Coherence Theorems in Two-Dimensional Category Theory

1 Introduction

Coherence theorems in category theory were originally defined roughly as any results which imply that some diagrams commute. In the introductory material of [3], G.M. Kelly argues that this definition can be extended to include his work in that paper: "It is reasonable - and probably very common - to use the term 'coherence theorem' for a result which, having found out something about D^* from a knowledge of its algebras, goes back to these algebras and deduces something useful about them." Here D^* refers to a 2-monad with the pseudo-algebras of any given 2-monad D as its algebras. Kelly goes on, in [3], to point out that such results may often imply that some diagrams commute. We can conclude theorems that allow us to deduce information about the algebras of a monad might be included under the heading of coherence results.

The purpose of this paper is to examine these senses of what it means to be a coherence result by presenting various results in 2-dimensional category theory in a manner accessible to anyone with a basic knowledge of ordinary, 1-dimensional category theory.

In Section 2, we define structures of two-dimensional category theory necessary for the results of Sections 3 and 4. Section 3 uses the Yoneda Lemma for Bicategories to show that every bicategory is biequivalent to a 2-category, tracing a path similar to that of Tom Leinster's *Basic Bicategories* [4]. Section 4 works through proving, by the same method in Section 1 of Joyal and Street's *Braided Tensor Categories* [2] that every free monoidal category is strictly tensor equivalent to the free strict monoidal category with the same generating category.

In Section 5, we take a step back and show applications of the main theorems of Sections 3 and 4, in the form of proving that some classes of diagrams commute.

In Section 6, we define terms involving 2-monads and their algebras and go on to state and give a sketch proof of the main theorem of Power's *A General Coherence Result* [5].

The paper concludes in Section 7 by examining applications of the main theorem of Section 6, particularly those resembling the results shown in Sections 3 and 4. Thanks particularly here to Dr. Richard Garner for his reformulation of a theorem in [1] and helpful conversation regarding this material.

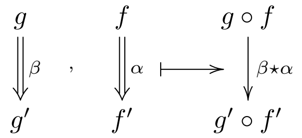
2 Basic Definitions for Two-Dimensional Category Theory

We begin by defining a 2-dimensional category, one of the stricter structures we will examine. Note that for clarity of expression we often describe higher dimensional categories in terms of n-cells, where 0-cells are objects and n-cells are morphisms between $(n-1)$-cells. Also, we use $\mathbf{1}$ for the one-object discrete category and 1 to denote identity 1-cells and 2-cells. Various subscripts will be appended to 1 and other morphisms, but in many cases, where context indicates sufficiently well what is meant, they will be left off to reduce clutter.

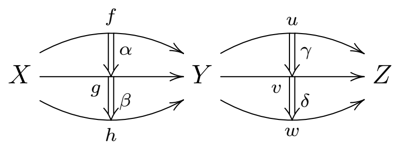
Definition 2.1. A *2-category* $\mathcal{C}$ is a category such that for any $X, Y \in \operatorname{ob} C$, $\mathcal{C}(X,Y)$ is also a category. The morphisms of each $\mathcal{C}(X,Y)$ are called 2-cells, and their operation of composition is called vertical composition. Further, for each $X, Y, Z \in \operatorname{ob}\mathcal{C}$, there is a functor $c_{XYZ} : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ which has its action on the morphisms of $\mathcal{C}$ given by composition in $\mathcal{C}$. The action of these functors on 2-cells, however, defines an additional type of composition of 2-cells, called horizontal composition:

$$c_{XYZ} : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$$

$$(g,f) \longmapsto g \circ f$$



The objects of $\mathcal{C}$, 2-cells of $\mathcal{C}$, and horizontal composition give the objects, morphisms, and composition of morphisms of a category; in particular, this means there are identities for horizontal composition. Further, observe that the functorality of c_{XYZ} ensures that horizontal composition distributes over vertical composition and vice versa, i.e. given 1-cells $f, g, h \in \mathcal{C}(X,Y)$, $u, v, w \in \mathcal{C}(Y, Z,)$ and 2-cells $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$, $\gamma : u \Rightarrow v$, $\delta : v \to w$, we have $(\delta \star \beta) \cdot (\gamma \star \alpha) = (\delta \cdot \gamma) \star (\beta \cdot \alpha)$. These compositions are typically shown diagramatically as below, suggesting the origin of the terms vertical and horizontal composition.



To gain an intuition for 2-categories, it is helpful to note that the archetypal 2-category is **CAT**, with 0-cells categories, 1-cells functors, and 2-cells natural transformations.

Next, we examine bicategories, which are weaker 2-categories. The main difference is that when we forget any structure involving 2-cells in a bicategory, we are not left with a category. Instead, in a bicategory, identity 1-cells and associativity of composition of 1-cells need only be present up to isomorphism. The data and axioms of a bicategory are accordingly weakened to accommodate this difference.

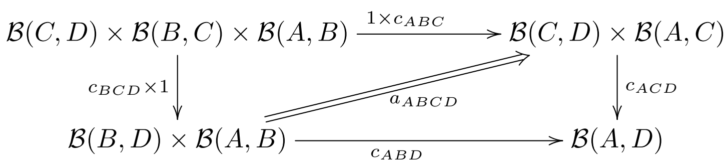
Definition 2.2. A *bicategory* $\mathcal{B}$ is a collection of 0-cells where for each pair of objects X and Y, $\mathcal{B}(X,Y)$ is a category. The objects of each category $\mathcal{B}(X,Y)$ are the 1-cells of $\mathcal{B}$, the morphisms are 2-cells, and their operation of composition is called vertical composition. For each $X, Y, Z \in \operatorname{ob}\mathcal{B}$, there is a functor which gives composition of 1-cells and horizontal composition of 2-cells:

$$c_{XYZ} : \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \longrightarrow \mathcal{B}(X,Z)$$

$$(g,f) \longmapsto g \circ f$$

$$(\beta,\alpha) \longmapsto (\beta \star \alpha)$$

We have the following natural isomorphism to describe the associativity of 1-cells in $\mathcal{B}$:

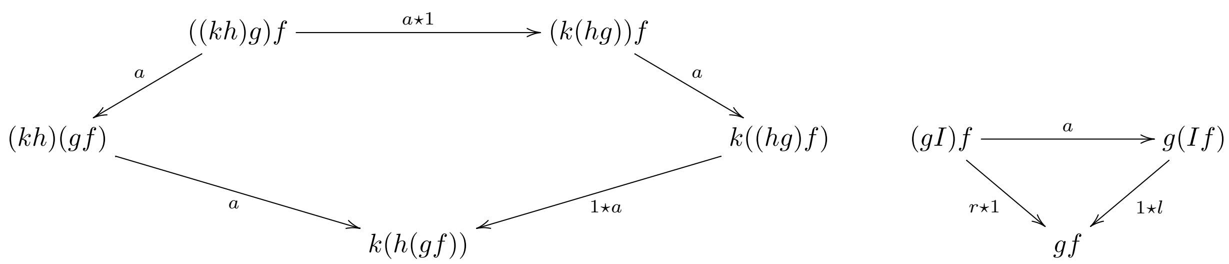


Also, for each object X, we have a functor which picks out a particular 1-cell from X to X:

$$I_X : \mathbf{1} \longrightarrow \mathcal{B}(X,X)$$

The image of I_X, denoted 1_X, is called the unit or the identity on X. However, unlike the strict identities of 2-categories, the identities in bicategories need only act identically up-to-isomorphism when composed with other 1-cells. The unitality of thse identity 1-cells, that is, the weak identity requirements, are given by the following natural isomorphisms:

Finally, any 1-cells f, g, h, k with appropriate domains and codomains satisfy axioms given by the following commutative diagrams:

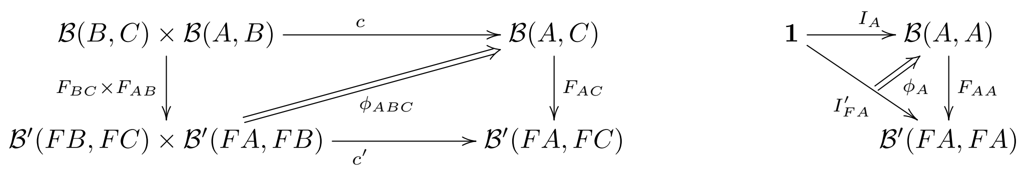


Just as a group may be viewed as a category with one object, a monoidal category $\mathcal{V}$ may be viewed as a bicategory $\tilde{\mathcal{V}}$ with one object, denoted $*$. The objects of $\mathcal{V}$ correspond to the 1-cells in $\tilde{\mathcal{V}}$, which all map from $*$ to itself, and the morphisms of $\mathcal{V}$ correspond to the 2-cells of $\tilde{\mathcal{V}}$. Then the data and axioms of the bicategory correspond with monoidal structure. For example, the functor I_* picks out the 1-cell of $\tilde{\mathcal{V}}$ corresponding to the unit of $\mathcal{V}$. Also, if $\mathcal{V}$ is a strict monoidal category, the data and axioms correspond in the same way to those of a 2-category with exactly one object. Further, we notice that $\tilde{\mathcal{V}}(*, *)$ is isomorphic to the monoidal category $\mathcal{V}$, and so we often think of the two interchangeably.

Now, we need ways of mapping between these 2-dimensional categorical structures analogous to functors mapping between categories.

Definition 2.3. A *morphism* F from $\mathcal{B}$ to $\mathcal{B}'$, bicategories, consists of:

- A function $F : \operatorname{ob}\mathcal{B} \to \operatorname{ob}\mathcal{B}'$
- For each pair $A, B \in \operatorname{ob}\mathcal{B}$, a functor $F_{AB} : \mathcal{B}(A, B) \to \mathcal{B}'(FA, FB)$
- For any three objects $A, B, C \in \operatorname{ob}\mathcal{B}$, natural transformations:



Given any $g \in \mathcal{B}(B, C)$ and $f \in \mathcal{B}(A, B)$, we refer to the component of ϕ_{ABC} at $g \times f$ as $\phi_{gf} : Fg \circ Ff \to F(g \circ f)$. The natural transformation ϕ_A consists of a single 2-cell in $\mathcal{B}'$ of the same name, $\phi_A : 1_{FA} \to F(1_A)$.

Any 1-cells f, g, h with appropriate domains and codomains satisfy an axioms given by the following commutative diagrams (note dashes have been added to indicate morphisms are in $\mathcal{B}'$):

$$\begin{array}{ccccc}
(Fh \circ Fg) \circ Ff & \xrightarrow{\phi\star 1} & F(h \circ g) \circ Ff & \xrightarrow{\phi} & F((h \circ g) \circ f) \\
\downarrow{\scriptstyle a'} & & & & \downarrow{\scriptstyle Fa} \\
Fh \circ (Fg \circ Ff) & \xrightarrow[1\star\phi]{} & Fh \circ F(g \circ f) & \xrightarrow[\phi]{} & F(h \circ (g \circ f))
\end{array}$$

$$\begin{array}{ccccc}
Ff \circ I'_{FA} & \xrightarrow{1\star\phi} & Ff \circ FI_A & \xrightarrow{\phi} & F(f \circ I_A) \\
\downarrow{\scriptstyle r'} & & & \swarrow{\scriptstyle Fr} & \\
Ff & & & &
\end{array}
\qquad
\begin{array}{ccccc}
I'_{FB} \circ Ff & \xrightarrow{\phi\star 1} & FI_B \circ Ff & \xrightarrow{\phi} & F(I_B \circ f) \\
\downarrow{\scriptstyle l'} & & & \swarrow{\scriptstyle Fl} & \\
Ff & & & &
\end{array}$$

In the case where the morphism F preserves associativity and unitality up to isomorphism, that is, for any $A, B, C \in \operatorname{ob}\mathcal{B}$, ϕ_A and ϕ_{ABC} are natural isomorphisms, we call F a homomorphism. When all the ϕ_A, ϕ_{ABC} are equalities, F is a strict homomorphism. If $\mathcal{B}$ and $\mathcal{B}'$ are 2-categories, a strict homomorphism is called a 2-functor.

