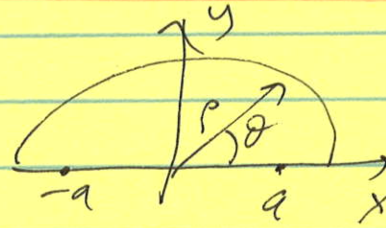
(c) Solve $\oint \frac{2a f(z)}{(z^2-a^2)} dz$ by contour integration

$$= \oint f(z)\,dz \left[\frac{1}{z-a} - \frac{1}{z+a}\right]$$

do a by same semi-circular contour

$$\boxed{= 2\pi i \left[\frac{f(a)}{2} - \frac{f(-a)}{2}\right] = 2\pi i \left[i v(a)\right]}$$

(b) $\oint \frac{2a f(z)}{(z^2-a^2)} dz$ over



$\rho > a$

again the integral on the semicircle $\to 0$ if $f(z) \to 0$ fast enough as $\rho \to \infty$

$$\Rightarrow = \int_{-\infty}^{\infty} \frac{f(x)}{x-a} dx - \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x+a}$$

$$= \int_{-\infty}^{\infty} \frac{2a f(x)}{(x^2-a^2)} dx = 2\pi i^2 v(a)$$

$$\to \int_{-\infty}^{\infty} \frac{2a u(x)\,dx}{x^2-a^2} = -2\pi v(a)$$

$$\boxed{\int_{0}^{\infty} \frac{2a u(x)\,dx}{x^2-a^2} = -\pi v(a)}$$

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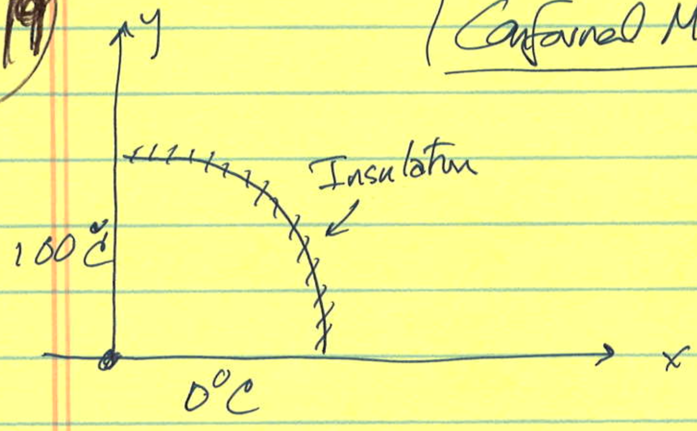
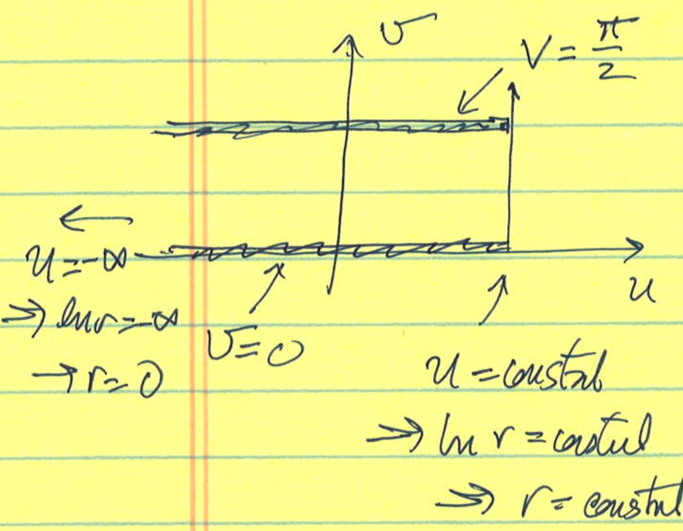
Conformal Mapping

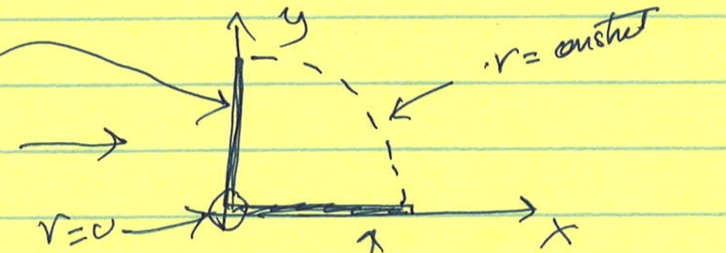


find $T(x,y)$

$$f(z) = \ln z = \ln r + \ln e^{i\theta} = \ln r + i\theta$$

$$= u + iv \Rightarrow \boxed{u = \ln r, \quad v = \theta}$$





(i) $\theta = \tan^{-1}\frac{y}{x} = 0 \Rightarrow$ i.e. y
x

(ii) $\theta = \tan^{-1}\frac{y}{x} = \frac{\pi}{2} \Rightarrow x = 0$

The $v(x,y)$ solution yields the correct BCs in the x-y plane.

In u-v plane

$$\boxed{T = \frac{100^\circ}{\left(\frac{\pi}{2}\right)} v = \frac{200^\circ}{\pi} v = \frac{200^\circ}{\pi}\theta = \frac{200^\circ}{\pi}\tan^{-1}\left(\frac{y}{x}\right)}$$

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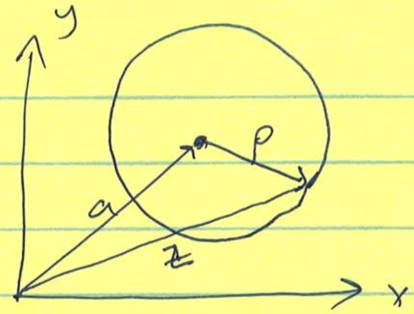

① $z = a + \rho e^{i\theta}$

② $f(z) = \underbrace{u(x,y) + iv(x,y)}_{\text{analytic}}$

Use Cauchy Integral Theorem.

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz,$$ $f(z)$ is analytic in and on C.

$$f(a) = \frac{1}{2\pi i}\oint \frac{u(x,y)+iv(x,y)}{z-a}\,dz$$

Let $z = a+\rho e^{i\theta} \Rightarrow dz = i\rho e^{i\theta}d\theta$

$$= \frac{1}{2\pi i}\oint \frac{u(r,\theta)+iv(r,\theta)}{\rho e^{i\theta}}\, i\rho e^{i\theta} d\theta$$

$$u(a)+iv(a) = \frac{i}{2\pi i}\oint \left(u(r,\theta)+iv(r,\theta)\right)d\theta$$

$$\Rightarrow \boxed{\begin{aligned} u(a) &= \frac{\oint u(r,\theta)\,d\theta}{2\pi} = \langle u \rangle \\ v(a) &= \frac{1}{\cancel{2\pi}}\oint \frac{v(r,\theta)\,d\theta}{2\pi} = \langle v \rangle \end{aligned}}$$