Homework 1 : Chapter 2: 5-66, 5-67, 6-1, 6-5, 10-11, 10-29, 16-1, 16-11

① Ch. 2, 5-66

a) Given $z = x + iy = \frac{i+2t}{t-i}$, find $x(t)$ and $y(t)$

$$= \frac{i+2t}{t-i}\left(\frac{t+i}{t+i}\right) = \frac{2t^2-1+i(2t+t)}{t^2+1}$$

$$x(t) = \frac{2t^2-1}{t^2+1}, \quad y(t) = \frac{3t}{t^2+1}$$

b) find $v(t)$ and $a(t)$

(i) $$V_z(t) = \frac{d}{dt}x + i\frac{d}{dt}y = \frac{4t}{t^2+1} - \frac{2t(2t^2-1)}{(t^2+1)^2} + i\left(\frac{3}{t^2+1} - \frac{6t^2}{(t^2+1)^2}\right)$$

$$= \frac{4t(t^2+1)-2t(2t^2-1)}{(t^2+1)^2} + i\left[\frac{3(t^2+1)-6t^2}{(t^2+1)^2}\right]$$

$$V_z(t) = \frac{6t}{(t^2+1)^2} - i\frac{3(t^2-1)}{(t^2+1)^2}$$

$$|V_z(t)| = \frac{1}{(t^2+1)^2}\sqrt{36t^2+9(t^4-2t^2+1)}$$

$$|V_z(t)| = \frac{3}{t^2+1} = V(t) \checkmark$$

(ii) $$\cancel{a_z(t) = \frac{d^2x}{dt^2} + i\frac{d^2y}{dt^2} = \left(\frac{6}{(t^2+1)^2} - \frac{12t(2t)}{(t^2+1)^3}\right) - i\left(\frac{6t}{(t^2+1)^2}\right.}$$

$$\cancel{\left. + i\left(\frac{3(t^2-1)}{(t^2+1)^3}(-2)2t\right)\right)}$$

ii) $$V_2(t) = \frac{6t}{(t^2+1)^2} - i\,\frac{3(t^2-1)}{(t^2+1)^2}$$

$$\frac{dV_2(t)}{dt} = \left\{\frac{6}{(t^2+1)^2} - \frac{12t\cdot 2t}{(t^2+1)^3}\right\} - i\left\{\frac{6t}{(t^2+1)^2} - \frac{12t(t^2-1)}{(t^2+1)^3}\right\}$$

$$= \left\{\frac{6(t^2+1) - 24t^2}{(t^2+1)^3}\right\} - i\left\{\frac{6t(t^2+1) - 12t(t^2-1)}{(t^2+1)^3}\right\}$$

$$= \left\{\frac{-6(3t^2-1)}{(t^2+1)^3}\right\} - i\left\{\frac{-6t^3+18t}{(t^2+1)^3}\right\}$$

$$= \left\{\frac{-6(3t^2-1)}{(t^2+1)^3}\right\} + i\left\{\frac{6t(3-t^2)}{(t^2+1)^3}\right\}$$

$$\left|\frac{dV_2}{dt}\right| = a(t) = \frac{1}{(t^2+1)^3}\sqrt{36(3t^2-1)^2 + 36t^2(-t^2+3)^2}$$

$$= \frac{6}{\left(\sqrt{t^2+1}\right)^3} \checkmark$$

② Chapter 2, 5-67

a) $z = x + iy = \dfrac{(1-it)}{(2t+i)}$

$$\dot{z} = \left(\frac{-i}{2t+i}\right) - \frac{(1-it)2}{(2t+i)^2}$$

$$= \frac{-i(2t+i) - 2(1-it)}{(2t+i)^2}$$

$$= \frac{-1}{(2t+i)^2} = \frac{-1}{(2t+i)^2}$$

$$\ddot{z} = + \frac{2\times 2}{(2t+i)^3} = \frac{4}{(2t+i)^3}$$

b) $V = |\dot{z}| = \left| - \dfrac{(2t-i)^2}{(4t^2+1)^2} \right| = \left| - \dfrac{(4t^2-1-4it)}{(4t^2+1)^2} \right|$

$$= \frac{1}{(4t^2+1)^2}\sqrt{(4t^2-1)^2 + 16t^2}$$

$$= \frac{1}{(4t^2+1)^2}\sqrt{16t^4 + 8t^2 + 1}$$

$$\boxed{V = \frac{1}{(4t^2+1)}}$$

c) $V|\ddot{z}| = \left| \dfrac{4(2t-i)^3}{(4t^2+1)^3} \right| = \left| \dfrac{4\left(8t^3 + 3[4t^2 i] + 3[2t(-1)] + i\right)}{(4t^2+1)^3} \right|$

$$= \frac{4}{(4t^2+1)^3} \left| 8t^3 - 6t + i(1-12t^2) \right|$$

$$= \frac{4}{(4t^2+1)^3}\sqrt{(64t^6 - 96t^4 + 36t^2) + (1 - 24t^2 + 144t^4)}$$

$$\underbrace{}^{(4t^2+1)^{3/2}}$$

$$= \frac{4}{(4t^2+1)^3}\sqrt{64t^6+48t^4+12t^2+1}$$

$$\boxed{a = \frac{4}{(4t^2+1)^{3/2}}}$$

(3) Chapter 3, 6-1

Prove that an absolutely convergent series of complex numbers converges. This means to prove that $\Sigma (a_n + i b_n)$ converges if $\Sigma \sqrt{a_n^2 + b_n^2}$ converges

