(12) Chap 14, 11-35

Show that the Cauchy Riemann equations in a general orthogonal curvilinear coordinate system are

$$\frac{1}{h_1}\frac{\partial u}{\partial x_1} = \frac{1}{h_2}\frac{\partial v}{\partial x_2}, \quad \frac{1}{h_1}\frac{\partial v}{\partial x_1} = -\frac{1}{h_2}\frac{\partial u}{\partial x_2}$$

Sol^n

for the sake of "concreteness," look at cylindrical coordinates first,

$$z = re^{i\theta} \qquad (1)$$

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

a) find $\frac{df(z)}{dz}$ by approaching the point along fixed θ direction $\Rightarrow d\theta = 0$

$$\Rightarrow \frac{df(z)}{dz} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + i\frac{\partial v}{\partial r}\frac{\partial r}{\partial z}$$

and $\frac{\partial r}{\partial z} = e^{-i\theta}$ from (1)

$$\Rightarrow \frac{df}{dz} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)e^{-i\theta}$$

b) find $\frac{df(z)}{dz}$ by approaching along path with fixed r

$$\Rightarrow \frac{df}{dz} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + i\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial z}$$

and $\frac{\partial \theta}{\partial z} = \frac{1}{ire^{i\theta}}$

$$\Rightarrow \frac{df}{dz} = \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{e^{-i\theta}}{ir}$$

c) for $\frac{df}{dz}$ to exist, we must have that

$$\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)e^{-i\theta} = \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{e^{-i\theta}}{ir}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

generalize to a curvilinear system; let $r \to x_1$ and

$\theta \to x_2$

and define $h_1^{-1} = \frac{\partial r}{\partial z}$ and

$h_2^{-1} = \frac{\partial \theta}{\partial z}$

$$\boxed{\Rightarrow \frac{1}{h_1}\frac{\partial u}{\partial x_1} = \frac{1}{h_2}\frac{\partial v}{\partial x_2} \text{ and } \frac{1}{h_1}\frac{\partial v}{\partial x_1} = -\frac{1}{h_2}\frac{\partial u}{\partial x_2}}$$

Show that u & v satisfy the Laplace equation

① $\underbrace{\left(\frac{h_2}{h_1}\right)\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}}_{a}$ and $\underbrace{\left(\frac{h_1}{h_2}\right)\frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}}_{a}$

$\underbrace{\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial u}{\partial x_1}\right) = \frac{\partial^2 v}{\partial x_1 \partial x_2}}_{A}$ and $\underbrace{\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial u}{\partial x_2}\right) = -\frac{\partial^2 v}{\partial x_2 \partial x_1}}_{B}$

add A and B

$$\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial u}{\partial x_2}\right) = 0$$

divide by $\frac{1}{h_1 h_2}$ to put it in the form of the Laplace eqn. $\nabla^2 u = 0$. We have

$$\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial u}{\partial x_2}\right)\right] = 0$$

which is the 2D Laplace equation (see 9.10 in text, p 527)

(3) <u>Chap 14, 2-24</u>

Is $\dfrac{y-ix}{x^2+y^2}$ analytic?

$$u(x,y) = \frac{y}{x^2+y^2}, \quad v(x,y) = \frac{-x}{x^2+y^2}$$

(i) $$\frac{\partial u}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \cancel{\frac{1}{(x^2+y^2)}} + \frac{2yx}{(x^2+y^2)^2}$$

$$= \cancel{\frac{x^2-y^2}{(x^2+y^2)^2}}$$

(ii) $$\frac{\partial v}{\partial x} = \frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

(i) $\Rightarrow \dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y}$

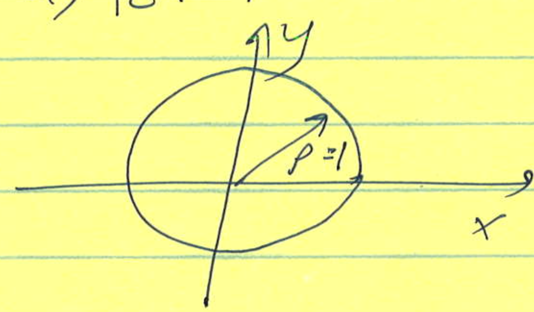
(ii) $\Rightarrow \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$