Furthermore, if $\mathcal{B}$ and $\mathcal{B}'$ are 1-object bicategories, and F is a (strict) homomorphism, F is a (strict) tensor functor F_{**} with domain and codomain monoidal categories $\mathcal{B}(*, *)$ and $\mathcal{B}'(*, *)$, respectively, along with a function sending the object of $\mathcal{B}$ to the object of $\mathcal{B}'$.

Next, we define the arrows between morphisms, that is, the structures analogous to natural transformations in **CAT**.

Definition 2.4. Given morphisms between bicategories, $F, G : \mathcal{B} \to \mathcal{B}'$, a *transformation* σ from F to G consists of

- For each $A \in \operatorname{ob}\mathcal{B}$, component 1-cells $FA \xrightarrow{\sigma_A} GA$
- Natural transformations

$$\begin{array}{ccc}
\mathcal{B}(A, B) & \xrightarrow{F_{AB}} & \mathcal{B}'(FA, FB) \\
\downarrow{\scriptstyle G_{AB}} & \nearrow\!\!\!\nearrow{\scriptstyle \sigma_{AB}} & \downarrow{\scriptstyle (\sigma_B)_*} \\
\mathcal{B}'(GA, GB) & \xrightarrow[(\sigma_A)^*]{} & \mathcal{B}'(FA, GB)
\end{array}$$

  where $(\sigma_A)^*$ and $(\sigma_B)_*$ respectively indicate the functors induced by precomposition with σ_A and postcomposition with σ_B. Given any $f \in \mathcal{B}(A, B)$, σ_f gives the individual 2-cell giving the component of σ_{AB} from $\sigma_A \circ Gf$ to $Ff \circ \sigma_B$.

Given 1-cells f and g with appropriate domains and codomains, transformations obey axioms indicated by the following commutative diagrams:

$$\begin{array}{ccccccccccc}
(Gg \circ Gf) \circ \sigma_A & \xrightarrow{a'} & Gg \circ (Gf \circ \sigma_A) & \xrightarrow{1\star\sigma_f} & Gg \circ (\sigma_B \circ Ff) & \xrightarrow{a'^{-1}} & (Gf \circ \sigma_B) \circ Ff & \xrightarrow{\sigma_g\star 1} & (\sigma_C \circ Fg) \circ Ff & \xrightarrow{a'} & \sigma_C \circ (Fg \circ Ff) \\
\downarrow{\scriptstyle \psi\star 1} & & & & & & & & & & \downarrow{\scriptstyle 1\star\phi} \\
G(g \circ f) \circ \sigma_A & & & & & \xrightarrow[\sigma_{gf}]{} & & & & & \sigma_C \circ F(g \circ f)
\end{array}$$

$$\begin{array}{ccccc}
I'_{GA} \circ \sigma_A & \xrightarrow{l'} & \sigma_A & \xrightarrow{r'^{-1}} & \sigma_A \circ I'_{FA} \\
\downarrow{\scriptstyle \psi\star 1} & & & & \downarrow{\scriptstyle 1\star\phi} \\
GI_A \circ \sigma_A & & \xrightarrow[\sigma_{I_A}]{} & & \sigma_A \circ FI_A
\end{array}$$

In the case where the σ_{AB}'s are natural isomorphisms, σ is a strong transformation. Furthermore, in the case where $\mathcal{B}$ and $\mathcal{B}'$ are 2-categories and F, G are 2-functors, if each σ_{AB} is an identity we have precisely the naturality condition needed for σ to be a 2-natural transformation. When $\mathcal{B}$ and $\mathcal{B}'$ are 1-object bicategories, a (strong) transformation σ is a tensor (iso)morphism $\sigma_{**} : G_{**} \to F_{**}$ along with an identity 1-cell on the object of $\mathcal{B}'$.

Definition 2.5. Given morphisms between bicategories, $F, G : \mathcal{B} \to \mathcal{B}'$ and transformations σ, $\sigma' : F \Rightarrow G$, a *modification* $\Gamma : \sigma \to \sigma'$ is given by component 2-cells $\Gamma_A : \sigma_A \to \sigma'_A : FA \to GA$ for each $A \in \operatorname{ob}\mathcal{B}$ such that the following diagram commutes:

$$\begin{array}{ccc} Gf \circ \sigma_A & \xrightarrow{1\star\Gamma_A} & Gf \circ \sigma'_A \\ {\scriptstyle \sigma_f}\downarrow & & \downarrow{\scriptstyle \sigma'_f} \\ \sigma_B \circ Ff & \xrightarrow[\Gamma_B\star 1]{} & \sigma'_B \circ Ff \end{array} \tag{1}$$

Next, we look at some examples of structures which we will use in the next section that involve combinations of these definitions.

Example 2.6. Given bicategories $\mathcal{B}$ and $\mathcal{B}'$, we may form a bicategory $[\mathcal{B}, \mathcal{B}']$ consisting of 0-cells homomorphisms from $\mathcal{B}$ to $\mathcal{B}'$, 1-cells strong transformations and 2-cells modifications. If $[\mathcal{B}, \mathcal{B}']$ were a 2-category, given any homomorphism $F : \mathcal{B} \to \mathcal{B}'$, the identity strong transformation $1_F : F \to F$ would have components strict identities $(1_F)_A : FA \to FA$. However, the objects FA are in $\mathcal{B}'$, a bicategory, and the objects of bicategories do not necessarily have strict identity morphisms, so $[\mathcal{B}, \mathcal{B}']$ is not necessarily a 2-category. However, when $\mathcal{B}'$ is a 2-category, we do have the required strict identity morphisms and $[\mathcal{B}, \mathcal{B}']$ is indeed a 2-category. For example, for any bicategory $\mathcal{B}$, $[\mathcal{B}, \mathbf{Cat}]$ and $[\mathcal{B}^{op}, \mathbf{Cat}]$ are 2-categories.

Example 2.7. For any bicategory $\mathcal{B}$, there is a Yoneda embedding $Y : \mathcal{B} \to [\mathcal{B}^{op}, \mathbf{Cat}]$ which we will show is a homomorphism. The function part of Y maps each object $B \in \mathcal{B}$ to $\mathcal{B}(-, B)$, a homomorphism from $\mathcal{B}^{op}$ to $\mathbf{Cat}$. The homomorphism $\mathcal{B}(-, B)$ takes each $A \in \mathcal{B}$ to $\mathcal{B}(A, B)$, which is a category by the definition of a bicategory.

The remaining data necessary to define the Yoneda embedding is given by functors

$$Y_{AB} : \mathcal{B}(A, B) \longrightarrow [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(-, A), \mathcal{B}(-, B))$$

$$\begin{array}{ccc} A & & \mathcal{B}(-, A) \\ {\scriptstyle f}\downarrow & \longmapsto & \Downarrow{\scriptstyle f\circ(-)} \\ B & & \mathcal{B}(-, B) \end{array}$$

and natural transformations

$$\begin{array}{ccc} \mathcal{B}(B, C) \times \mathcal{B}(A, B) & \xrightarrow{c_{ABC}} & \mathcal{B}(A, C) \\ {\scriptstyle Y_{BC}\times Y_{AC}}\downarrow & \overset{\phi_{ABC}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle Y_{AC}} \\ [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(-, B), \mathcal{B}(-, C)) \times [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(-, A), \mathcal{B}(-, B)) & \xrightarrow{c'_{ABC}} & [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(-, A), \mathcal{B}(-, C)) \end{array}$$

Given morphisms $f \in \mathcal{B}(A, B)$ and $g \in \mathcal{B}(B, C)$, we follow the above diagram in both directions, arriving at transformations $\mathcal{B}(-, A) \Rightarrow \mathcal{B}(-, C)$:

$$
\begin{array}{ccc}
g \times f & & \\
\downarrow & & \\
(g \circ (-)) \times (f \circ (-)) & \longmapsto & g \circ (f \circ (-))
\end{array}
\qquad\qquad
\begin{array}{ccc}
g \times f & \longmapsto & g \circ f \\
 & & \downarrow \\
 & & (g \circ f) \circ (-)
\end{array}
$$

Going around the left and bottom of the diagram, the transformation we arrive at has components given by postcomposition with f followed by postcompostion with g, and in the other direction, the transformation has components given by postcomposition with $g \circ f$. So, ϕ_{gf} is a modification from this first composition to the second. Since these transformations are between functors mapping into **Cat**, their components at an object $D \in \mathcal{B}$ are functors $\mathcal{B}(D, A) \to \mathcal{B}(D, C)$. The component of the modification ϕ_{gf} at D, $(\phi_{gf})_D$, is then a natural transformation between these two functors. Given a morphism $h \in \mathcal{B}(D, A)$, the component of $(\phi_{gf})_D$ at h, which maps from $g \circ (f \circ h)$ to $(g \circ f) \circ f$, is given by the component at $g \times f \times h$ of a_{DABC} from the defintion of $\mathcal{B}$ as a bicategory.

Further, by the definition of a bicategory, a_{DABC} is a natural isomorphism, hence all its components are isomorphisms. So, since the components of $(\phi_{gf})_D$ are isomorphisms, $(\phi_{gf})_D$ is a natural isomorphism. Working our way back up, then, ϕ_{ABC} is an isomorphism because 1-cells in $\mathcal{B}$ associate up to isomorphism. Moreover, the Yoneda embedding is in fact a homomorphism because $[\mathcal{B}, \mathbf{Cat}]$ inherits its composition structure from $\mathcal{B}$.

As we will see, the main theorems of Sections 3 and 4 tell us something about diagrams commuting in categorical structures by demonstrating the existence of sufficiently strong comparison to stronger structures. In Section 3, the Yoneda embedding will demonstrate the existence of such a comparison. Next, we define these means of comparison used in the next two sections.

Definition 2.8. Two objects A and B in a bicategory $\mathcal{B}$ are *internally equivalent* if there exist 1-cells $f : A \longrightarrow B$, $g : B \longrightarrow A$ and isomorphisms $1 \longrightarrow g \circ f$ in $\mathcal{B}(A, A)$ and $f \circ g \longrightarrow 1$ in $\mathcal{B}(B, B)$.

For example, an equivalence of categories is an internal equivalence in **CAT**. An internal equivalence in $[\mathcal{B}, \mathbf{CAT}]$ between homomorphisms $F, G : \mathcal{B} \longrightarrow \mathbf{CAT}$ consists of strong transformations $\sigma : F \Rightarrow G$, $\tau : G \Rightarrow F$ such that we have isomorphisms $1 \to \tau \circ \sigma$ in $[\mathcal{B}, \mathbf{CAT}](F, F)$ and $\sigma \circ \tau \to 1$ in $[\mathcal{B}, \mathbf{CAT}](G, G)$.

Definition 2.9. Given a monoidal categories $\mathcal{V}$, $\mathcal{W}$ and (strict) tensor functors $F : \mathcal{V} \to \mathcal{W}$, $G : \mathcal{W} \to \mathcal{V}$, they form a *(strict) tensor equivalence* if there are tensor isomorphisms $\theta : 1_{\mathcal{V}} \to G \circ F$, $\phi : F \circ G \to 1_{\mathcal{W}}$.

Definition 2.10. Bicategories $\mathcal{B}$ and $\mathcal{B}'$ are *biequivalent* if we have a pair of homomorphisms $F : \mathcal{B} \to \mathcal{B}'$, $G : \mathcal{B}' \to \mathcal{B}$ with an equivalence $1 \to G \circ F$ inside the bicategory $[\mathcal{B}, \mathcal{B}]$ and an equivalence $F \circ G \to 1$ inside $[\mathcal{B}', \mathcal{B}']$.

Furthermore, just as an equivalence of categories can also be given as a functor which is fully faithful and essentially surjective, a biequivalence between bicategories $\mathcal{B}$ and $\mathcal{B}'$ can be given by a homomorphism $F : \mathcal{B} \to \mathcal{B}'$ where F is surjective up to equivalence on objects, and F_{AB} is an equivalence for any $A, B \in \mathcal{B}$.

3 Coherence Theorem for Bicategories

We begin this section by proving the Yoneda Lemma for Bicategories using a statement similar to that of [6].

