HW #6

Problems 10.2.7, 10.3.12, 10.8.6, 10.8.11, 10.10.8, 10.11.10, 10.11.12

35. 10.2.7: Show that the direct product of a vector and a 3rd rank tensor is a 4th rank tensor.

(I) (a) $u'_k = \sum_{i=1}^{3} a_{ki} u_i \leftarrow$ Vector

(b) $T'_{lmn} = \sum_{o=1}^{3} \sum_{p=1}^{3} \sum_{q=1}^{3} a_{lo} a_{mp} a_{nq} T_{opq}$

$$u'_k T'_{lmn} = \sum_{i=1}^{3} a_{ki} u_i \sum_{o=1}^{3} \sum_{p=1}^{3} \sum_{q=1}^{3} a_{lo} a_{mp} a_{nq} T_{opq}$$

insert

$$= \sum_{i=1}^{3} \sum_{o=1}^{3} \sum_{p=1}^{3} \sum_{q=1}^{3} a_{ki} a_{lo} a_{mp} a_{nq} u_i T_{opq}$$

$(u_i T_{opq})$

transforms as a 4th rank tensor

(II) (a) $T'_{kl} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ki} a_{lj} T_{ij}$

(b) $U'_{mn} = \sum_{o=1}^{3} \sum_{p=1}^{3} a_{mo} a_{np} U_{op}$

all and product of op

$$T'_{kl} U'_{mn} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ki} a_{lj} T_{ij} \sum_{o=1}^{3} \sum_{p=1}^{3} a_{mo} a_{np} U_{op}$$

36. <u>10.3.12</u>

Given $X_{ijkl} A_{kl} = B_{ij}$ and A_{kl} & B_{ij} are tensors, show that X_{ijkl} is a tensor.

<u>Solution</u>

① $X'_{\alpha\beta\gamma\delta} A'_{\gamma\delta} = B'_{\alpha\beta}$ transform $B_{ij} \rightarrow B'_{\alpha\beta}$

② $X'_{\alpha\beta\gamma\delta} A'_{\gamma\delta} = a_{\alpha i} a_{\beta j} B_{ij}$

③ $X'_{\alpha\beta\gamma\delta} A'_{\gamma\delta} = (a_{\alpha i} a_{\beta j})[X_{ijkl} A_{kl}]$ from given

$= (a_{\alpha i} a_{\beta j} X_{ijkl})[a_{\gamma k} a_{\delta l} A'_{\gamma\delta}]$ inverse $a^{-1}_{kl} = a_{lk}$, for rotation

$\Rightarrow [X'_{\alpha\beta\gamma\delta} - a_{\alpha i} a_{\beta j} X_{ijkl} a_{\gamma k} a_{\delta l}] A'_{\gamma\delta} = 0$

$[X'_{\alpha\beta\gamma\delta} - a_{\alpha i} a_{\beta j} a_{\gamma k} a_{\delta l} X_{ijkl}] \underline{A'_{\gamma\delta}} = 0$

arbitrary

$$\Rightarrow \boxed{X'_{\alpha\beta\gamma\delta} = a_{\alpha i} a_{\beta j} a_{\gamma k} a_{\delta l} X_{ijkl}}$$

$\rightarrow X_{ijkl}$ transforms like a Cartesian tensor

37. 10.8.6

find ds^2, scale factors, $d\vec{s}$, volume (or area) element, $\hat{a}$, $\hat{e}$ for the parabolic coordinates,

$$\begin{cases} x = \frac{1}{2}(u^2 - v^2) \\ y = uv \\ z = z \end{cases}$$

Sol^n

$\rightarrow$ $dx = u\,du - v\,dv$

a) $dy = u\,dv + v\,du$

$dz = dz$

and $ds^2 = u^2du^2 + v^2dv^2 - 2uv\,\cancel{du\,dv}$

$+ u^2dv^2 + v^2du^2 + 2\cancel{uv}\,dv\,du$

$+ dz^2$

$$\boxed{ds^2 = (u^2+v^2)(du^2+dv^2) + dz^2}$$

b) $d\vec{s} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$= du\underbrace{\left[\hat{i}u + \hat{j}v\right]}_{\hat{e}_u\sqrt{u^2+v^2}} + dv\underbrace{\left[-\hat{i}v + \hat{j}u\right]}_{\vec{e}_v\sqrt{v^2+u^2}} + dz\underbrace{\left[\hat{k}\right]}_{\hat{e}_z}$$

