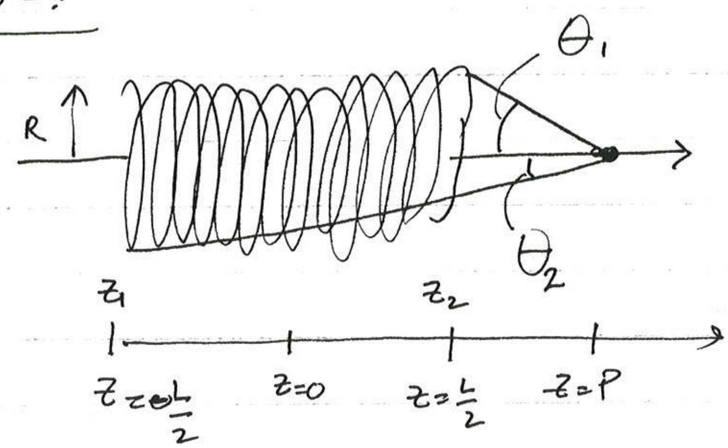
HW #2

S.11.1

Prob 5.11



Find the $\vec{B}$-field on the axis of a tightly wound solenoid consisting of $\underline{N}$ turns per unit length wrapped around a cylindrical tube of radius $\underline{R}$ and carrying current $\underline{I}$.

Example 6 $\Longrightarrow$ $B_z = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$ (5.34)

for the axis of a circular loop.

J = current per unit length along z = NI = $\left[\frac{A}{m}\right]$

$$d\vec{B}_s = \frac{\mu_0 I dz R^2}{2(R^2+z^2)^{3/2}}\hat{z}$$

$$\rightarrow \vec{B}_s = \frac{\mu_0 NIR^2}{2}\int_{z_1}^{z_2}\frac{dz}{(R^2+z^2)^{3/2}}\ ;$$

let: ① $\frac{z}{R} = \tan\theta$

$dz = R\,d\theta/\cos^2\theta$

② $\frac{R}{\sqrt{R^2+z^2}} = \cos\theta$

$$= \frac{\mu_0 NIR^2}{2}\int_{R\tan\theta_1}^{R\tan\theta_2}\frac{R\,d\theta}{\cos^2\theta}\frac{\cos^3\theta}{R^3}$$

$$= \frac{\mu_0 NI}{2}\left(\sin\theta\right)_{R\tan\theta_1}^{R\tan\theta_2}$$

$$= \frac{\mu_0 NI}{2}\left[\frac{-z_1}{\sqrt{R^2+z_1^2}} + \frac{z_2}{\sqrt{R^2+z_2^2}}\right]$$

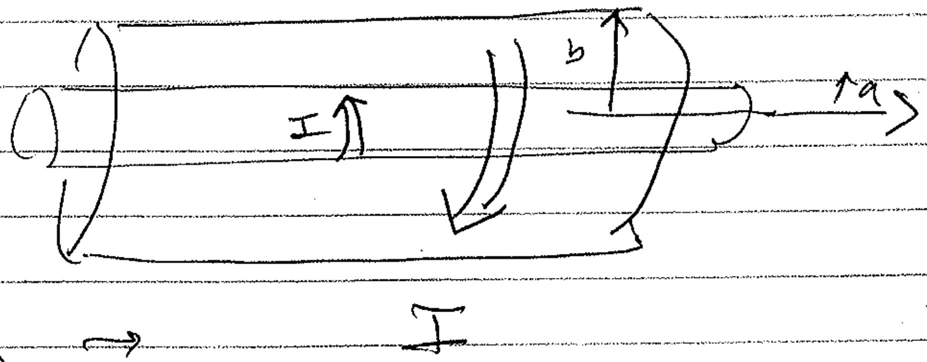
$$= \frac{\mu_0 NI}{2}\left[\frac{L/2}{\sqrt{R^2+L^2/4}} - \frac{-L/2}{\sqrt{R^2+L^2/4}}\right]$$