Proposition 3.1. Given a homorphism $T : \mathcal{B} \to \mathbf{Cat}$ of bicategories we have the following equivalence in $[\mathcal{B}, \mathbf{CAT}]$:

$$[\mathcal{B}, \mathbf{Cat}](\mathcal{B}(?, -), T) \to T$$

$$\text{or dually, } [\mathcal{B}^{op}, \mathbf{Cat}](\mathcal{B}(-, ?), T) \to T$$

Proof. The homomorphism $[\mathcal{B}, \mathbf{Cat}](\mathcal{B}(?, -), T)$ is defined to send an object $X \in \mathcal{B}$ to $[\mathcal{B}, \mathbf{Cat}](\mathcal{B}(X, -), T)$ the category of transformations from $\mathcal{B}(X, -)$ to T, which we know to be a category by the definition of $[\mathcal{B}, \mathbf{Cat}]$ as a bicategory.

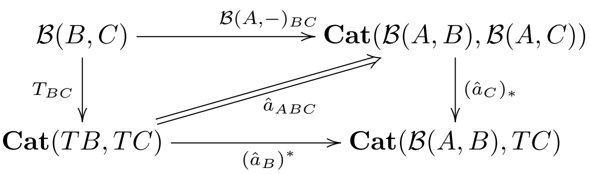
We demonstrate this equivalence by constructing strong transformations σ and τ where each component gives an equivalence of categories. We begin by constructing these transformations.

For any $A \in \mathcal{B}$, let

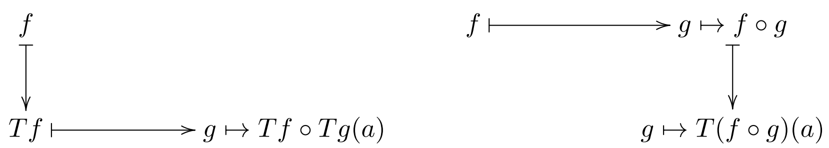
$$\begin{array}{rcl} \sigma_A : [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(A, -), T) & \longrightarrow & TA \\ \alpha & \longmapsto & \hat{\alpha} = \alpha_A(I_A) \\ \alpha \xrightarrow{\Gamma} \beta & \longmapsto & \alpha_A(I_A) \xrightarrow{(\Gamma_A)_{I_A}} \beta_A(I_A) \end{array}$$

$$\begin{array}{rcl} \tau_A : TA & \longrightarrow & [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(A, -), T) \\ a & \longmapsto & \hat{a} : \mathcal{B}(A, -) \Longrightarrow T \\ & & (\hat{a})_B : \mathcal{B}(A, B) \longrightarrow B \\ & & f \longmapsto Tf(a) \\ a \xrightarrow{h} b & \longmapsto & \hat{a} \xrightarrow{\hat{h}} \hat{b} \quad \hat{h}_B : \hat{a}_B \Longrightarrow \hat{b}_B, \quad (\hat{h}_B)_f = Tf(h) \end{array}$$

First, we show these maps are well-defined by demonstrating that $\hat{a}$ is a strong transformation from $\mathcal{B}(A, -)$ to T. So, we want to find natural isomorphisms:



Following the above diagram in both directions for an arbitrary $f \in \mathcal{B}(B, C)$, we get:



That is, we arrive at two functors from $\mathcal{B}(A, B)$ to TC, whose respective actions on an arbitrary $g \in \mathcal{B}(A, B)$ are given above. The component of $\hat{a}_{ABC}$ at f is a natural transformation between those functors. Let the component of this natural transformation at g be given by the component of ϕ_{fg} at a, where ϕ_{fg} is the component at $g \times f$ of ϕ_{ABC}, the natural transformation from the definition of T a homomorphism. Since T is a homomorphism, ϕ_{ABC} is a natural isomorphism, hence $\hat{a}_{ABC}$ is an isomorphism, as desired.

Next, to show that, for any $A \in \mathcal{B}$, σ_A and τ_A given an equivalence, we need to exhibit natural isomorphisms:

$$\eta : 1_{[\mathcal{B}, \mathbf{Cat}](\mathcal{B}(A, -), T)} \longrightarrow \tau_A \circ \sigma_A \quad \text{and} \quad \epsilon : \sigma_A \circ \tau_A \longrightarrow 1_{TA}$$

First we show ϵ:

Observe that $\sigma_A \circ \tau_A$ is a functor from TA to itself sending $a \in TA$ to $TI_A(a)$

By the definition of T a homomorphism, there is a natural isomorphism $\phi_A^{-1} : TI_A \to I_{TA}$. Since I_{TA} is the unit on $TA \in \mathbf{Cat}$, a 2-category, it is simply the identity functor. So, we have an isomorphism $\phi_A^{-1} : \sigma_A \circ \tau_A \to 1_{TA}$, as required.

Next, η:

Observe

$$\begin{aligned} \tau_A \circ \sigma_A : [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(A,-),T) &\longrightarrow [\mathcal{B}, \mathbf{Cat}](\mathcal{B}(A,-),T) \\ \alpha : \mathcal{B}(A,-) \Rightarrow T \quad &\longmapsto \quad \hat{\hat{\alpha}} : \mathcal{B}(A,-) \Rightarrow T \\ \alpha_B : \mathcal{B}(A,B) \to TB \quad & \quad\quad \hat{\hat{\alpha}}_B : \mathcal{B}(A,B) \to TB \\ f \mapsto \alpha_B(f) \quad & \quad\quad f \mapsto Tf(\alpha_A(I_A)) \end{aligned}$$

Since α is a strong transformation, we have a natural isomorphism α_{AB}:

$$\begin{array}{ccc} \mathcal{B}(A,B) & \xrightarrow{\mathcal{B}(a,-)_{AB}} & \mathbf{Cat}(\mathcal{B}(A,A),\mathcal{B}(A,B)) \\ {\scriptstyle T_{AB}}\downarrow & \overset{\alpha_{AB}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle (\alpha_B)_*} \\ \mathbf{Cat}(TA,TB) & \xrightarrow[(\alpha_A)^*]{} & \mathbf{Cat}(\mathcal{B}(A,A),TB) \end{array}$$

Going both way around the diagram, we have:

$$\begin{array}{ccc} f & \longmapsto & g \mapsto f \circ g \\ & & \downarrow \\ & & g \mapsto \alpha_B(f \circ g) \end{array} \qquad \begin{array}{ccc} f & & \\ \downarrow & & \\ Tf & \longmapsto & g \mapsto Tf \circ \alpha_A(g) \end{array}$$

In particular, for $g = I_A$, we see we have an isomorphism between $\alpha_B(f \circ I_A)$ and $I_A \mapsto Tf \circ \alpha_A(I_A)$. Also, $f \circ I_A$ is isomrphic to f by the definition of a bicategory. Combining these, we have an isomorphism $(\alpha_f)_{I_A} \circ \alpha_B(r_f^{-1}) : \alpha_B(f) \to Tf \circ \alpha_A(I_A)$, as desired.

Thus we have constructed the equivalence. □

Theorem 3.2. Every bicategory is biequivalent to a 2-category.

Proof. Given a bicategory $\mathcal{B}$, we claim that $\mathcal{B}$ is biequivalent to the full image of the Yoneda embedding Y, that is, the subcategory S of $[\mathcal{B}^{op}, \mathbf{Cat}]$ with objects $\{YB | B \in \text{ob}\,\mathcal{B}\}$ and for any $C, D \in \text{ob}\,\mathcal{B}$, morphisms $S(YC, YD) = [\mathcal{B}^{op}, \mathbf{Cat}](YC, YD)$. To demonstrate the desired biequivalence, we show that Y', the Yoneda embedding with codomain restricted to S, satisfies the alternative criteria for biequivalence given at the end of Definition 2.10.

First, Y' is surjective up to equivalence on objects since, by construction, it satisfies the stronger condition of being surjective on objects. Then, for any $A, B \in \mathcal{B}$, $Y'_{AB} = Y_{AB}$ is the component at A of the equivalence given by the Yoneda Lemma for Bicategories applied to the case where $T = \mathcal{B}(-, B)$. And as we saw in the proof of Proposition 3.1, the components of the equivalence are equivalences of categories. □

Corollary 3.3. Every monoidal category is tensor equivalent to a strict monoidal category.

Proof. Let $\mathcal{V}$ be a monoidal category. We view it as a 1-object bicategory $\tilde{\mathcal{V}}$ and apply Theorem 3.2. In this case, the Yoneda embedding Y consists of a function sending the one object of $*$ of $\tilde{\mathcal{V}}$ to a single object in $[\tilde{\mathcal{V}}^{op}, \mathbf{Cat}]$ and a tensor functor $Y_{**} : \tilde{\mathcal{V}}(*,*) \to [\tilde{\mathcal{V}}, \mathbf{Cat}](\tilde{V}(-,*), \tilde{\mathcal{V}}(-,*))$. The homomorphism $\tilde{\mathcal{V}}(-,*) : \tilde{\mathcal{V}} \to \mathbf{Cat}$ sends the single object of $\tilde{\mathcal{V}}$ to $\tilde{\mathcal{V}}(*,*)$, which is the category $\mathcal{V}$, and sends each $f \in \mathcal{V}(*,*)$ to a functor $f \otimes (-) : \mathcal{V} \to \mathcal{V}$ which sends objects of $\mathcal{V}$ to their tensor product with f on the left.

We observe S, the full image of $\tilde{\mathcal{V}}$ in $[\tilde{V}, \mathbf{Cat}]$, is a one object 2-category, which can be viewed as a strict monoidal category. By definition, the biequivalence given by Y' consists of another homomorphism as well, call it $Z' : S \to \tilde{\mathcal{V}}$, such that we have an equivalence $1 \to Z' \circ Y'$ inside $[\tilde{\mathcal{V}}, \tilde{\mathcal{V}}]$ and an equivalence $Y' \circ Z' \to 1$ in $[S, S]$.

The equivalence inside $[\tilde{\mathcal{V}}, \tilde{\mathcal{V}}]$ consists of isomorphisms $1 \to Z' \circ Y' \circ 1 = Z' \circ Y'$, $Z' \circ Y' \circ 1 = Z' \circ Y' \to 1$, which are each given by an identity 1-cell on $*$, the object of $\tilde{\mathcal{V}}$, and tensor isomorphisms $1 \Rightarrow Z'_{**} \circ Y'_{**}$, $Z'_{**} \circ Y'_{**} \Rightarrow 1$. The equivalence in $[S, S]$ is analogous. So, along with some additional data, the biequivalence gives us a tensor equivalence between Y'_{**} and Z'_{**}. □

Next, we examine the construction used to prove Corollary 3.3 more closely to see why the tensor equivalence is not necessarily a strict tensor equivalence.

As discussed in the above proof, given a monoidal category $\mathcal{V}$, the codomain of the tensor functor Y_{**} is the strict monoidal category $[\tilde{\mathcal{V}}, \mathbf{Cat}](\tilde{V}(-,*), \tilde{\mathcal{V}}(-,*))$, which has objects strong transformations from $\mathcal{V}(-,*)$ to itself. These strong transformations by definition each consist of a functor $F : \mathcal{V} \to \mathcal{V}$ and a natural isomorphism relating $F_\star \circ Y_{**}$ and $F^\star \circ Y_{**}$ where $F_\star$ and $F^\star$ are the functors induced by postcomposition and precomposition with F, respectively. Incidentally, in [2], Joyal and Street prove Corollary 3.3 by constructing precisely this strict monoidal category and the tensor functor given by Y_{**} and then demonstrating Y_{**} is fully faithful.

Given any object $V \in \mathcal{V}$, Y_{**} maps it to the functor $V \otimes -$, which maps from $\mathcal{V}$ to $\mathcal{V}$ by tensoring objects of $\mathcal{V}$ on the left with V, accompanied by a natural isomorphism which has as its component at X a natural isomorphism $(X \otimes -) \otimes V \cong V \otimes (X \otimes -)$. For Y' to be a strict tensor functor, we would need the following diagram to commute:

$$\begin{array}{ccc} \mathcal{V}(*,*) \times \mathcal{V}(*,*) & \xrightarrow{\quad c \quad} & \mathcal{V}(*, ast) \\ \downarrow{\scriptstyle Y_{**} \times Y_{**}} & & \downarrow{\scriptstyle Y_{**}} \\ [\mathcal{V}, \mathbf{CAT}](\mathcal{V}(-,*), \mathcal{V}(-,*)) \times [\mathcal{V}, \mathbf{CAT}](\mathcal{V}(-,*), \mathcal{V}(-,*)) & \xrightarrow{\quad c' \quad} & [\mathcal{V}, \mathbf{CAT}](\mathcal{V}(-,*), \mathcal{V}(-,*)) \end{array}$$

If we start with (V, W) in the upper left, following it around the left and bottom of the diagram yields a strong transformation with components given by $V \otimes (W \otimes -)$. Following (V, W) around the top and right of the diagram, we instead get a strong transformation with components given by $((V \otimes W) \otimes -)$. So, the above diagram only commutes up to isomorphism since the tensor product in $\mathcal{V}$ is only necessarily associative up to isomorphism. Thus Y' does not give a strict tensor functor in Corollary 3.3.

