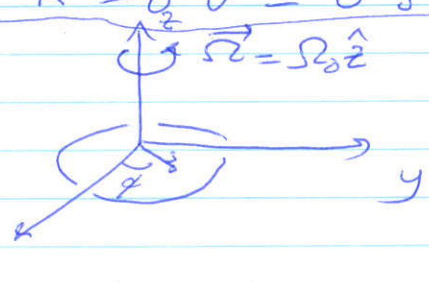
HW 1 Solutions

5.6

a) A phonograph record w/ uniform surface charge σ rotates at angular velocity Ω. Find $\vec{K}$

$$\boxed{\vec{K} = \sigma \vec{v} = \sigma \vec{\Omega} \times \vec{r} = \sigma \Omega_0 s \hat{\phi}}$$

where $\vec{\Omega} = \Omega_0 \hat{z}$



$\vec{v} = \vec{\Omega} \times \hat{r}$

b) A uniformly charged sphere of a radius R and charge Q spins at $\vec{\Omega} = \Omega_0 \hat{z}$, find $\vec{J}$

$$\vec{J} = \rho \vec{v} = \rho \vec{\Omega} \times \vec{r} = \rho (0, 0, \Omega_0) \times (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

$$= \rho(-\Omega_0 r\sin\theta\sin\phi, \Omega_0 r\sin\theta\cos\phi, 0)$$

$$= -\rho r \Omega_0 \sin\theta \underbrace{(\sin\phi, -\cos\phi, 0)}_{-\hat{\phi}}$$

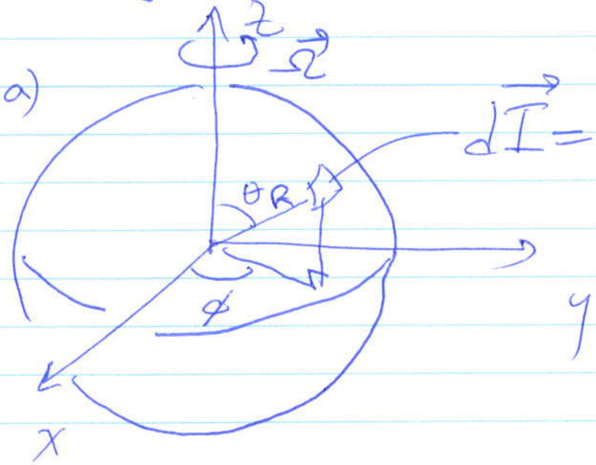
$$\boxed{\vec{J} = \rho r \Omega_0 \sin\theta \hat{\phi}}$$

where $\rho = \dfrac{3QR^3}{4\pi}$

5.12

Find the magnetic field at the center of an uniformly charged spherical shell of radius R and charge Q spinning w/ $\vec{\Omega} = \Omega_0 \hat{z}$

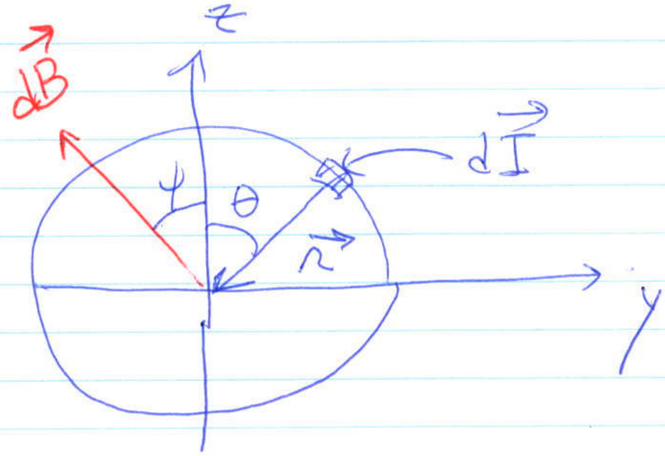
a)



$$d\vec{I} = \vec{K}\,dS = \underbrace{(\sigma\Omega_0 R\sin\theta\,\hat{\phi})}_{\vec{K}}\underbrace{\left(\overset{R\sin\theta\,d\phi\,R\,d\theta}{\cancel{\Omega_0 R\sin\theta}}\right)}_{dS}$$

$$\boxed{d\vec{I} = \sigma\Omega_0 R^3 \sin^2\theta\,d\theta\,d\phi\,\hat{\phi}}$$

b) Biot-Savart law



$$d\vec{B} = \frac{d\vec{I}\times\hat{r}}{r^2}\frac{\mu_0}{4\pi}$$

(i) By symmetry, the current element at $-y$ produces a field that cancels horizontal components $\Longrightarrow$ dB has only a z-component

$$dB_z = \hat{z}\cdot d\vec{B} = \frac{(\sigma\Omega_0 R^3\sin^2\theta\,d\theta\,d\phi)\,R}{R^3}\cos\psi\,\frac{\mu_0}{4\pi}$$