No, it is not analytic

(14) Chapter 14, 3-20

$$\oint_C \frac{\cosh z\,dz}{2\ln 2 - z} \text{ if } C \text{ is the circle (a) } |z|=1 \quad \text{(b) } |z|=2$$

a) $|z|=1$



$2\ln 2 - z = 0$ when

$z = 2\ln 2 = 1.386$

> 1

$\Rightarrow$ no singular point in region $\Rightarrow \oint_C = 0$

by Cauchy Theorem

b) $|z|=2 \Rightarrow$ a pole at $z = 2\ln 2$

$\oint_C \frac{\cosh z\,dz}{2\ln 2 - z}$; find residue at $2\ln 2$

$\Rightarrow (z-2\ln 2)\left[\frac{\cosh z}{(2\ln 2 - z)}\right], \; z \to 2\ln 2$

$b_1 = -\cosh(2\ln 2) = -\cosh(\ln 2^2)$

$= -\frac{1}{2}\left[\underset{4}{e^{\ln 2^2}} + \underset{\frac{1}{4}}{e^{-\ln 2^2}}\right]$

$= -\frac{1}{2}\left(2^2 + \frac{1}{2^2}\right)$

$b_1 = -17/8$

$$\rightarrow \boxed{\oint \frac{\cosh z\,dz}{2\ln 2 - z} = -\frac{17}{4}\pi i}$$

(15) Chapter 14, 4-4

find the first few terms for the Laurent series for $f(z) = \dfrac{1}{z(z-1)(z-2)^2}$

a) $0 < |z| < 1$

$$f(z) = \frac{1}{z(z-1)(z-2)^2}$$

$$= \frac{a}{z} + \frac{b}{(z-1)} + \frac{c}{(z-2)^2} \left(= -\frac{1}{4z} + \frac{1}{z-1} + \frac{1}{2(z-2)^2} \right)$$

$$= -\frac{1}{4z} - \left(1 + z + z^2 + z^3 + \cdots\right) + \frac{1}{8}\left(1 + z + \frac{3z^2}{4} + \frac{z^3}{2} + \frac{5z^4}{16} + \cdots\right)$$

$$f(z) = \underbrace{-\frac{1}{4z}}_{\text{converges if } |z| \neq 0} + \underbrace{\left[-\frac{7}{8} - \frac{7}{8}z - \frac{29}{32}z^2 - \frac{15}{16}z^3 - \cdots\right]}_{\text{converges if } |z| < 1}$$

$$\Rightarrow \boxed{b_1 = -\frac{1}{4}}$$, finally

b) $1 < |z| < 2$

$$f(z) = \frac{1}{z(z-1)(z-2)^2} = \frac{1}{z}\left[\frac{1}{z-1} + \frac{1}{(z-2)^2}\right]$$

$$= \frac{1}{z}\left(\frac{1}{z}\frac{1}{1-\frac{1}{z}}\right) + \frac{1}{z}\frac{1}{4(1-\frac{z}{2})^2}$$

$$= \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{4z}\left(1 + 2\left(\frac{z}{2}\right) + 3\left(\frac{z}{2}\right)^2 + 4\left(\frac{z}{2}\right)^3 + \cdots\right)$$

$$= \underbrace{\left\{\frac{1}{4z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots\right\}}_{|z|>1,\ \text{converges}} + \underbrace{\left\{\frac{1}{4} + \frac{3}{8}\left(\frac{z}{2}\right) + \frac{1}{2}\left(\frac{z}{2}\right)^2 + \frac{5}{8}\left(\frac{z}{2}\right)^2 + \cdots\right\}}_{|z|<2,\ \text{converges}}$$

c) $2 < |z|$

$$f(z) = \frac{1}{z(z-1)(z-2)^2}$$

$$\frac{a}{z} + \frac{b}{(z-1)} + \frac{c}{(z-2)^2} = \frac{1}{z(z-1)(z-2)^2}$$

$$a(z-1)(z-2)^2 + bz(z-2)^2 + z(z-1)c = 1$$

(i) $z = 1 \Rightarrow -b = 1$

(ii) $z = 2 \Rightarrow 2c = 1$

(iii) $z = 0 \Rightarrow -4a = 1$

$$f(z) = -\frac{1}{4z} - \frac{1}{(z-1)} + \frac{1}{2(z-2)^2}$$

$$= -\frac{1}{4z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{2z^2}\left(1 + 2\left(\frac{2}{z}\right) + 3\left(\frac{2}{z}\right)^2 + 4\left(\frac{2}{z}\right)^3 + 5\left(\frac{2}{z}\right)^4 + \cdots\right)$$

$$= \underbrace{-\left(\frac{5}{4z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)}_{\text{converges for } |z|>1} + \frac{1}{8}\underbrace{\left(\left[\frac{2}{z}\right]^2 + 2\left[\frac{2}{z}\right]^3 + 3\left[\frac{2}{z}\right]^4 + \cdots\right)}_{\text{converges for } |z|>2}$$