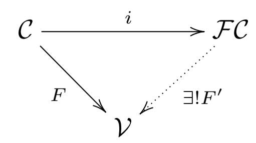
The main theorem of the next section shows a strict tensor equivalence.

4 Free Monoidal Categories

The main theorem of this section demonstrates a strict tensor equivalence between the free monoidal category and a free strict monoidal category generated by the same category.

Definition 4.1. Given a category $\mathcal{C}$, the *free monoidal category* $\mathcal{FC}$ has objects built inductively starting with the objects of $\mathcal{C}$, an additional object I, the unit; we then include an object $M \otimes N$ for every $M, N \in \operatorname{ob} \mathcal{FC}$. Then, the arrows of $\mathcal{FC}$ are equivalence classes of arrows built up from arrows in $\mathcal{C}$ and the morphisms a, l and r from the definition of a monoidal category by means of tensoring, substituting, inverting and composing. The equivalence relation is given by the axioms of a monoidal category.

Consider the functor $i : \mathcal{C} \to \mathcal{FC}$ that includes objects of $\mathcal{C}$ into $\mathcal{FC}$ and sends arrows to the equivalence classes that respectively contain them. Given any functor from $\mathcal{C}$ to a monoidal category $F : \mathcal{C} \to \mathcal{V}$, there exists a unique strict tensor functor $F' : \mathcal{FC} \to \mathcal{V}$ so that the following diagram commutes:

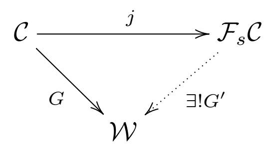


(2)

The uniqueness is clear by the construction of $\mathcal{FC}$ and the strictness of F'. This result is what we consider to be the universal property of $\mathcal{FC}$.

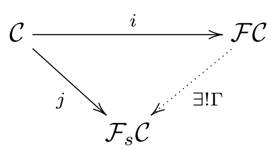
Definition 4.2. Given a category $\mathcal{C}$, the *free strict monoidal category* $\mathcal{F}_s\mathcal{C}$ has objects words $A_1 A_2 \dots A_n$ where the A_i are objects of $\mathcal{C}$ and arrows words $f_1 f_2 \dots f_n$ with domain and codomain $\operatorname{dom}(f_1) \operatorname{dom}(f_2) \dots \operatorname{dom}(f_n)$ and $\operatorname{cod}(f_1) \operatorname{cod}(f_2) \dots \operatorname{cod}(f_n)$. The tensor product is concatenation, which suggests notationally the associativity present in strict monoidal categories. The unit of $\mathcal{F}_s\mathcal{C}$ is the concatenation of an empty selection of objects and, as such, acts identically when concatenated with other objects.

Denote the inclusion of $\mathcal{C}$ into $\mathcal{F}_s\mathcal{C}$ by $j : \mathcal{C} \to \mathcal{F}_s\mathcal{C}$. We have a universal property of $\mathcal{F}_s\mathcal{C}$ similar to that of $\mathcal{FC}$: given any strict monoidal category $\mathcal{W}$ and a functor $G : \mathcal{C} \to \mathcal{W}$, there exists a unique strict tensor functor $G : \mathcal{F}_s\mathcal{C} \to \mathcal{W}$ so that the following diagram commutes:



(3)

Now, use the universal property of $\mathcal{FC}$ as follows to produce the strict tensor functor $\Gamma : \mathcal{FC} \to \mathcal{F}_s\mathcal{C}$:



(4)

Any object $A \in \mathcal{FC}$ is of the form $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ with parentheses added in to denote in what order the tensoring occurs, where $A_i \in \mathcal{C} \cup \{I\}$ where I is the unit of $\mathcal{FC}$. By the definition of Γ, we arrive at ΓA by simply exchanging the tensors of A for concatenation to get $A_1 A_2 \cdots A_n$ and then removing all the A_i's that are equal to I.

Theorem 4.3. For any category $\mathcal{C}$, $\Gamma : \mathcal{F}\mathcal{C} \to \mathcal{F}_s\mathcal{C}$ is a strict tensor equivalence.

This equivalence can be demonstrated by displaying a functor from $\mathcal{F}_s\mathcal{C}$ to $\mathcal{F}\mathcal{C}$ and then showing that its composition with Γ in either order is naturally isomorphic (as a tensor functor) to the appropriate identity functor. In [2], Joyal and Street take the alternate route of showing Γ is fully faithful and essentially surjective on objects. We will show this proof method first, beginning by constructing the remaining machinery to get through their argument that Γ is faithful.

Lemma 4.4. Given categories $\mathcal{A}$, $\mathcal{X}$ and functors $S, T : \mathcal{A} \to \mathcal{X}$, there exists a category $Eq(S,T)$, a functor $P : Eq(S,T) \to \mathcal{A}$, and a natural isomorphism $\sigma : SP \Rightarrow TP$ such that for any functor $F : \mathcal{C} \to \mathcal{A}$ and natural isomorphism $\tau : SF \Rightarrow TF$, there exists a unique functor $K : \mathcal{C} \to Eq(S,T)$ such that $PK = F$ and $\sigma K = \tau$. That is, there is a unique functor that factors both F through P and σ through τ. Furthermore, if $\mathcal{A}$, $\mathcal{X}$ are monoidal and S, T are tensor functors, there exists a unique monoidal structure on $Eq(S,T)$ such that P is a strict tensor functor and σ is a tensor isomorphism.

Proof. Let $\mathcal{A}$, $\mathcal{X}$ be categories, and S, T be functors from $\mathcal{A}$ to $\mathcal{X}$.

We construct the category $Eq(S,T)$ with objects pairs (A,h) where $A \in \operatorname{ob}\mathcal{A}$ and $h : SA \to TA$ is an isomorphism. The morphisms of $Eq(S,T)$, $f : (A,h) \to (A',h')$, are arrows $f : A \to A'$ in $\mathcal{A}$ such that $\begin{array}{ccc} SA & \xrightarrow{h} & TA \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SA' & \xrightarrow{h'} & TA' \end{array}$ commutes.

We construct the functor

$$\begin{array}{rcl} P : Eq(S,T) & \longrightarrow & \mathcal{A} \\ (A,h) & \longmapsto & A \\ (A,h) \xrightarrow{f} (A',h') & \longmapsto & A \xrightarrow{f} A' \end{array}$$

The diagram $Eq(S,T) \xrightarrow{P} \mathcal{A} \overset{S}{\underset{T}{\rightrightarrows}} \mathcal{X}$ commutes up to natural isomorphism $\sigma : SP \Rightarrow TP$ with components $\sigma_{(A,h)} = h$. The naturality of σ comes from the construction of morphisms in $Eq(S,T)$.

Next, suppose we are given a category $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{A}$, and a natural isomorhism $\tau : SF \Rightarrow TF$. We will next show there exists a unique functor $K : \mathcal{C} \to Eq(S,T)$ such that $\sigma K = \tau$ and

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle K}\downarrow & \searrow^{F} & \\ Eq(S,T) & \xrightarrow[P]{} & \mathcal{A} \end{array} \quad \text{commutes.} \tag{5}$$

Construct K as follows:

$$\begin{array}{rcl} K : \mathcal{C} & \longrightarrow & Eq(S,T) \\ C & \longmapsto & (FC, \tau_C) \\ C \xrightarrow{f} C' & \longmapsto & (FC,\tau_C) \xrightarrow{Ff} (FC', \tau_{C'}) \end{array}$$

Note K is well defined as a functor into $Eq(S,T)$, that is $\begin{array}{ccc} SFC & \xrightarrow{\tau_C} & TFC \\ {\scriptstyle SFf}\downarrow & & \downarrow{\scriptstyle TFf} \\ SFC' & \xrightarrow[\tau_{C'}]{} & TFC' \end{array}$ commutes by naturality of τ.

By construction, then, (5) commutes.

Observe that $\tau = \sigma K : SPK \to TPK$, that is, for any $C \in \mathcal{C}$, $\sigma_{KC} = \sigma_{(FC,\tau_C)}$.

Now, to demonstrate the uniqueness of K, suppose we have a functor $K' : \mathcal{C} \to Eq(S,T)$ such that

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle K'}\downarrow & \searrow{\scriptstyle F} & \\ Eq(S,T) & \xrightarrow{P} & \mathcal{A} \end{array}$$

commutes.

Since this diagram commutes, for any $C \in \mathcal{C}$, $K'C = (FC, \eta_C)$ where η_C is some isomorphism $SFC \xrightarrow{\cong} TFC$.

Further, $PK'f = Ff$ for any $f : C \to C'$ in $\mathcal{C}$, so $K'F = Ff : (FC, \eta_C) \to (FC', \eta_{C'})$ such that

$$\begin{array}{ccc} SFC & \xrightarrow{\eta_C} & TFC \\ {\scriptstyle SFf}\downarrow & & \downarrow{\scriptstyle TFf} \\ SFC' & \xrightarrow{\eta_{C'}} & TFC' \end{array}$$

commutes by the construction of $Eq(S,T)$.

Thus the η_C's are the components of a natural isomorphism $\eta : SF \to TF$ and $\sigma K' = \eta$.

Thus the map $K' : \mathcal{C} \to Eq(S,T)$ is uniquely determined by the choice of natural isomorphism $SF \Rightarrow TF$.

Now, suppose $\mathcal{A}$, $\mathcal{X}$ are monoidal categories and S, T are tensor functors.

First, suppose we have a monoidal structure on $Eq(S,T)$ such that P is a strict tensor functor and σ is an isomorphism of tensor functors. Since P is a strict tensor functor, we know $(A,h) \otimes (A',h') = (A \otimes A', h'')$ for some $h'' : S(A \otimes A') \xrightarrow{\cong} T(A \otimes A')$.

Since $\sigma : SP \Rightarrow TP$ is an isomorphism of tensor functors,

$$\begin{array}{ccc} SP(A,h) \otimes SP(A',h') & \xrightarrow{\phi_s} & SP((A,h) \otimes (A',h')) \\ {\scriptstyle \sigma_{(A,h)} \otimes \sigma_{(A',h')}}\downarrow & & \downarrow{\scriptstyle \sigma_{(A,h)\otimes(A'h')}} \\ TP(A,h) \otimes TP(A',h') & \xrightarrow[\phi_T]{} & TP((A,h) \otimes (A',h')) \end{array}$$

commutes, where ϕ_S and ϕ_T are the natural isomorphisms from the definitions of S and T respectively as tensor functors. The above diagram simplifies to

$$\begin{array}{ccc} SA \otimes SA' & \xrightarrow{\phi_S} & S(A \otimes A') \\ {\scriptstyle h \otimes h'}\downarrow & & \downarrow{\scriptstyle h''} \\ TA \otimes TA' & \xrightarrow[\phi_T]{} & T(A \otimes A') \end{array}$$

So $h'' = \phi^{-1} \circ (h \otimes h') \circ \phi_T$

Let I be the object of $\mathcal{A}$ which is part of the unit object of $Eq(S,T)$. We see I must be the unit object of $\mathcal{A}$ since P is a strict tensor functor. Since S and T are tensor functors, we have isomorphisms $I_\mathcal{X} \xrightarrow{\phi_{S,0}} SI$, $I_\mathcal{X} \xrightarrow{\phi_{T,0}} TI$ where $I_\mathcal{X}$ is the unit of $\mathcal{X}$.

Since $\sigma : SP \Rightarrow TP$ is an isomorphism of tensor functors,

$$\begin{array}{ccc} I & \xrightarrow{\phi_{S,0}} & SI \\ {\scriptstyle \phi_{T,0}}\downarrow & \swarrow{\scriptstyle \sigma_I} & \\ TI & & \end{array}$$

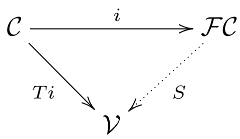
commutes, so the unit object of $Eq(S,T)$ is $(I, \phi_{S,0}^{-1} \circ \phi_{T,0})$.

