *and let* v *be a vertex. Then* $\#(G) = \#(G - v)$ *if and only if, in* G*,* v *is a pendant adjacent to a vertex which is also adjacent to some other pendant.*

Proof. Let G and v be as in the hypothesis. Suppose $\#(G) = \#(G - v)$. By Lemma 4.8, $\deg_G(v) = 1$ and $\#(G - v - v_1) = 0$ where v_1 is the one vertex adjacent to v in G. Since $\#(G - v) \neq 0$ and $\#(G - v - v_1) = 0$, removing v_1 from $G - v$ isolates a vertex, hence v_1 is adjacent to some vertex u which is a pendant in $G - v$. Since G is bipartite, u cannot be adjacent to v in G, so u is a pendant in G as well.

The converse implication follows as a direct application of Lemma 4.8. □

In the next section we will use the general graph theory results we have just proved to show that there are no bipartite graphs G such that $\#(G) = 19$ or $\#(G) = 37$.

5 Missing numbers 19 and 37

We are now going to prove the main graph theory results. We begin by making some observations about the problem and proving some lemmas.

If $G_1, \ldots, G_k$ are the connected components of G, then $\#(G) = \#(G_1) \cdots \#(G_k)$. In particular, if $\#(G) = p$ for a prime p, then G must have a connected component with p edge coverings. Thus, we can restrict our consideration to connected graphs.

Remark 5.1. *Let $\#(G) \neq 0$. If G' is a subgraph of G, then $\#(G') \leq \#(G)$.*

The next result is valid for bipartite graphs only. We call a vertex v in a graph G an *isolating* vertex if $\#(G-v) = 0$; a vertex that is not isolating is *nonisolating.*

Lemma 5.2. *Let $G = \{\mathcal{V}, \mathcal{E}\}$ be a bipartite graph and let v be a nonisolating vertex such that $\deg_G(v) = m$. Let v share edges in G with precisely the vertices $v_1, \ldots, v_m \in \mathcal{V}$. Let $\{v_1, \ldots, v_m\} \supseteq V' \supseteq V \supsetneq \emptyset$. Then $\#(G - v - V) = 0$ implies $\#(G - v - V') = 0$.*

Proof. Since v is a nonisolating vertex and $\#(G - v - V) = 0$, removing the set V of vertices from G isolates some vertex w. Because G is bipartite and w shares edges with vertices in V, w does not share an edge with v, so $w \notin V'$. Hence w is an isolated vertex in $G - v - V'$ as well. $\square$

This next lemma uses (4.4) to make some bounds for the value of $\#(G - v)$ based on the value of $\#(G)$.

Lemma 5.3. *Let $G = \{\mathcal{V}, \mathcal{E}\}$ be a linked graph where $v \in \mathcal{V}$, $\deg_G(v) = m$. If v is a nonisolating vertex, then*

$$\frac{\#(G)}{3^m - 1} \leq \#(G - v) \leq \frac{\#(G)}{2^m - 1} \tag{5.1}$$

Proof. By Remark 5.1, we have the following inequality:

$$\begin{aligned} &\sum_{k=1}^{m} 2^{m-k} \sum_{1 \leq i_1 < \cdots < i_k \leq m} \#(G - v - v_{i_1} - \cdots - v_{i_k}) \\ &\leq \sum_{k=1}^{m} 2^{m-k} \binom{m}{k} \#(G - v) = (3^m - 2^m)\#(G - v). \end{aligned} \tag{5.2}$$

We apply Lemma 4.8 to (5.2):

$$(2^m - 1)\#(G - v) \leq \#(G) \leq (2^m - 1)\#(G - v) + (3^m - 2^m)\#(G - v) \quad \square$$

These lemmas will help us to prove the main results of this paper.

Theorem 5.4. *There is no bipartite graph G such that $\#(G) = 19$.*

Proof. Suppose, to the contrary, that $G = \{\mathcal{V}, \mathcal{E}\}$ is a connected bipartite graph such that $\#(G) = 19$. We will use (4.4) to obtain a contradiction.