$$\Rightarrow \boxed{h_u = \sqrt{u^2+v^2},\ h_v = \sqrt{u^2+v^2},\ h_z = 1}$$

c) transformation matrix

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} du \\ dv \\ dz \end{pmatrix}$$

to get the volume element find the Jacobian,

$$\begin{vmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{vmatrix} = u^2+v^2$$

$$\Rightarrow \boxed{dx\,dy\,dz = (u^2+v^2)\,du\,dv\,dz}$$

38. 10.8.11

Find $\vec{v}$ and $\vec{a}$ for parabolic cylindrical coordinates

a) $d\vec{s} = du\,(\hat{i}u + \hat{j}v) + dv\left[-\hat{i}v + \hat{j}u\right] + dz\,\hat{k}$

$$\Rightarrow \boxed{\vec{v} = \frac{d\vec{s}}{dt} = \underbrace{(\hat{i}u + \hat{j}v)}_{\hat{e}_u}\frac{du}{dt} + \underbrace{(-\hat{i}v + \hat{j}u)}_{\hat{e}_v}\frac{dv}{dt} + \underbrace{\hat{k}}_{\hat{e}_z}\frac{dz}{dt}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (\hat{i}\dot{u} + \hat{j}\dot{v})\frac{du}{dt} + (\hat{i}u + \hat{j}v)\frac{d^2u}{dt^2}$$
$$+ (-\hat{i}\dot{v} + \hat{j}\dot{u})\frac{dv}{dt} + (-\hat{i}v + \hat{j}u)\frac{d^2v}{dt^2} + \hat{k}\frac{d^2z}{dt^2}$$

or

b) $d\vec{s} = \hat{e}_u\,du + \hat{e}_v\,dv + \hat{e}_z\,dz$

$$\Rightarrow \vec{v} = \hat{e}_u\frac{du}{dt} + \hat{e}_v\frac{dv}{dt} + \hat{e}_z\frac{dz}{dt}$$

$\hat{k}$ is time-independent

and

$$\vec{a} = \frac{d\vec{v}}{dt} = \hat{e}_u\frac{d^2u}{dt^2} + \frac{d\hat{e}_u}{dt}\frac{du}{dt} + \hat{e}_v\frac{d^2v}{dt^2} + \frac{d\hat{e}_v}{dt}\frac{dv}{dt} + \hat{e}_z\frac{d^2z}{dt^2}$$
$$= \hat{e}_u\frac{d^2u}{dt^2} + (\hat{i}\dot{u} + \hat{j}\dot{v})\frac{du}{dt} + \hat{e}_v\frac{d^2v}{dt^2} + (-\hat{i}\dot{v} + \hat{j}\dot{u})\frac{dv}{dt}$$
$$+ \hat{e}_z\frac{d^2z}{dt^2}$$

39. 10.10.8

Using $ds^2 = g_{ij} dx^i dx^j$ show that g_{ij} is a second rank, covariant tensor

Sol^n

$$ds^2 = g_{ij} dx^i dx^j = g'_{\mu\nu} dx^{\mu\prime} dx^{\nu\prime}$$

scalar (invariant)

$$\Rightarrow g_{ij} dx^i dx^j = g'_{\mu\nu} dx^{\mu\prime} dx^{\nu\prime}$$

$$= g'_{\mu\nu} \left[\frac{\partial x'_\mu}{\partial x_i} dx^i\right]\left[\frac{\partial x'_\nu}{\partial x_j} dx^j\right]$$

$$\Rightarrow 0 = \underbrace{\left[g_{ij} - g'_{\mu\nu} \frac{\partial x'_\mu}{\partial x_i}\frac{\partial x'_\nu}{\partial x_j}\right]}_{0} dx^i dx^j$$

$$\Rightarrow \qquad g_{ij} = \frac{\partial x'_\mu}{\partial x_i}\frac{\partial x'_\nu}{\partial x_j} g'_{\mu\nu}$$

$$\frac{\partial x_i}{\partial x'_\mu} g_{ij} = \frac{\partial x_i}{\partial x'_\mu}\frac{\partial x'_\mu}{\partial x_i}\frac{\partial x'_\nu}{\partial x_j} g_{\mu\nu}$$

$$\boxed{\frac{\partial x_j}{\partial x'_\nu}\frac{\partial x_i}{\partial x'_\mu} g_{ij} = g_{\mu\nu}}$$