So, we have shown the uniqueness of the monoidal structure in $Eq(S,T)$ since it has been completely determined by the assumption that P is a tensor functor and σ is an isomorphism of tensor functors. For existence, we can simply define the monoidal structure on $Eq(S,T)$ to be given by tensor product $(A,h) \otimes (A',h') = (A \otimes A', \phi^{-1} \circ (h \otimes h') \circ \phi_T)$ and unit $(I, \phi_{S,0}^{-1} \circ \phi_{T,0})$ where I is the unit object of $\mathcal{A}$. This is a well-defined monoidal structure, and we already know it satisfies the appropriate diagrams to show that P is a strict tensor functor and σ is an isomorphism of tensor functors. $\square$

Now that we have shown the presence of these weak equalizer structures in **CAT** and in the category of monoidal categories and tensor functors, we can show the following result (Proposition 1.5 in [2]):

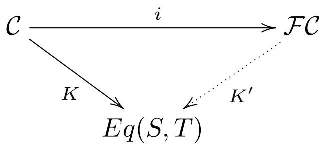
Proposition 4.5. Given a monoidal category $\mathcal{V}$, every tensor functor $T : \mathcal{FC} \to \mathcal{V}$ is isomorphic to a strict tensor functor $S : \mathcal{FC} \to \mathcal{V}$.

Proof. Given a tensor functor $T : \mathcal{FC} \to \mathcal{V}$, by the universal property from $\mathcal{FC}$ free, there exists a unique strict tensor functor $S : \mathcal{FC} \to \mathcal{V}$ such that



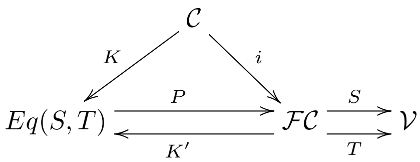
. commutes.

By Lemma 4.4, there is a unique tensor functor $K : \mathcal{C} \to Eq(S,T)$ such that $PK = i$ and $\sigma K : Si \Rightarrow Ti$ is an identity, where $\sigma : SP \Rightarrow TP$ is the isomorphism constructed in the proof of the lemma. Using the universal property of $\mathcal{FC}$ again, we have a unique strict tensor functor K' such that



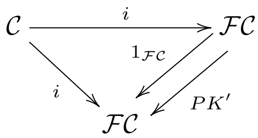
commutes.

So far, we have constructed



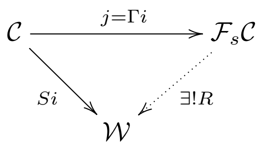
We know P is strict by Lemma 4.4 and K' is strict by construction, so PK' is strict as well.

By construction, $PK = i$ and $K'i = K$, so $PK'i = i$ and we have $PK' = 1$ by the uniqueness part of the universal property of $\mathcal{FC}$ applied to the following diagram:



Since σ is an isomorphism, $\sigma K' : S \Rightarrow T$ is an isomorphism, as desired. $\square$

To show that Γ of Theorem 4.3 is faithful, then, we first observe from Corollary 3.3 that there exists a faithful tensor functor into a strict monoidal category, $T : \mathcal{FC} \to \mathcal{W}$. Further, by Proposition 4.5, there exists a strict tensor functor $S : \mathcal{FC} \to \mathcal{W}$ such that $S \cong T$. So, S is also faithful. By the universal property of $\mathcal{F}_s\mathcal{C}$, there exists a unique strict tensor functor R such that



commutes.

Since $R\Gamma i = Si$, by the universal property of $\mathcal{FC}$, $R\Gamma = S$. Thus Γ must be faithful.

To argue that Γ is full and essentially surjective, we need one last construction: an extension of $i : \mathcal{C} \to \mathcal{FC}$ to $i' : \mathcal{F}_s\mathcal{C} \to \mathcal{FC}$. Viewing $\mathcal{F}_s\mathcal{C}$ as the disjoing union of categories $\mathcal{C}^n$ for all $\mathbb{N} \cup \{0\}$, we define i' inductively as follows, where i'_n is the restriction of i' to $\mathcal{C}^n$:

$$i'_0 = I, \qquad i'_1 = i, \qquad i'_{n+1} = (\mathcal{C}^{n+1} = \mathcal{C}^n \times \mathcal{C} \xrightarrow{i'_n \times i} \mathcal{FC} \times \mathcal{FC} \xrightarrow{\otimes} \mathcal{FC})$$

That is, $i'(A_1A_2\dots A_n) = (\cdots(A_1 \otimes A_2) \otimes \cdots \otimes A_n)$ where $A_i \in \mathcal{C}$ for all i.

Noting that $\Gamma \circ i' = 1_{\mathcal{F}_s\mathcal{C}}$, we see Γ is surjective on $\mathcal{F}_s\mathcal{C}$. Also, if i' is essentially surjective, $\Gamma \circ i' = 1_{\mathcal{F}_s\mathcal{C}}$ implies that Γ is full. That is, given any $f : A \to B$, $A, B \in \mathcal{F}_s\mathcal{C}$, then $\Gamma \circ i' \circ f = f$, so the function, $\mathcal{F}_s\mathcal{C}(A, B) \to \mathcal{F}_s\mathcal{C}(A, B)$ given by $\Gamma \circ i'$ is onto, and thus so is the map $\mathcal{FC}(i'A, i'B) \to \mathcal{F}_s\mathcal{C}(A, B)$ given by Γ. However, for any function $C, D \in \mathcal{FC}$, the map $\mathcal{FC}(C, D) \to \mathcal{F}_s\mathcal{C}(\Gamma C, \Gamma D)$ given by Γ is not necessarily surjective. However, this map is surjective if there exist isomorphisms $\omega : i'X \to C$, $\zeta : D \to i'Y$ for some $X, Y \in \mathcal{F}_s\mathcal{C}$, since given any $g : \Gamma C \to \Gamma D$, we have $\Gamma\zeta \circ g \circ \Gamma\omega : \Gamma i'X \to \Gamma i'Y$ so there exists $h : X \to Y$ such that $\Gamma h = \Gamma\zeta \circ g \circ \Gamma\omega$, hence $\Gamma\zeta^{-1} \circ \Gamma h \circ \Gamma\omega^{-1} = \Gamma(\zeta^{-1} \circ h \circ \omega^{-1}) = g$.

To show that i' is essentially surjective, since $\Gamma \circ i' = 1_{\mathcal{F}_s\mathcal{C}}$, it is sufficient to simply observe that any two objects in $\mathcal{FC}$ that have the same image under Γ are isomorphic by means of associativity and inverse right and left unitality isomorphisms.

This completes the first proof that Γ is a tensor equivalence. However, much of the machinery needed to demonstrate that Γ is an equivalence by constructing a pseudoinverse has already been established. As noted above, $\Gamma \circ i' = 1_{\mathcal{F}_s\mathcal{C}}$. In the other direction, as discussed, $i' \circ \Gamma$ takes an object of $\mathcal{FC}$, expressable as objects of $\mathcal{C} \cup \{I\}$ tensored together in some order, removes instances of the unit and rearranges the association of the tensor products to go from left to right. Given an arbitrary morphism $f : A \to B$ in $\mathcal{FC}$, by definition f can be built from arrows in $\mathcal{C}$ and morphisms a, l, and r from the definition of a monoidal category by means of tensoring, substituting, inverting and composing. Applying $i' \circ \Gamma$ to f removes the instances of a, l, and r and replaces the tensoring with concatenation, and then puts the tensoring back in from left to right. We give the components of a natural isomorphism $i' \circ \Gamma \Rightarrow 1_{\mathcal{FC}}$ by combining associativity and unitality isomorphisms. Then naturality comes out of the consistency of the definition of the isomorphism with the definition of $i' \circ \Gamma$.

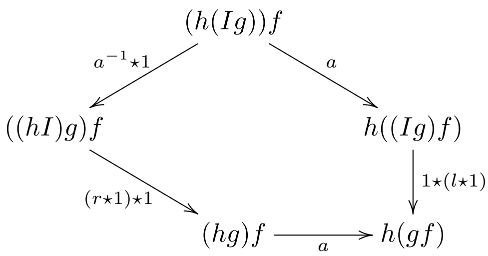
5 Commuting Diagrams

In this section, we show some classes of diagrams commute by using the coherence results we have proven. That is, now we are seeing why these theorems are called coherence results.

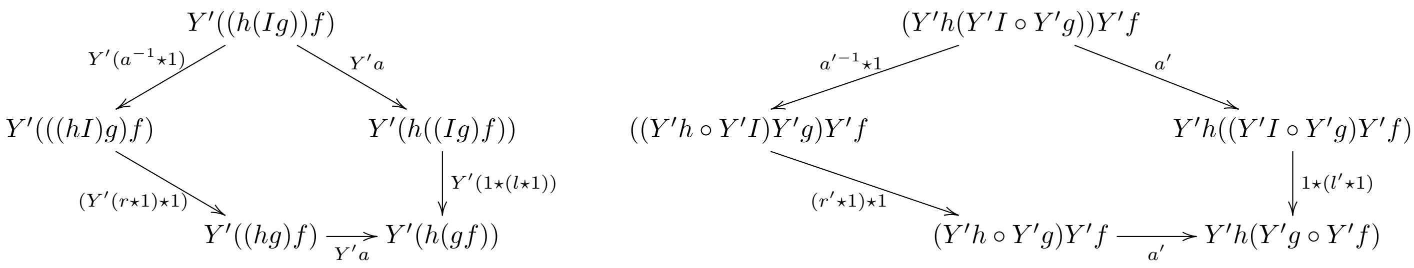
First, we look at a result coming from Theorem 3.2. The discussion is similar to that given in [4].

Application 5.1. Given a bicategory $\mathcal{B}$, any diagram built out of a, l, and r, that is, the 2-cells giving associativity and left, right composition with unit elements, commutes.

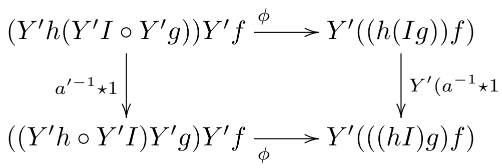
Given 1-cells f, g, h with appropriate domains and codomains, we show an example of such a diagram below:



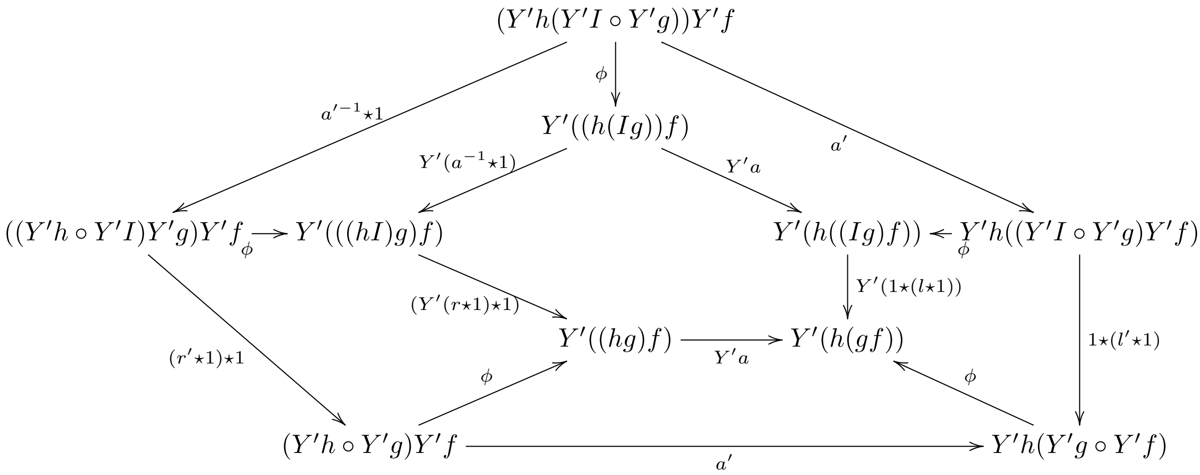
Next, we bring the above diagram into a 2-category using the restricted Yoneda Lemma Y' from the proof of Theorem 3.2. First, we apply Y' to the diagram above. Next we apply Y' individually to the 1-cells of the diagram and combine them in the same order as above within the 2-category which is the codomain of Y'. We show these diagrams below, using a dash to indicate associativity and left and right unitality morphisms in this 2-category.