$$\boxed{\vec{B}_s = \frac{\mu_0 N I L}{2\sqrt{R^2 + L^2/4}}\,\hat{z}}$$

if $L \to \infty$,

$$\boxed{\vec{B}_s \simeq \mu_0 N I \,\hat{z}}$$

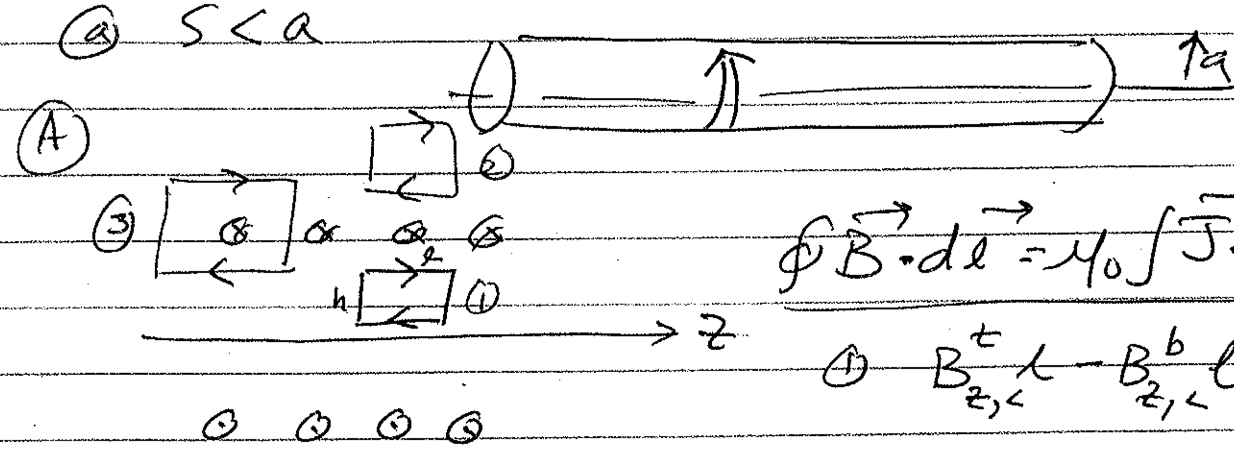
Prob 5.15



find $\vec{B}$ in (i) $s<a$, (ii) $a<s<b$, (iii) $b<s$

Sol^n

ⓐ $s<a$



$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{s}$

① $B^t_{z,<}\, l - B^b_{z,<}\, l = 0$

$\Rightarrow B_{z,<}$ is constant

② $B^t_{z,>}\, l - B^b_{z,>}\, l = 0$

and $B_{z,>} \to 0$ at ∞

$\Rightarrow B_{z,>} = 0$ $\quad n_1 l$

③ $\cancel{B^t_{z,>}}^{\,0}\, l - B^b_{z,<}\, l = N\mu_0 I$

$$\boxed{\vec{B}_{z,<} = -n_1 \mu_0 I \hat{z}}$$

Ⓑ Outer cylinder contributes, $\vec{B}^{outer}_{z,<} = +n_2 \mu_0 I \hat{z}$

$$\Rightarrow \boxed{\vec{B}_{Total} = \mu_0 I (n_2 - n_1)\hat{z} \quad , s<a}$$

(B) Between, $a < s < b$

$$\vec{B}_{Total} = \vec{B}_{inner} + \vec{B}_{outer}$$

$$= 0 + n_2 \mu_0 I \hat{z}$$

$$\boxed{\vec{B}_{Total} = \mu_0 n_2 I \hat{z} \quad , a < s < b}$$

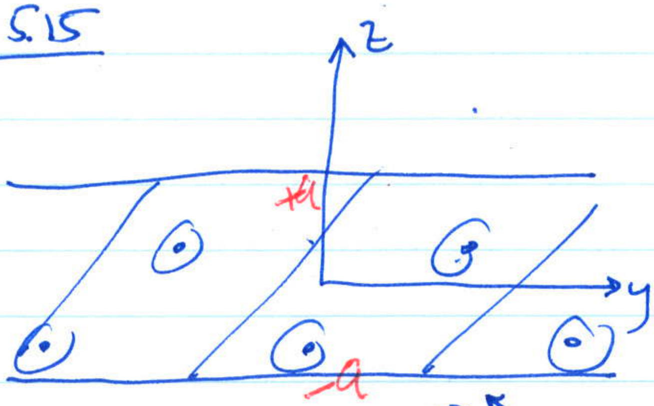
(c) Outside, $b < s$

$$\vec{B}_{Total} = \vec{B}_{inner} + \vec{B}_{outer}$$

$$= 0 + 0$$

$$\boxed{\vec{B}_{Total} = 0 \quad , b < s}$$

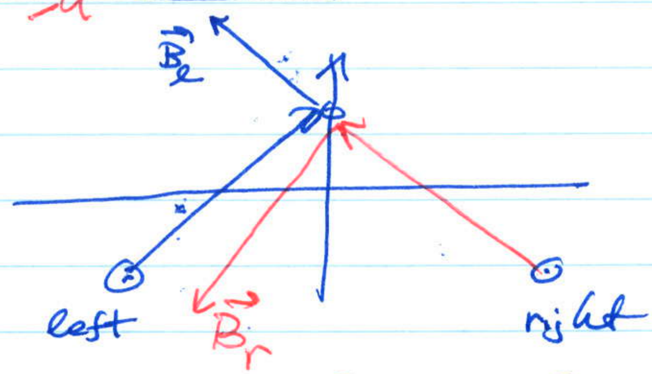
Prob 5.15



$\vec{J} = J_0 \hat{x}$

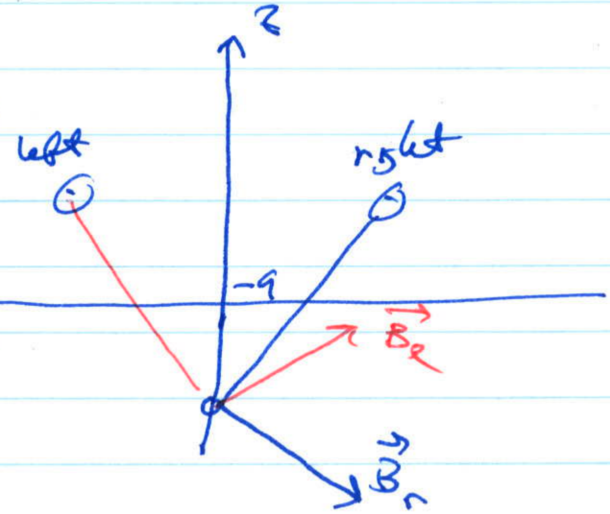
find $\vec{B}$ everywhere

a) for $z > a$



⇒ only B_y survives, B_z cancels. also $B_x = 0$ b/c $\vec{J}$ is in x direction

b) for $z < a$

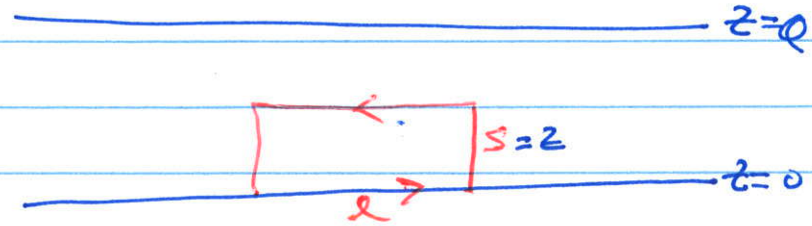


⇒ only B_y survives but is in opposite direction compared to the $z > a$ field.