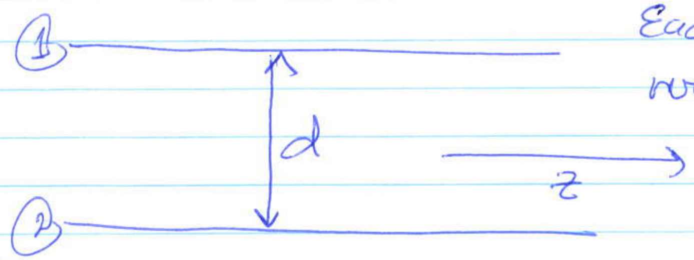
$$\cos\psi = \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\Rightarrow B_z = \frac{\mu_0}{4\pi}\sigma\Omega_0 R\int_0^{\pi}\int_0^{2\pi}\underbrace{\sin^3\theta\,d\theta}_{\sin^2\theta\;\sin\theta\,d\theta}\,d\phi$$

$$= \frac{\mu_0}{2}\sigma\Omega_0 R\int_0^{\pi}(1-\cos^2\theta)\,d(-\cos\theta)$$

$$= -\frac{\mu_0}{2}(\sigma\Omega_0 R)\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} = \frac{2\mu_0}{3}\sigma\Omega_0 R$$

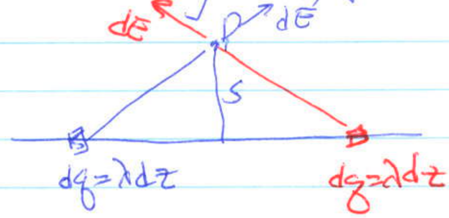
5.13



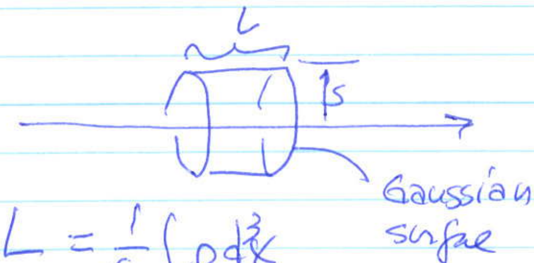
Each wire has $\lambda \left[\frac{Q}{L}\right]$ and moves w/ $\vec{V} = V_0 \hat{z}$

(i) Q: How fast must the wires move to cancel the electrostatic attraction?

A: find $\vec{E}$ of an infinite line charge



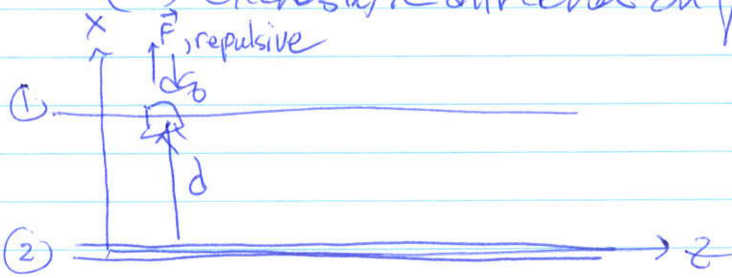
By indicated symmetry, there is only a radial field. Use Gauss's law



$$\oint \vec{E}\cdot d\vec{S} = E_s 2\pi s L = \frac{1}{\epsilon_0}\int \rho \, d^3x$$

$$= \frac{\lambda L}{\epsilon_0} \Rightarrow \boxed{E_s = \frac{\lambda}{2\pi\epsilon_0 s}}$$