Clearly the diagram on the right commutes since associativity and unitality morphisms are identities in 2-categories. The definition of a homomorphism gives us natural isomorphism with which to compare the arrows of the diagrams above. For example, the following diagram commutes since Y' is a homomorphism; the arrows ϕ are natural isomorphisms with appropriate domains built up from ϕ_f's and ϕ_A's.



Now, comparing all the arrows in our diagrams like this, we have the following:



The outer diagram commutes and each individual square constructed with the ϕ's commutes, as discussed. So, by a diagram chase, the inner diagram commutes. The inner diagram is simply Y' applied to our original diagram, so is this strong enough information to tell us our original diagram commutes? That is, if $Y'\alpha = Y'\beta$ for some 1-cells α and β, does $\alpha = \beta$? Well, the component functors of Y' are faithful, so yes, our original diagram commutes.

Just as Theorem 3.2 implies Corollary 3.3 any diagram in a monoidal category built out of associativity and right and left unitality arrows commutes as well.

Next, we examine diagrams in a more specific type of monoidal category:

Application 5.2. Let Λ be a small discrete category. Any diagram in the free monoidal category $\mathcal{F}\Lambda$ commutes.

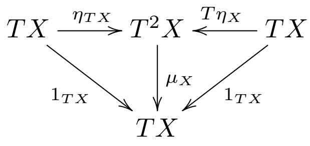
By definition, any arrow in $\mathcal{F}\Lambda$ is built out of identity, associativity, and right and left unitality morphisms, so any diagram in $\mathcal{F}\Lambda$ commutes by an argument parallel to that given in the previous application.

However, in [2], rather than examining the morphisms in $\mathcal{F}\Lambda$, Joyal and Street present a more hands-off argument using Theorem 4.3, which is cute enough to be worth mentioning as well. The form of the argument is also parallel to that used in the previous application, executed by applying the strict tensor equivalence Γ of Theorem 4.3 to any diagram in $\mathcal{F}\Lambda$. However, here all diagrams commute in the target of Γ, $\mathcal{F}\Lambda$, because it is a discrete category.

6 Algebras of 2-Monads

In the first part of this essay, we examined results which we considered coherence theorems in the sense that they allow us to prove that classes of diagrams commute. Now we move on to examining coherence theorems in the extended sense mentioned in the introduction. The main theorem given in this section is the primary result of [5]. It shows that if a 2-monad has a particular quality, then each of its pseudo-algebras are equivalent, in a sense we will define, to a strict algebra. We begin by presenting the definitions necessary for the statement of this theorem.

Definition 6.1. Given a 2-category $\mathcal{K}$, a *2-monad* $\mathbb{T}$ on $\mathcal{K}$ consists of a 2-functor $T : \mathcal{K} \to \mathcal{K}$ and 2-natural transformations $\mu : T^2 \Rightarrow T$, $\eta : 1 \Rightarrow T$ such that for any $X \in \operatorname{ob}\mathcal{K}$ the diagrams (familiar from the definition of a monad)



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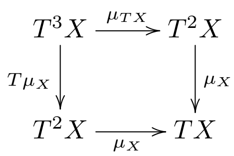
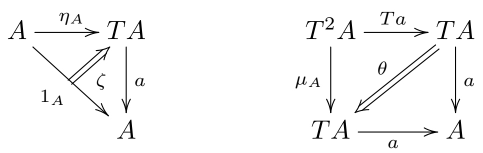


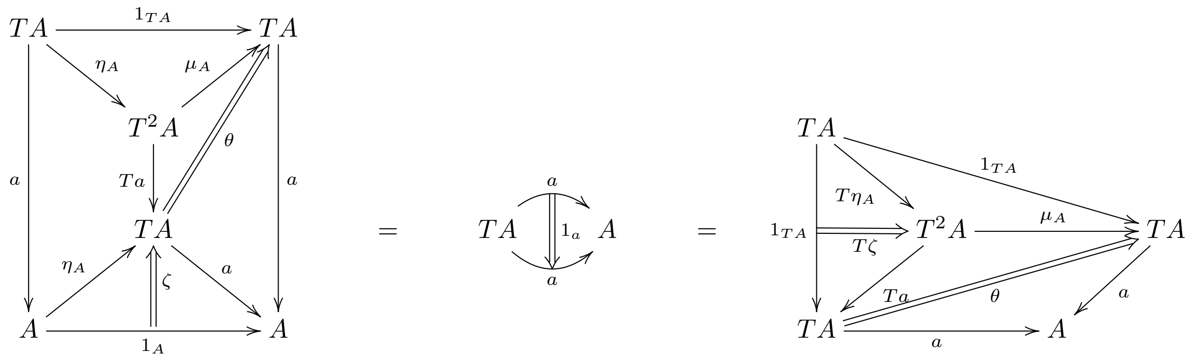
commute.

Definition 6.2. Given a 2-category $\mathcal{K}$ and two monads $\mathbb{D} = (D, \eta, mu)$ and $\mathbb{E} = (E, \eta', \mu')$ on $\mathcal{K}$, a *2-monad map* is a 2-natural transformation $\tau : D \Rightarrow E$ such that $\mu' \cdot \tau^2 = \tau \cdot \mu$ and $\eta' = \tau \cdot \eta$, where τ^2 is the 2-natural transformation from D^2 to E^2 induced by τ.

Definition 6.3. Given a 2-category $\mathcal{K}$, a *pseudo-$\mathbb{T}$-algebra* is given by an object A in $\mathcal{K}$, a morphism $a : TA \to A$ and invertible 2-cells ζ and θ:



such that



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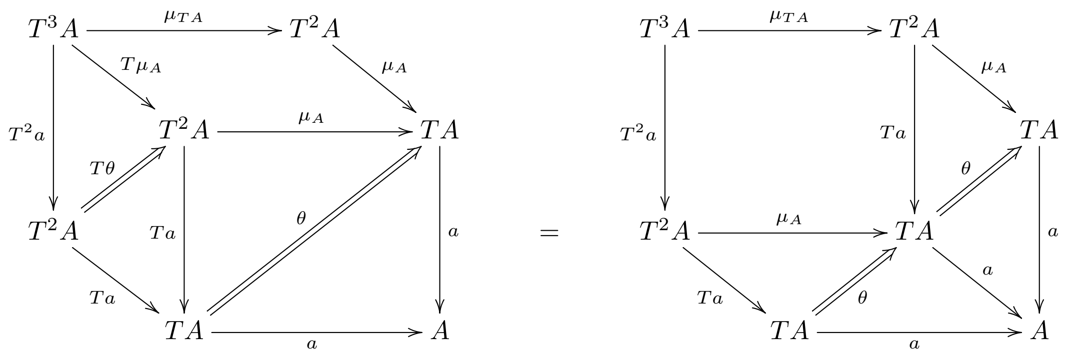
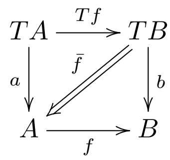


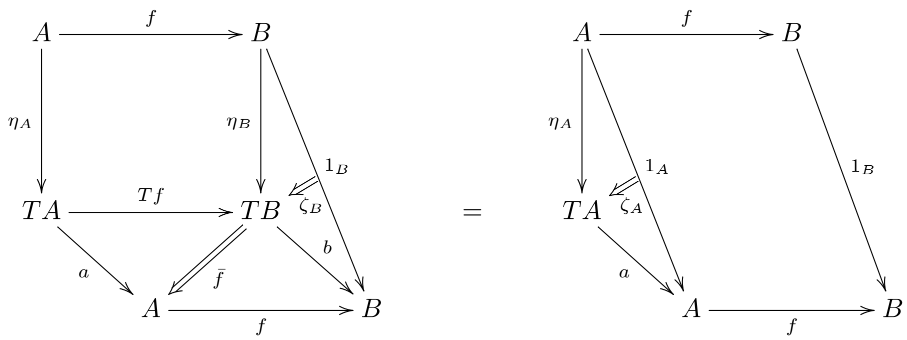
When ζ and θ are identities, the pseudo-$\mathbb{T}$-algebra is a $\mathbb{T}$-*algebra*, which we may refer to as a *strict* $\mathbb{T}$-*algebra* for emphasis.

The equivalence of pseudo-algebras and algebras in the main theorem of this section occurs in the 2-category $\mathbf{Ps}-\mathbb{T}-\mathbf{Alg}$, which has objects pseudo-$\mathbb{T}$-algebras, 1-cells pseudo-$\mathbb{T}$-algebra morphisms, and its 2-cells are algebra 2-cells. We define these last two as follows:

Definition 6.4. Given pseudo-$\mathbb{T}$-algebras $a := (A, a, \theta_A, \zeta_A)$ and $b := (B, b, \theta_B, \zeta_B)$, a morphism of pseudo-$\mathbb{T}$-algebras from a to b is given by a 1-cell f and an invertible 2-cell $\bar{f}$



such that



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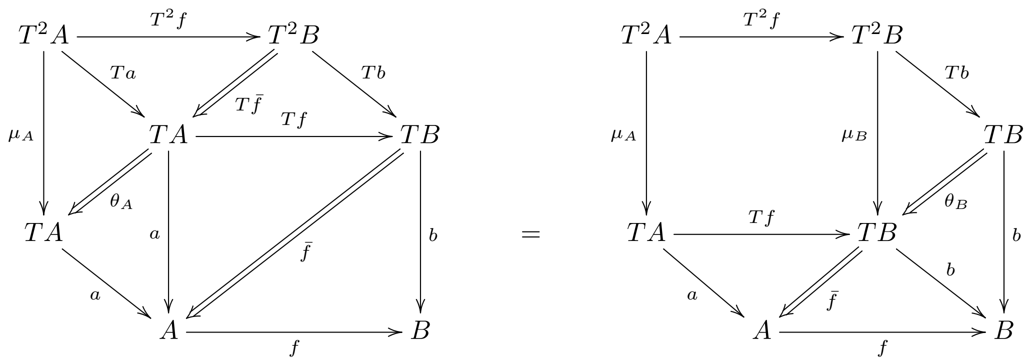
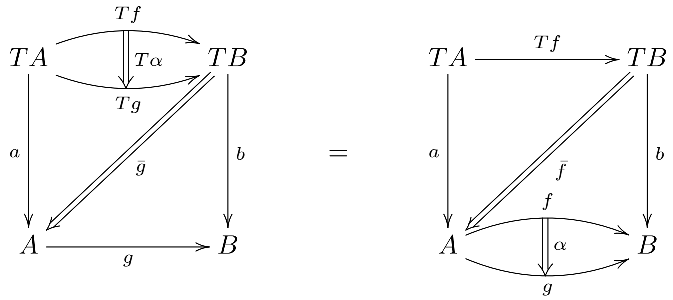


Definition 6.5. Given pseudo-$\mathbb{T}$-algebras as above and algebra morphisms $(f, \bar{f})$, $(g, \bar{g})$, an *algebra 2-cell* $\alpha : (f, \bar{f}) \Rightarrow (f, \bar{g})$

is a 2-cell $\alpha : f \to g$ in K such that



An *equivalence of pseudo-$\mathbb{T}$-algebras* $a := (A, a, \theta_A, \zeta_A)$, $b := (B, b, \theta_B, \zeta_B)$ in $\mathbf{Ps} - \mathbb{T} - \mathbf{Alg}$ is given by morphisms of pseudo-$\mathbb{T}$-algebras $f : A \to B$, $g : B \to A$ such that there are invertible algebra 2-cells $\alpha : g \circ f \Rightarrow 1_A$, $\beta : 1_B \Rightarrow f \circ g$ where 1_A and 1_B are identity morphisms on the algebras a,b.

Theorem 6.6 below deals with the case when our 2-category is $\mathbf{Cat}_g^X$ where $\mathbf{Cat}_g$ is the 2-category with objects categories, 1-cells functors, and 2-cells natural isomorphisms and X is a set. So, we may think of the objects of $\mathbf{Cat}_g^X$ as collections of categories indexed by X and the morphisms as collections of functors indexed by X.