By Corollary 4.2, the number of edge coverings of a graph containing vertices which are all pendants or pendant neighbors is a power of 2. Since 19 is not a power of 2, G contains amicable vertices. Let $v \in \mathcal{V}$ such that v has maximal degree m among amicable vertices, where v is adjacent to $v_1, \ldots, v_m$. By Lemma 5.3, we know $\#(G) = 19 \geq (2^m - 1)\#(G - v)$. Since $\#(G - v) > 0$ and v is

not a pendant, we have $2 \le m \le 4$. We consider these possible values of m as different cases.
CASE 1. Suppose $m = 4$. By Lemma 5.3, we have $\#(G - v) = 1$. However, since $19 < (2^4 - 1) + 2^3$, (4.4) shows $\#(G - v_1 - v_i) = 0$ for all $1 \le i \le 4$. Then by Lemma 5.2, all the terms of (4.4) other than $\#(G - v)$ must be 0, implying 15=19.
CASE 2. Suppose $m = 3$. By Lemma 5.3, $\#(G - v) = 1$ or 2. Examine (4.4) in this case. If $\#(G - v) = 2$, then since 19 is odd, we must have $\#(G - v - v_1 - v_2 - v_3)$ odd, and thus nonzero, since all the other terms have even coefficients. Then the contrapositive of Lemma 5.2 implies that $\#(G - v - v_i) \ne 0$ and $\#(G - v - v_i - v_j) \ne 0$ for all $i, j \in \{1, 2, 3\}$. However, all the terms of (4.4) are then nonzero and sum to more than 19.

Now suppose $m = 3$ and $\#(G - v) = 1$. By Remark 5.1, every subgraph of $G - v$ has 1 or 0 edge coverings. Also, we see that $\#(G - v - v_1 - v_2 - v_3) = 0$ since the sum on the right of (4.4) is odd. Further, some of the terms on the right hand side of (4.4) must also be 0 so that their sum does not exceed 19, forcing us into two possibilities:

(i) $\#(G - v - v_i) = 0$ for some $i \in \{1, 2, 3\}$

(ii) $\#(G - v - v_i) \ne 0$ for each i and $\#(G - v - v_j - v_k) = 0$ for all $i, j, k \in \{1, 2, 3\}$ such that $j \ne k$

In situation (i), Lemma 5.2 tells us that $\#(G - v - v_i - v_j) = \#(G - v - v_i - v_k) = 0$, forcing the terms on the right hand side of (4.4) to sum to at most 17. In situation (ii), observe that $\#(G - v - v_i) = \#(G - v - v_j) = 1$ and $\#(G - v - v_i - v_j) = 0$ implies that removing either v_i or v_j from $G - v$ does not isolate any vertices, but removing both isolates some vertex. Thus in situation (ii) for each unordered pair (i, j) where $i, j \in \{1, 2, 3\}$ and $i \ne j$, there exists some vertex $w_{i,j}$ in $G - v$ that shares edges with exactly v_i and v_j. So, $G - v$ has a subgraph which consists of a 6-cycle consisting of $v_1, w_{1,2}, v_2, w_{2,3}, v_3, w_{3,1}$. A 6-cycle with distinct vertices has 18 edge coverings, which by Remark 5.1 contradicts $\#(G - v) = 1$.
CASE 3. Suppose $m = 2$. A direct application of Corollary 4.7 implies that a Fibonacci number greater than 1 divides $\#(G)$ or that $\#(G)$ is a Lucas number, a contradiction.

Therefore, there are no bipartite graphs that have exactly 19 edge coverings. □

Next we will prove the major new result of this paper. The proof is longer than that of Theorem 5.4, but uses many of the same methods.

Theorem 5.5. *There is no bipartite graph G such that $\#(G) = 37$.*

Proof. We will proceed in a manner similar to the proof of Theorem 5.4, using (4.4) to find our contradictions.

Suppose there exists a connected bipartite graph $G = \{\mathcal{V}, \mathcal{E}\}$ such that $\#(G) = 37$.

By Corollary 4.2, the number of edge coverings of a graph containing vertices which are all pendants or pendant neighbors is a power of 2. Since 37 is not a power of 2, G contains amicable vertices. Let $v \in \mathcal{V}$ such that v has maximal degree m among amicable vertices, where v is adjacent to $v_1, \ldots, v_m$.

Lemma 5.3 shows that $m \leq 5$. Further, since v is not a pendant, we know $2 \leq m \leq 5$. We now consider the cases.

Case 1. Suppose $m = 5$. Then by Lemma 5.3 we must have $\#(G - v) = 1$. If $\#(G - v - v_i) \neq 0$ for some i, then the terms on the right hand side of (4.4) will sum to more than 37. If $\#(G - v - v_i) = 0$ for all $1 \leq i \leq 5$, then by Lemma 5.2 all the remaining terms on the right hand side of (4.4) will be 0, forcing the terms to sum to less than 37.