$\Rightarrow g_{ij}$ transforms like a 2^nd rank, covariant tensor

4a. Prob 10.11.1

Show that the elements of R_{ij} (rotation matrix) are the elements of a Cartesian tensor

① $r_i = R_{ij} r_j$

② in transformed frame, look at

$$r'_\alpha = R'_{\alpha\beta} r'_\beta$$

$$(a_{\alpha i} r_i)$$

$$(a_{\alpha i}[R_{ij} r_j])$$

$$(a_{\alpha i} R_{ij} R_{\beta j} r'_\beta)$$

$$\Rightarrow (R_{\alpha i} R_{ij} R_{\beta j} r'_\beta) = R'_{\alpha\beta} r'_\beta$$

$$(R_{\alpha i} R_{ij} R_{\beta j} - R'_{\alpha\beta}) \times r'_\beta = 0$$

$$\Rightarrow \text{if } r'_\beta \neq 0$$

(a) $\Rightarrow R'_{\alpha\beta} = (R_{\alpha i} \underbrace{R_{ij}) R_{\beta j}}$

$R_{ij} R_{\beta j} = R_{\beta j} R_{ij}$ (as can be verified by substitution)

$R'_{\alpha\beta} = (R_{\alpha i} R_{\beta j} R_{\beta j}) \Rightarrow R_{ij}$ transforms correctly

(b) or use Result of Problem 1 directly $\Rightarrow$

$\Rightarrow R'_{\alpha\beta} = R_{\alpha i} R_{ij} R_{\beta j}$ ← similarity transformation

$\Rightarrow R_{ij}$ a 2nd rank tensor

#1. Prob 10.11.12

a) $$\begin{pmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3} \end{pmatrix} = J = \text{Jacobian matrix}$$

and

$J = |J| = 1$; Jacobian in Cartesian coordinates

$\neq 1$, non-Cartesian systems

Using (10.1) & (10.2)

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

and

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & -r\sin\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

find $J = |J|$

b) $$J = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$|J| = 0 + r^2\cos^2\theta\sin\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi$$
$$+ r^2\sin\theta\cos^2\theta\sin^2\phi + r^2\sin^3\theta\cos^2\phi$$
$$= r^2\Big[\sin\theta\big(\cos^2\theta\,\cancel{\tan}\overset{}{\cos^2\phi} + \sin^2\theta\sin^2\phi\big)$$
$$+ \sin\theta\big(\cos^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi\big)\Big]$$
$$= r^2\sin\theta\left[\cos^2\theta + \sin^2\theta\right]$$
$$\boxed{|J| = r^2\sin\theta = J}$$

$$T_{kl}'U_{mn}' = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{o=1}^{3}\sum_{p=1}^{3} a_{ki}a_{lj}a_{mo}a_{np}\left(T_{ij}T_{op}\right)$$

quantity transforms as a 4^{th} rank tensor

III (a) $T_{ab\cdots}' = \sum_{\alpha,\beta\cdots} a_{a\alpha}a_{b\beta}a_{c\gamma}\cdots T_{\alpha\beta\gamma\cdots}$

m indices, m indices

(b) $U_{AB\cdots}' = \sum_{A,B\cdots} a_{AA}a_{BB}a_{CC}\cdots U_{ABC\cdots}$

n indices, n indices

$$\Rightarrow T_{ab\cdots}'U_{AB\cdots}' = \sum_{\alpha,\beta\cdots,A,B,\cdots}\left(a_{a\alpha}a_{b\beta}a_{c\gamma}\cdots\right)\left(a_{AA}a_{BB}\cdots\right)\left(T_{\alpha\beta\gamma\cdots}U_{ABC\cdots}\right)$$

( ) transforms as an $(n+m)$ rank tensor