Theorem 6.6. Let $\mathbb{T} = (T, \eta, \mu)$ be a 2-monad on $\mathbf{Cat}_g^X$. Suppose that for any morphism $f : A \to B$ in $\mathbf{Cat}_g^X$ such that for each $x \in X$, the component functor $f_x : A_x \to B_x$ is a bijection on objects, the morphism $Tf_x : TA_x \to TB_x$ is a bijection on objects for all $x \in X$ as well. Then every pseudo-$\mathbb{T}$-algebra is equivalent in $\mathbf{Ps} - \mathbb{T} - \mathbf{Alg}$ to a strict $\mathbb{T}$-algebra.

In [5] Power goes on to show this quick corollary, which gives Theorem 6.6 in a nicer form:

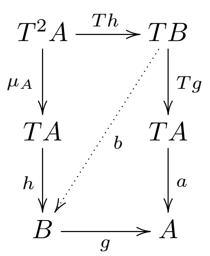
Corollary 6.7. The statement of Theorem 6.6 holds if we replace $\mathbf{Cat}_g^X$ with $\mathbf{Cat}^X$.

Proof. A 2-monad $\mathbb{T}$ on $\mathbf{Cat}^X$ restricts nicely to a 2-monad $\mathbb{T}_g$ on $\mathbf{Cat}_g^X$, and an equivalence of pseudo-$\mathbb{T}_g$-algebras is also an equivalence of pseudo-$\mathbb{T}$-algebras. □

Next we give a summary of Power's argument in [5] to prove Theorem 6.6

Sketch Proof of Theorem 6.6. Power's proof is constructive. Given a pseudo-$\mathbb{T}$-algebra in $\mathbf{Cat}_g^X$, i.e. an object $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ where each $\mathcal{A}_x$ is a category and a morphism $a : T\mathcal{A} \to \mathcal{A} = (a_x : (TA)_x \to A_x)_{x \in X}$ satisfying the necessary axioms, Power constructs the desired strict algebra by factoring each component functor a_x as $a_x = g_x \circ h_x$ for g_x fully faithful and h_x bijective on objects and inducing the desired strict algebra on $\mathcal{B} = (\mathcal{B}_x)_{x \in X}$ where $\mathcal{B}_x = \operatorname{dom} g_x$. Note $\mathcal{B}_x$ has the same object set, up to isomorphism, as $\mathcal{A}_x$.

Power uses this factorization to construct the strict algebra $b : T\mathcal{B} \to \mathcal{B}$ by inducing a map across the diagonal of the following diagram, which commutes up to the natural isomorphism θ_A from the definition of a pseudo-algebra:



Fulfilling the necessary axioms to show the algebra is strict and that g is a morphism of algebras forming part of the equivalence between a and b comes from the construction of b and the fact that a is a pseudo-$\mathbb{T}$-algebra. □

Theorem 6.6 has the same ring to it as Theorems 3.2 and 4.3; we have a means of comparison between a structure and a stronger structure, in this case an equivalence in $\mathbf{Ps} - \mathbb{T} - \mathbf{Alg}$ between a pseudo-algebra and an algebra.

7 Applications of Theorem 6.6

The hypothesis of Theorem 6.6 may seem very specific, but in the following examples of where we apply the theorem, we can see how strong the result really is; in fact it yields results similar to those shown in Sections 3 and 4.

There is a class of 2-monads on **Cat** called *flexible* which Power references briefly in the introduction of [5] as among the 2-monads to which Theorem 6.6 applies.

Proof that any pseudo-algebra of a flexible 2-monad $\mathbb{T}$ is isomorphic to a strict $\mathbb{T}$-algebra predated the writing of [5]. However, among the flexible 2-monads on **Cat** is the 2-monad for which monoidal categories are the algebras and the result that any pseudo-algebra is isomorphic to an algebra gives us a result similar to Theorems 3.2 and 4.3. So we give a brief discussion here of flexible 2-monads leading up to a result involving this particular 2-monad.

Given a 2-category $\mathcal{K}$, there are 2-categories $2\mathbf{Mnd}(\mathcal{K})$ and $2\mathbf{Mnd}_\psi(\mathcal{K})$ consisting of objects 2-monads, 1-cells, respectively, 2-monad maps and their up-to-isomorphism counterparts, 2-monad pseudomorphisms, and 2-cells modifications. There is an adjunction between 2-functors $(-)' \dashv I : 2\mathbf{Mnd}(\mathcal{K}) \to 2\mathbf{Mnd}_\psi(\mathcal{K})$ where I is an inclusion and $(-)'$ takes a monad $\mathbb{D}$ in $2\mathbf{Mnd}_\psi(\mathcal{K})$ to a monad $\mathbb{D}'$ in $2\mathbf{Mnd}(\mathcal{K})$ which has as its algebras the pseudo-algebras of $\mathbb{D}$.

We take as our definition of flexibility two equivalent characterizations which are given in Theorem 4.4 of [1]. They are reformulated as follows, where the 2-monad map q_D refers to the $\mathbb{D}$ component of the counit of the adjunction $(-)' \dashv I$.

Theorem 7.1. Given a 2-monad $\mathbb{D}$ on a 2-category $\mathcal{K}$, the following conditions are equivalent:

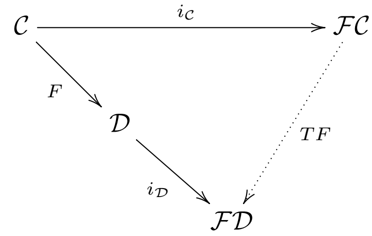
1. $\mathbb{D}$ is flexible
2. The 2-monad map $q_D : \mathbb{D}' \to \mathbb{D}$ is a retract equivalence in $2\mathbf{Mnd}(\mathcal{K})$ (i.e. there exists a 2-monad map $p_D : \mathbb{D} \to \mathbb{D}'$ such that $q_D p_D = 1_D$ and there is a natural isomorphism $p_D q_D \cong 1_{D'}$).
3. There exists a 2-monad $\mathbb{E}$ and 2-monad maps f, g $\mathbb{D} \overset{f}{\hookrightarrow} \mathbb{E}' \overset{g}{\twoheadrightarrow} \mathbb{D}$ such that $gf = 1_{\mathbb{D}}$.

The characterization of flexibility (2) above seems like it might be easier to use since it only involves the 2-monad in question and the closely related dashed version. However, since (2) clearly implies (3), (3) is a more general criterion, and we will find (3) to be easier to use when demonstrating that our specific monad is flexible as we can make a convenient choice for the second 2-monad $\mathbb{E}$ in order to facilitate the construction of the monic and epic 2-monad maps.

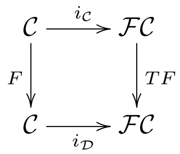
Application 7.2. Any category with a pseudo-$\mathbb{T}$-algebra structure is a monoidal category, where $\mathbb{T}$ is the 2-monad on **Cat** with monoidal categories as its strict algebras.

First, we construct the flexible 2-monad on **Cat**, $\mathbb{T} = (T, \eta, \mu)$ which has monoidal categories as its algebras. After defining it, we will verify that it has the correct algebras, prove it is flexible, and then show the above result.

The 2-functor T sends a category $\mathcal{C}$ to the free monoidal category generated on it, $\mathcal{FC}$ and a functor $F : \mathcal{C} \to \mathcal{D}$ to the strict tensor functor $TF : \mathcal{FC} \to \mathcal{FD}$ induced by the universal property of $\mathcal{FC}$:

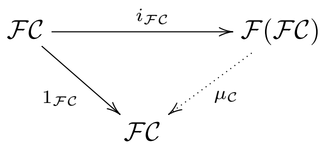


The unit of of the monad, η, has components $\eta_{\mathcal{C}} = i_{\mathcal{C}}$. Given a functor $F : \mathcal{C} \to \mathcal{D}$ the naturality square



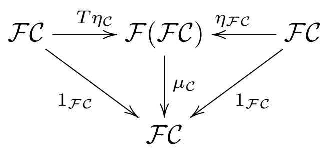
commutes since it is precisely the definition of TF.

We again use the universal property of the free monoidal categories to define the components of μ:



The naturality of μ follows from its definition.

Finally, we want to verify the axioms for a 2-monad, i.e. the following diagrams commute for any category $\mathcal{C}$:



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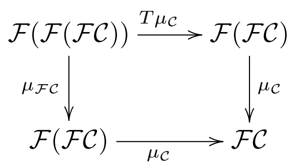
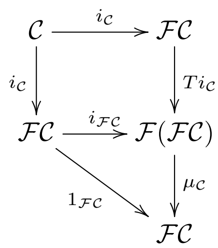
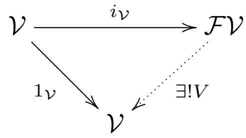


For the unit laws diagram, the right hand side is the definition of $\mu_{\mathcal{C}}$. For the left hand side, consider the following commutative diagram, which has the definition of $T\eta_{\mathcal{C}}$ on top and the definition of $\mu_{\mathcal{C}}$ below:

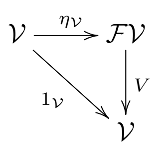


By the uniqueness part of the universal property of $\mathcal{FC}$, $\mu_{\mathcal{C}} \circ Ti_{\mathcal{C}} = 1_{\mathcal{FC}}$, and thus the left hand side of the unit laws diagram commutes. The associativity law diagram follows by a similar argument.

Now, we will show the algebras for $\mathbb{T} = (T, \eta, \mu)$ are in fact what we intended. Suppose $\mathcal{V}$ is a monoidal category. Define a strict tensor functor V using the universal property of a free monoidal category:



To show V gives a $\mathbb{T}$-algebra, we need to verify the following diagrams commute:



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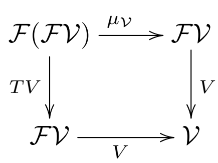
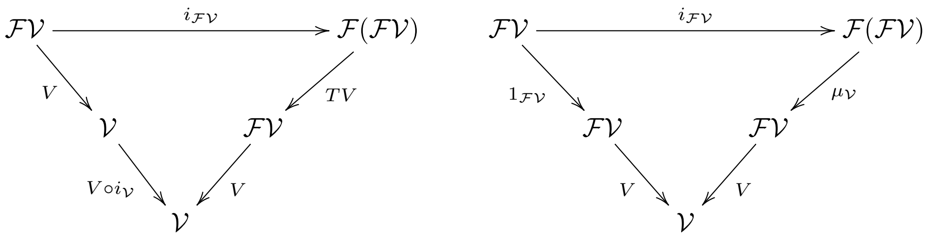
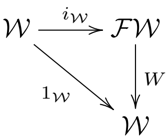


The diagram on the left is the definition of V. For the diagram on the right, consider the following commutative diagrams which are the definitions of μ_V and TV, respectively, with V appended:



Observing $V \circ i_{\mathcal{V}} = 1_{\mathcal{V}}$, the uniqueness part of the universal property shows that $V \circ TV = V \circ \mu_V$, hence $V : \mathcal{F}\mathcal{V} \to \mathcal{V}$ is an algebra for $\mathbb{T}$.

In the other direction, suppose a category $\mathcal{W}$ is an algebra for $\mathbb{T}$, that is, there is a functor $W : \mathcal{F}\mathcal{W} \to \mathcal{W}$ such that



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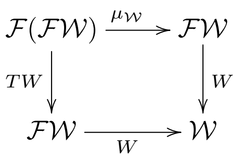

commute. (6)

We then observe that there is a monoidal structure on $\mathcal{W}$ carried in by W. Given $A, B \in \operatorname{ob}\mathcal{W}$, let $A \otimes B = W(A \otimes_{\mathcal{F}} B)$ where $\otimes_{\mathcal{F}}$ is the tensor product in $\mathcal{F}\mathcal{W}$ and let the unit in $\mathcal{W}$ be $I = W(I_{\mathcal{F}}$ where $I_{\mathcal{F}}$ is the unit in $\mathcal{F}\mathcal{W}$. Then we have the appropriate isomorphisms and commutative diagrams from the structure of $\mathcal{F}\mathcal{W}$ and the algebra axioms to satisfy the data and axioms for $\mathcal{W}$ to be a monoidal category.