Case 2. Suppose $m = 4$. By Lemma 5.3, we have $\#(G-v) = 1$ or $\#(G-v) = 2$. We examine each of these subcases.

(i) Suppose $\#(G-v) = 2$. We find a contradiction here with the same argument used for the $m = 5$ case.

(ii) Suppose $\#(G - v) = 1$. By Remark 5.1, $\#(G - v - V) = 1$ or 0 for any nonempty $V \subseteq \{v_1, v_2, v_3, v_4\}$. Examining the right hand side of (4.4), we see that $\#(G - v - v_i) = 0$ for at least two values of i. Otherwise, the terms will sum to more than 37. However, if $\#(G - v - v_i) = 0$ for three or all four values of i, Lemma 5.2 shows that for any V such that $|V| \geq 2$, we will have $\#(G - v - V) = 0$, forcing the terms on the right hand side of (4.4) to sum to less than 37. So, $\#(G - v - v_i) = 1$ for exactly two values of i, without loss of generality, assume the values of i are 1 and 2. Lemma 5.2 shows $\#(G-v-V) = 0$ for all $|V| \geq 2$ except for $V = \{1, 2\}$. Whether $\#(G - v - v_1 - v_2)$ is 1 or 0, the terms on the right hand side of (4.4) will sum to less than 37.

Case 3. Suppose $m = 3$. Lemma 5.3 shows that $2 \leq \#(G - v) \leq 5$. We consider the subcases.

(i) First, suppose $\#(G-v) = 5$. We arrive at a contradiction using an argument parallel to the one used for the $m = 5$ case.

(ii) Suppose $\#(G - v) = 4$. Observe all the terms on the right hand side of (4.4) are even except for the $\#(G - v - v_1 - v_2 - v_3)$ term. However, in order for the terms on the right hand side of (4.4) to sum to an odd value, $\#(G - v - v_1 - v_2 - v_3)$ must be odd, and thus nonzero. By Lemma 5.2, every term on the right hand side of (4.4) must be nonzero, forcing the terms on the right hand side of (4.4) to sum to more than 37, a contradiction.

(iii) Suppose $\#(G - v) = 3$. Observe all the other terms on the right hand side of (4.4), except for $\#(G-v-v_1-v_2-v_3)$ are even. So, for the terms to sum to 37, it must be that $\#(G-v-v_1-v_2-v_3)$ is even. If $\#(G-v-v_1-v_2-v_3) \geq 2$, then Remark 5.1 and Lemma 5.2 show that the other graphs included in the right hand side of (4.4) must all have at least 2 edge coverings, forcing the terms to sum to more than 37. So $\#(G-v-v_1-v_2-v_3) = 0$. Using Remark 5.1 and Lemma 5.2 again, $\sum_{i=1}^{3} \#(G - v - v_i) \geq \sum_{1 \leq j < k \leq 3} \#(G - v - v_j - v_k)$. The two possible choices of values for these sums which agree with (4.4) are

$$\sum_{i=1}^{3} \#(G-v-v_i) = 3 \quad \text{and} \quad \sum_{1\le j<k\le 3} \#(G-v-v_j-v_k) = 2$$
$$\text{or} \sum_{i=1}^{3} \#(G-v-v_i) = 4 \quad \text{and} \quad \sum_{1\le j<k\le 3} \#(G-v-v_j-v_k) = 0.$$