Sol^n

a series $\Sigma(a_n + i b_n)$ is absolutely convergent

$\Rightarrow \Sigma |a_n + i b_n|$ is bounded

$= \Sigma |z_n|$

$= |z_1| + |z_2| + |z_3| + \cdots$

By the triangle inequality

$$\Sigma |z_n| > |(z_1 + z_2 + z_3 + z_3)|$$

$$= |\Sigma a_n + i \Sigma b_n|$$

$$= \sqrt{(\Sigma a_n)^2 + (\Sigma b_n)^2}$$

$\Rightarrow \Sigma a_n$ & Σb_n are bounded

(5) Chapter 2, 10-11

find roots of $(-8)^{\frac{1}{3}}$

Sol^{n} Use $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$ or $z^{\frac{1}{n}} = \left[r(\cos\theta + i\sin\theta)\right]^{\frac{1}{n}}$

(i) $z^{\frac{1}{3}} = (-8)^{\frac{1}{3}} = 8^{\frac{1}{3}}(\cos\theta + i\sin\theta)^{\frac{1}{3}}$

$\left.\begin{array}{l}-8 \text{ is real} \\ \text{and} < 0\end{array}\right\} \Rightarrow \theta = \pi + 2k\pi$

$$(-8)^{1/3} = 8^{1/3} e^{i\left(\frac{\pi + 2k\pi}{3}\right)}$$

roots are $\boxed{\begin{array}{l} r = 2\left(\underset{k=0}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}},\ \underset{k=1}{\cos\pi},\ \underset{k=2}{\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}}\right)\end{array}}$

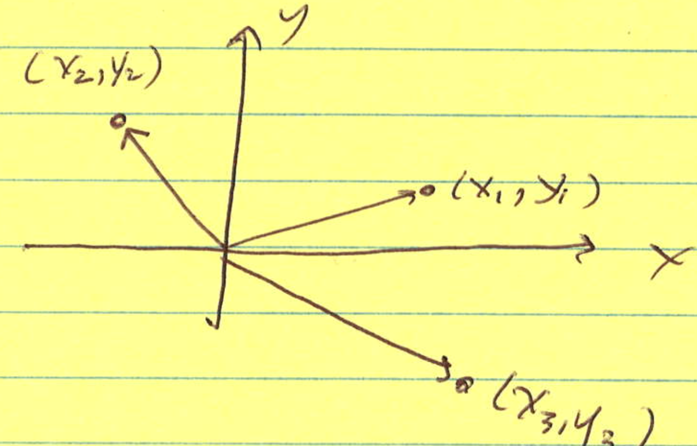
note that $k=3 \Rightarrow \theta = \frac{7\pi}{3}$ which is the same as $\frac{\pi}{3}$ and is not an unique root

(6) Chapter 2, 10.29

Show that the center of mass of three identical particles situated at z_1, z_2, z_3 is $\frac{z_1+z_2+z_3}{3}$

Sol

C-of-M is ?



the C-of-M is defined as the point about which

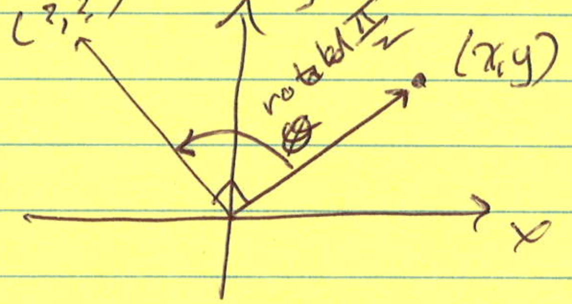
$$\sum_{i=1}^{3} m_i (z_i - z_{com}) = 0$$

$$\sum_{i=1}^{3} m_i z_i - 3m z_{com} = 0$$

$$z_{com} = \frac{z_1+z_2+z_3}{3}$$

⑦ Chapter 2, 16-1

a) Show that if the line through the origin and the point z is rotated $\pi/2$ about the origin, it becomes the line through the origin and the point iz.



rotate vector, not the axes so that

$$R^{v} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

note that if we rotated the axes rather than the vector by θ then

$$R^{axes} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Anyway

$$\begin{pmatrix} x' \\ y' \end{pmatrix}_{\text{rotated}} = \begin{pmatrix} \cancelto{0}{\cos\frac{\pi}{2}} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cancelto{0}{\cos\frac{\pi}{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} -y \\ x \end{pmatrix} \Rightarrow \boxed{z' = -y + ix}$$

$$= iz \checkmark$$