(i) Electrostatic attraction on parts of line 1

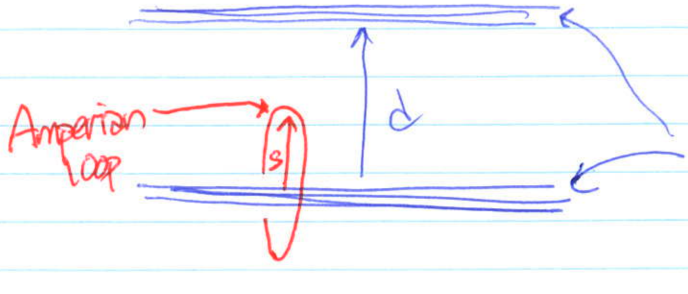


$$dF = \lambda dz \, E_s$$

$$= \lambda dz \frac{\lambda}{2\pi\epsilon_0 s}$$

$$\Rightarrow \boxed{\frac{dF}{dz} = \frac{\lambda^2}{2\pi\epsilon_0 s} \left[\text{force per length}\right]}$$

(ii) find Lorentz force



$$\vec{I} = \lambda V_0 \hat{z} \Rightarrow \int \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S}$$

$$B_\phi 2\pi s = \mu_0 \lambda V_0$$

$$\Rightarrow \boxed{B_\phi = \frac{\mu_0 \lambda v_0}{2\pi s}}$$

find Lorentz force on line element dz of wire (1)

$$d\vec{F} = \lambda v_0 \hat{z}\, dz \times \frac{\mu_0 \lambda v_0}{2\pi s}\hat{\phi}$$

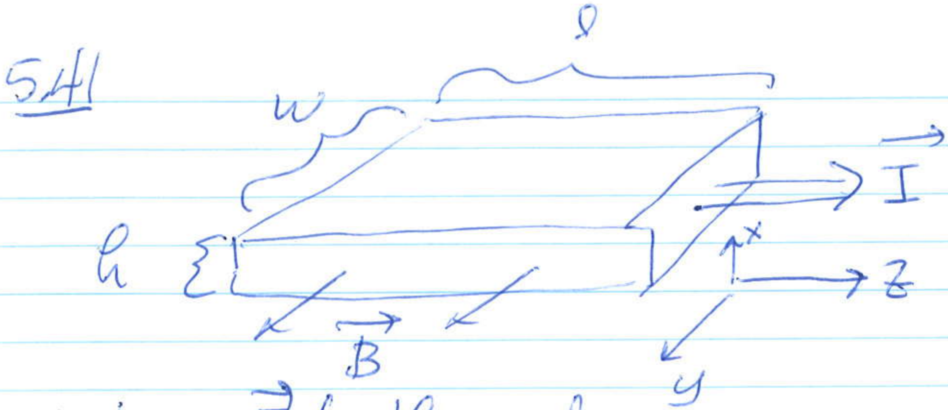
$$\boxed{\frac{d\vec{F}}{dz} = \frac{(\lambda v_0)^2}{2\pi}\mu_0 \frac{1}{s}(-\hat{s})}$$; pulls wires together

(ii) Total force per length dz = 0 when

$$\underbrace{\frac{\lambda^2 \hat{s}}{2\pi\epsilon_0 s}}_{dF_e} - \underbrace{\frac{(\lambda v_0)^2}{2\pi}\frac{\mu_0}{s}\hat{s}}_{dF_B} = 0$$

$$\boxed{v_0^2 = \frac{1}{\epsilon_0 \mu_0} = c^2}$$

5.41



a) Ignore $\vec{E}$ for the moment.

$$\vec{F}_B = q\vec{v}\times\vec{B} \Rightarrow d\vec{F}_B = I d\vec{z}\times \vec{B}_0 = I d\vec{z}\times B_0\hat{y}$$
$$= IB_0 dz(-\hat{x})$$