(a) Assume $\sum_{i=1}^{3} \#(G-v-v_i) = 3$ and $\sum_{1\le j<k\le 3} \#(G-v-v_j-v_k) = 2$. By Remark 5.1 and Lemma 5.2, we must have $\#(G-v-v_i) = 1$ for all $1 \le i \le 3$ and $\#(G-v-v_j-v_k) = 1$ for two choices of j,k such that $1 \le j < k \le 3$, and $\#(G-v-v_j-v_k) = 0$ for the third choice. Without loss of generality, let $\#(G-v-v_1-v_2) = 0$. Then there must exist some vertex $z \in \mathcal{V}-v$ that is adjacent only to v_1 and v_2 in $G-v$. Applying Lemma 5.3 to $G-v$ and $G-v-v_1$, we see $\deg_{G-v}(v_1) = 2$, so there exists some other vertex, $z' \in \mathcal{V}-v$ that is adjacent to v_1 in $G-v$. Since G and hence all its subgraphs are bipartite, we have $z \ne v_3 \ne z'$ and $z \ne v_2 \ne z'$. So, $G-v-v_3$ contains the subgraph $G' = \{\{v_1,v_2,z,z'\},\{(v_2,z),(z,v_1),(v_1,z')\}\}$. Note that $\#(G') = 2$ by inspection. This contradicts our assumption $\#(G-v-v_3) = 1$ by Remark 5.1.
(b) Assume $\sum_{i=1}^{3} \#(G-v-v_i) = 4$ and $\sum_{1\le j<k\le 3} \#(G-v-v_j-v_k) = 0$. Since $\#(G-v) = 3$, Remark 5.1 shows that $\#(G-v-v_i) \le 3$ for all $1 \le i \le 3$. By (4.4) for some i, we must have $\#(G-v-v_i) > 1$. Further, for some $j \ne i$ we must also have $\#(G-v-v_j) \ne 0$. Without loss of generality, suppose $\#(G-v-v_1) > 1$ and $\#(G-v-v_2) > 0$. Since $\#(G-v-v_1-v_2) = 0$, there exists some $u \in \mathcal{V}-v$ such that u is adjacent to only v_1 and v_2 in $G-v$. It follows from Lemma 5.3 that $\deg_{G-v}(v_1) = 1$, so v_1 is only adjacent to u in $G-v$. If $\deg_{G-v}(v_2) = 1$ as well, then Corollary 4.10 shows that $\#(G-v-v_1) = \#(G-v-v_2) = \#(G-v) = 3$, contradicting $\sum_{i=1}^{3} \#(G-v-v_i) = 4$. Then Lemma 5.3 gives us $\deg_{G-v}(v_2) = 2$ and $\#(G-v-v_2) = 1$. Let w be the second vertex to which v_2 is adjacent in $G-v$. The only pendant vertex adjacent to u in $G-v$ is v_1, so by Corollary 4.10 we know $\#(G-v-v_1) \ne \#(G-v)$, hence $\#(G-v-v_1) = 2$, meaning $\#(G-v-v_3) = 1$. However, observe $G' = \{\{v_1,v_2,u,w\},\{(v_1,u),(u,v_2),(v_2,w)\}\}$ is a subgraph of $\#(G-v-v_3)$ and $\#(G') = 2$. This contradicts $\#(G-v-v_3) = 1$ by Remark 5.1. So we have completed our contradiction of the $m = 3$, $\#(G-v) = 3$ case.
(iv) The last subcase is $\#(G-v) = 2$. Examining (4.4), we see that every term on the right hand side is even except for $\#(G-v-v_1-v_2-v_3)$. So, $\#(G-v-v_1-v_2-v_3)$ must be odd and thus nonzero. By Lemma 5.2, all the terms on the right hand side of (4.4) must be nonzero. Applying Remark 5.1, we conclude that $2 = \#(G-v) \ge \#(G-v-v_i) \ge 1$ for all $1 \le i \le 3$. Lemma 5.3 then shows that $\deg_{G-v}(v_i) = 1$ for all $1 \le i \le 3$. For the terms on the right hand side of (4.4) not to sum to more than 37, we must have $\#(G-v-v_i) = 1$ for at least two values of i. Without loss of generality, assume $\#(G-v-v_1) = \#(G-v-v_2) = 1$. Since $\deg_{G-v}(v_1) = 1$, there exists some $u \in \mathcal{V}$ that is adjacent to v_1 in $G-v$. Since $\#(G-v-v_1) \ne 0$, it must be the case that $\deg_{G-v}(u) > 1$. So u is also adjacent to some vertex

w in $G - v$. Since $\#(G - v - v_1) < \#(G - v)$, Corollary 4.10 shows that u cannot be adjacent to any other pendants in $G - v$. So w is not a pendant in $G - v$ and $v_2 \neq w \neq v_3$. It follows that w is adjacent to some vertex $y \neq u$ in $G - v$. Since G (and thus $G - v$) is bipartite, $v_2 \neq y \neq v_3$. So, we know $G' = \{\{v_1, u, w, y\}, \{(v_1, u), (u, w), (w, y)\}\}$ is a subgraph of $\#(G - v - v_2)$ with $\#(G') = 2$. This contradicts $\#(G - v - v_2) = 1$ by Remark 5.1. Thus we have shown $m \neq 3$.
Case 4. Suppose $m = 2$. A direct application of Corollary 4.7 implies that a Fibonacci number greater than 1 divides $\#(G)$ or that $\#(G)$ is a Lucas number, a contradiction. □

We remark that the condition of being bipartite is essential for these results; non-bipartite graphs with exactly 19 or 37 edge coverings were shown in Examples 4.9 and 4.5, respectively.

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