For example, consider $I \otimes A$ and A. In $\mathcal{F}\mathcal{W}$, $I_{\mathcal{F}} \otimes_{\mathcal{F}} A \cong A$, so by the functorality of W, $W(I_{\mathcal{F}} \otimes_{\mathcal{F}} A) \cong W(A)$. By the commutative diagram on the above left, $W(A) = A$. By applying the second commutative diagram to $I_{\mathcal{F}} \otimes_{\mathcal{F}\mathcal{F}} A$ (where $\otimes_{\mathcal{F}\mathcal{F}}$ is the tensor in $\mathcal{F}(\mathcal{F}\mathcal{W})$) we have $W(I_{\mathcal{F}} \otimes_{\mathcal{F}} A) = W(I \otimes_{\mathcal{F}} A)$. By definition, $W(I \otimes_{\mathcal{F}} A) = I \otimes A$. Hence $A \cong I \otimes A$.

Further, we can see that W must be a strict tensor functor, consistent with the previous proof that monoidal categories are among the algebras of $\mathbb{T}$.

Our definition of $\mathbb{T}$ shows that it satisfies the hypothesis of Theorem 6.6; if $F : \mathcal{C} \to \mathcal{D}$ is a functor bijective on objects, TF is bijective on objects as well since TF is a strict tensor functor and acts on objects of $\mathcal{C}$ included into $\mathcal{F}\mathcal{C}$ the same way that F does.

However, in order to demonstrate that the monad $\mathbb{T}$ is also flexible, we take a detour to examine a 2-monad which will allow us to fulfill condition (3) of Theorem 7.1.

This 2-monad on **Cat**, $\mathbb{D} = (D, \tilde{\eta}, \tilde{\mu})$, has strict monoidal categories as its strict algebras. We can construct of $\mathbb{D}$ in a way similar to that for the 2-monad for monoidal categories. The functor part of $\mathbb{D}$ takes a category to the free strict monoidal category it generates and the unit and multiplication 2-natural transformations are induced by the universal property of free strict monoidal categories. Given a category $\mathcal{C}$, we can think of the free strict monoidal category $\mathcal{F}_s\mathcal{C}$ as a 1-object 2-category with 1-cells objects of $\mathcal{C}$ and composition of 1-cells given by concatenating these objects. The 2-cells are then the concatenated 1-cells from $\mathcal{C}$.

The pseudo-$\mathbb{D}$-algebras are categories $\mathcal{C}$ and functors $F : \mathcal{F}_s\mathcal{C} \to \mathcal{C}$ such that the following diagrams commute up to isomorphism:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\tilde{\eta}_\mathcal{C}} & \mathcal{F}_s\mathcal{C} \\ & \searrow^{1_\mathcal{C}} \;\cong & \downarrow F \\ & & \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_s(\mathcal{F}_s\mathcal{C}) & \xrightarrow{DF} & \mathcal{F}_s\mathcal{C} \\ \tilde{\mu}_\mathcal{C}\downarrow & \cong & \downarrow F \\ \mathcal{F}_s\mathcal{C} & \xrightarrow[F]{} & \mathcal{C} \end{array}$$

The pseudo-$\mathbb{D}$-algebras must also satisfy the coherence conditions from the definition of a pseudo-algebra. Keeping in mind that $\mathcal{F}_s\mathcal{C}$ can be viewed as the disjoint union of categories $\mathcal{C}^n$ for all $n \in \mathbb{N} \in \cup\{0\}$, we can think of F as a collection of functors $\otimes_n : \mathcal{C}^n \to \mathcal{C}$, and re-express the pseudo-algebra conditions as follows:

- For all $C \in \mathcal{C}$, there is an isomorphism $\otimes_1 C \cong C$.
- For any $n \in \mathbb{N}$, $\{m_1, \ldots m_n\} \in \mathbb{N}$, isomorphisms $\otimes_n(\otimes_{m_1}(C_{11}, \ldots, C_{1m_1}), \ldots, \otimes_{m_n}(C_{n1}, \ldots, C_{nm_n})) \cong \otimes_{\sum_{1\leq i\leq n} m_i}(C_{11}, \ldots C_{1m_1}, C_{21}, \ldots, C_{nm_n})$ where each $C_{ij} \in \mathrm{ob}\,\mathcal{C}$.

The monad $\mathbb{D}'$ has as its algebras the pseudo-algebras of $\mathbb{D}$, and so the functor D' freely endows any category with the structure given above. That is, given a category $\mathcal{C}$, the objects of $D'\mathcal{C}$ are built up inductively starting with the objects of $\mathcal{C}$ and then including $\otimes_n(C_1, \ldots, C_n)$ for any $n \in \mathbb{N}$ and $C_i \in D\mathcal{C}$. The arrows of $D'\mathcal{C}$ are also built up inductively, starting with the arrows in $\mathcal{C}$, adding the arrows for the isomorphisms given above, and including $\otimes_n(f_1, \ldots, f_n)$ for any $n \in \mathbb{N}$ and f_i's 1-cells of $D\mathcal{C}$ as well as any possible compositions.

Now we can describe the components of the 2-natural transformation portions of the 2-monad maps f and g which fulfill condition (3) of Theorem 7.1 to show that $\mathbb{T}$ is indeed flexible. That is, given any category $\mathcal{C}$ we have

$$T\mathcal{C} = \mathcal{F}\mathcal{C} \xhookrightarrow{f_\mathcal{C}} D'\mathcal{C} \xrightarrow{g_\mathcal{C}} \mathcal{F}\mathcal{C} = T\mathcal{C}$$

Given any object of $\mathcal{F}\mathcal{C}$, the functor $f_\mathcal{C}$ replaces the unit with $\otimes_0$ and replaces tensor products with the unit as one of their arguments with $\otimes_1$ and then replaces remaining tensor products with $\otimes_2$. For example, $f_\mathcal{C}$ replaces $C \otimes I$ with $\otimes_1(C\otimes_0)$, $I \otimes C$ with $\otimes_1(\otimes_0 C)$ and $C \otimes C'$ with $\otimes_2(CC')$. Then the functor $g_\mathcal{C}$ replaces $\otimes_0$ with the unit, adding a tensor product when necessary, and $\otimes_n(C_1C_2\cdots C_n)$ with $(\cdots((C_1 \otimes C_2) \otimes \cdots \otimes C_n)$ for any $n \geq 1$ (the same as the action of i' on $C_1C_2\cdots C_n$. For example, $g_\mathcal{C}$ would map $\otimes_3(C_1 \otimes_0 C_2C_3)$ to $((C_1 \otimes I) \otimes C_2) \otimes C_3)$. By construction $f_\mathcal{C}$ and $g_\mathcal{C}$ are respectively injective and surjective, as required.

Now that we have shown $\mathbb{T}$ is flexible, we examine what it means that each of its pseudo-algebras is isomorphic in $\mathbf{Ps}-\mathbb{T}-\mathbf{Alg}$ to a strict $\mathbb{T}$-algebra. Since the monads we are looking at are on **Cat**, this isomorphism means, along with coherence conditions, that for any category $\mathcal{C}$ which is part of a pseudo-algebra, there exists a category $\mathcal{D}$ with the strict algebra structure which is isomorphic to it, that is, there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = 1_\mathcal{C}$ and $F \circ G = 1_\mathcal{D}$. In the case we examined above, where the algebras of $\mathbb{T}$ are monoidal categories, this means that for any category $\mathcal{V}$ with a pseudo-$\mathbb{T}$-algebra structure, $\mathcal{V}$ is in fact monoidal since we can bring the monoidal structure across the isomorphism.

As mentioned in the introduction to [5], the 2-monad $\mathbb{D}$ for strict monoidal categories is not flexible. So, the following application, which features a generalization of $\mathbb{D}$, is a case where Theorem 6.6 improves our knowledge of equivalences of pseudo-algebras and algebras beyond what is given to us by flexibility.

Application 7.3. There is a monad on $\mathbf{Cat}^{X\times X}$ to which Theorem 6.6 applies which has algebras 2-categories and pseudo-algebras bicategories.

In the previous example, we examined the 2-monad $\mathbb{D}'$ on $\mathbf{Cat}$ which has strict monoidal categories as its strict algebras.

This construction can be extended to the case where we put a free structure on collections of categories indexed over $X \times X$ for a set X, i.e. objects in $\mathbf{Cat}^{X\times X}$. There is a 2-monad $\mathbb{T} = (T, \eta, \mu)$ where T maps any $\mathcal{A} \in \mathbf{Cat}^{X\times X}$ to the free 2-category with objects elements of X, 1-cells built inductively starting with the objects of the categories $\mathcal{A}_{xy}$ and composing them by means of concatenation, and 2-cells built similarly out of the morphisms in the categories $\mathcal{A}_{xy}$, any equations among the 2-cells resulting from the relationships within and between the categories $\mathcal{A}_{xy}$. For example, given any $x, y \in X$, 1-cells from x to y include the objects of $\mathcal{A}_{xy}$, the objects of $\mathcal{A}_{xy}$ precomposed with objects of $\mathcal{A}_{xx}$, objects of $\mathcal{A}_{xz}$ postcomposed with objects of $\mathcal{A}_{zy}$, etc. Also, of course, we can view $T\mathcal{A}$ as an object of $\mathbf{Cat}^{X\times X}$ with some extra structure, just as we can view a strict monoidal category is a 1-object 2-category or a category with extra structure. From this perspective, $(TA)_{xy}$ is the category with objects as we described the 1-cells from x to y above.

The action of T on morphisms of $\mathbf{Cat}^{X\times X}$ is again similar to the strict monoidal case. Given a morphism $F \in \mathbf{Cat}^{X\times X}(\mathcal{A}, \mathcal{B})$, that is, an $X \times X$-indexed collection of functors $F_{xy} : \mathcal{A}_{xy} \to \mathcal{B}_{xy}$, the functor $TF_{xy} : (T\mathcal{A})_{xy} \to (T\mathcal{B})_{xy}$ just takes each object in $(T\mathcal{A})_{xy}$, applies the appropriate functor of F to each of its component objects from the categories of $\mathcal{A}$ and preserves their order of concatenation. For example, given $A_{xx} \in \mathcal{A}_{xx}$, $A_{xy} \in \mathcal{A}_{xy}$, $TF_{xy}(A_{xy}A_{xx}) = (F_{xy}A_{xy})(F_{xx}A_{xx})$.

So, if every component functor of F is bijective on objects, so is each component of TF, hence T satsifies the hypothesis of Theorem 6.6. Next, we examine what the conclusion of Theorem 6.6 implies about algebras in this case.

In the case of the 2-monad $\mathbb{D}$ whose algebras are strict monoidal categories, itsalgebras and pseudo-algebras are discussed in the previous application. In fact, $\mathbf{Ps} - \mathbb{D} - \mathbf{Alg}$ is equivalent to the category of monoidal categories. Generalizing to $\mathbb{T}$, an algebra is an object of $\mathbf{Cat}^{X\times X}$ with a 2-categorical structure when we consider the objects to be elements of X. A pseudo-$\mathbb{T}$-algebra has n-fold tensor products similar to the monoidal case, and $\mathbf{Ps} - \mathbb{T} - \mathbf{Alg}$ is also equivalent to the 2-category of bicategories with object set X. Then by Theorem 6.6, each pseudo-$\mathbb{T}$-algebra is equivalent to a $\mathbb{T}$-algebra in $\mathbf{Ps} - \mathbb{T} - \mathbf{Alg}$. The data of these equivalences in $\mathbf{Ps} - \mathbb{T} - \mathbf{Alg}$ includes functors and isomorphisms demonstrating equivalences between the category parts of the algebras along with additional data obeying coherence conditions. Combining these, any bicategory with a set of objects is biequivalent to a 2-category with the same object set.

This last application of Theorem 6.6 then implies results about diagrams commuting the same way that Theorem 3.2 does. So, we can see very clearly why Theorem 6.6 can be called a coherence theorem, which nicely illustrates the definitions of coherence theorems discussed in the introduction. However, the coherence theorems included in this essay all imply diagrams commute by demonstrating the existence of a sufficiently strong means of comparison between a categorical structure and a stronger structure of the same type. This characteristic gives us perhaps a more visceral sense of what a coherence theorem looks like than that provided by the definitions given in the introduction.

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