$\boxed{\text{downward if } q>0}$

b) In equilibrium, we get

$$\uparrow \uparrow \uparrow \uparrow E_0\hat{x}$$
$+ + + + + + +$

where $\vec{F}_B + \vec{F}_e = \vec{I}\times\vec{B} + q\vec{E} = 0$

$$IB_0(-\hat{x}) + qE_0\hat{x} = 0 \Rightarrow qvB_0 - qE_0 = 0$$

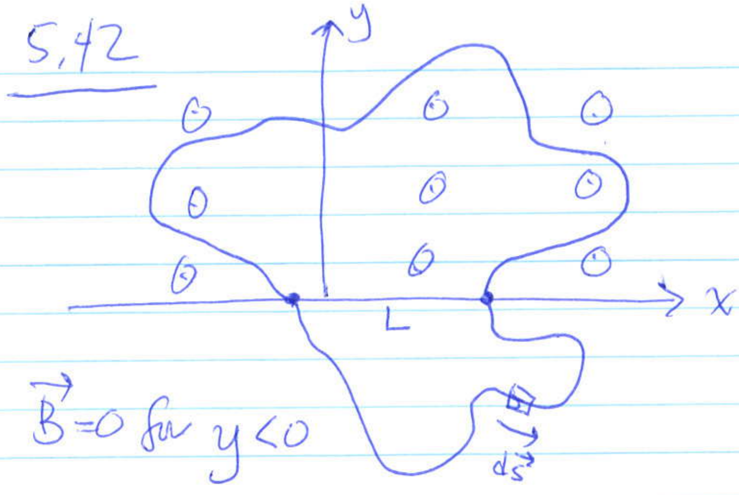
$$\boxed{E_0 = B_0 v}$$

$$\Rightarrow \boxed{\Delta V = -B_0 v h}$$

c) if $q<0$, $\vec{F}_B$ forces $-$ charges downward, because $I>0$. Because $I>0 \Rightarrow$ if $q<0$, $v<0$ and charges move in the ~~opposite~~ same sense of (b)

Because $q(<0)\downarrow \Rightarrow \vec{E} = -E_0\hat{x} \Rightarrow \boxed{\Delta V = +B_0 v h}$

5.42



$\vec{B} = B_0 \hat{z}$, out of paper

Show that the net force on the loop is

$$I B_0 L \hat{y}$$

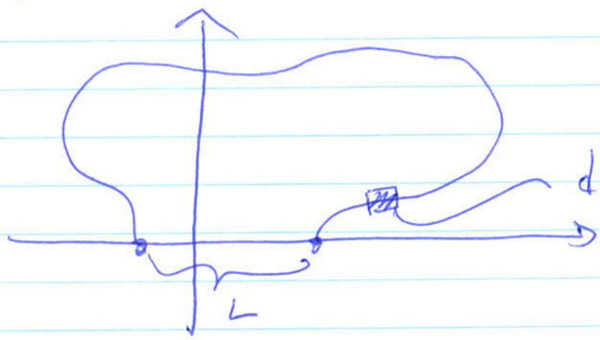
independent of the shape of the loop

Demonstration

$$d\vec{F} = I\, d\vec{s} \times B_0 \hat{z} = I B_0 \underbrace{\left(d\vec{s} \times \hat{z} \right)}$$

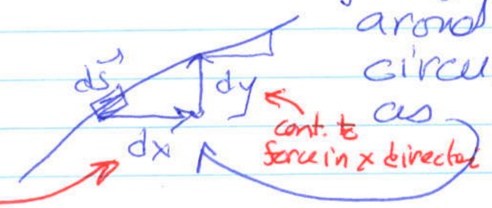
Solve in Cartesian coordinates

and we need only compute $(d\vec{s} \times \hat{z})$

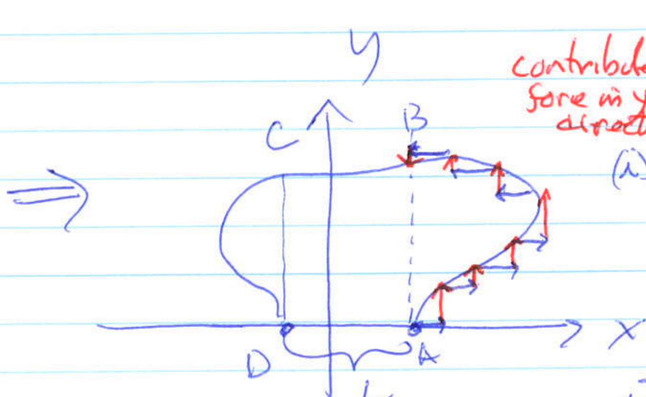


$d\vec{s} = \hat{\imath}\, dx + \hat{\jmath}\, dy$

so that we can integrate around the circuit as



$\Rightarrow$

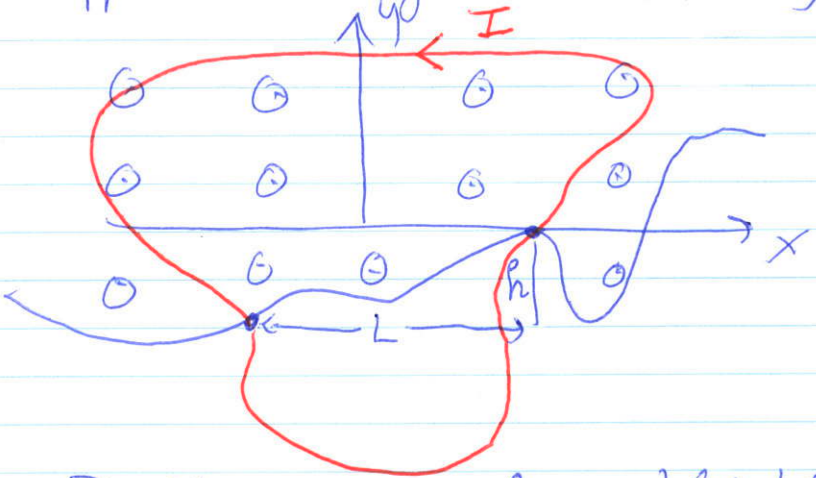


(i) We see that $\sum_i dx_i$ from $A \to B$ sum to ~~sum~~ zero. The same will be true for $C \to D$.

(ii) We see that the sum of $\sum_j dy_j$ from $A \to D$ is zero.

$\Rightarrow$ the only contribution to $(d\vec{s} \times \hat{z})$ comes from the path $A \to D$ and is of length L.

Suppose the $\vec{B}$ region has boundary



By the same argument we that the force in the $\hat{y}$ direction (due to path integral in dx) is still

$$I B_0 L$$

Now, by the same argument that there will be a force in $\hat{x}$ (or $-\hat{x}$) direction (due to the path integral in dy) will be

$$I B_0 h$$

5.45

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}$$ for a monopole. Suppose a particle of mass m and charge q_e moves in this field.

a) Find the acceleration

$$\vec{F} = q_e \vec{V}\times\vec{B} \longrightarrow \vec{a} = \frac{q_e}{m}\vec{V}\times\vec{B}$$

$$= \boxed{\frac{\mu_0}{4\pi}\frac{q_e q_m}{m r^2}(\vec{V}\times\hat{r})}$$

b) Show that $V^2 = |\vec{V}\cdot\vec{V}|$ is constant

take $\vec{V}\cdot\vec{F}$. Because $\vec{V}\times\vec{B}$ is a vector $\perp$ to both $\vec{V}$ & $\vec{B} \Rightarrow (\vec{V}\times\vec{B})\cdot\vec{V} = 0$

$$\Rightarrow \vec{V}\cdot\vec{F} = \vec{V}\cdot q_e(\vec{V}\times\vec{B}) = 0$$

and therefore,

$$\vec{V}\cdot\vec{F} = \vec{V}\cdot m\frac{d\vec{V}}{dt} = m\frac{d}{dt}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) = 0$$

and $\boxed{V^2 = |\vec{V}\cdot\vec{V}| = \text{constant}}$

c) Show $\vec{Q} = m(\vec{r}\times\vec{v}) - \frac{\mu_0}{4\pi}q_e q_m\hat{r}$ is conserved

take $$\frac{d\vec{Q}}{dt} = m\left[\cancel{\frac{d\vec{r}}{dt}\times\vec{v}}^{\,0} + \vec{r}\times\frac{d\vec{v}}{dt}\right] - \frac{\mu_0}{4\pi}q_e q_m\frac{d\hat{r}}{dt}$$

$$= m\left[\vec{r}\times\left(\frac{\mu_0}{4\pi}\frac{q_e q_m}{m}\right)\left(\vec{v}\times\frac{\hat{r}}{r^2}\right)\right] - \frac{\mu_0 q_e q_m}{4\pi}\left[\frac{\vec{v}}{r} - \frac{\vec{r}(\vec{v}\cdot\vec{r})}{r^3}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left[\vec{v}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{v}\cdot\vec{r})\right] - \frac{\mu_0}{4\pi}\frac{q_e q_m}{r}\left[\vec{v} - \frac{\vec{r}(\vec{v}\cdot\vec{r})}{r^2}\right]$$

$$= 0$$