c) ⇒ at $z = 0$, $B_y = 0$ (by symmetry)

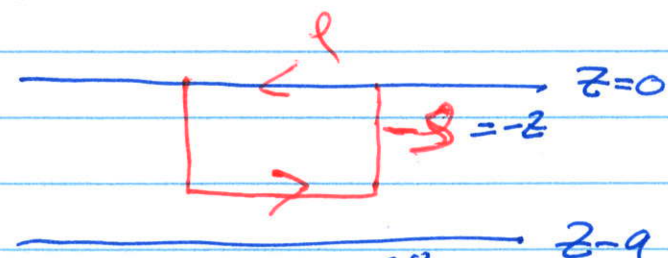
$\boxed{\text{find } \vec{B} \text{ using Ampere's law, } \oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S}}$

for $0 < z < a$



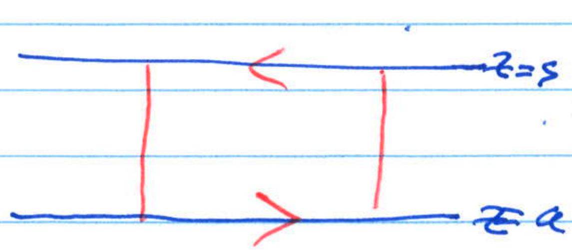
$$-B_y(s)\,l + \cancel{B_y(0)}^{0}\,l = \mu_0 J l s \Rightarrow \boxed{B_y(s) = -\mu_0 J_0 s}$$

for $-a < z < 0$



$$+B_y(-s)\,l + \cancel{B_y(0)}^{0}(-l) = +\mu_0 J_0 l s \Rightarrow \boxed{B_y(-s) = \mu_0 J_0 s}$$

for $a < z < s$



$$\Rightarrow -B_y(s)\,l + B_y(a)\,l = 0 \Rightarrow \boxed{B_y(s) = B_y(a)}$$

for $-a > z > -s$

$$\Rightarrow \boxed{B_y(-s) = B_y(-a)}$$

write this in a more sensible way using z, then

$$B_y(z) = \begin{cases} -\mu_0 J_0 a & z > a \\ -\mu_0 J_0 z & -a < z < a \\ \mu_0 J_0 a & z < -a \end{cases}$$

Prob 5.22

Suppose monopoles existed. How would you modify Maxwell's equations and the force law?

Current Maxwell & force law

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad \vec{\nabla}\cdot\vec{B} = 0$ & $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

$\underbrace{\vec{\nabla}\times\vec{E} = 0 \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}}_{\text{electrostatics \& magnetostatics}}$

a) Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \;,\; \vec{\nabla}\cdot\vec{B} = \boxed{c_1\rho_m}$$

$$\vec{\nabla}\times\vec{E} = \boxed{c_2\vec{J}_m} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

c_1 and c_2 are not clear (yet). For example, later we write

$\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$. Then $\vec{\nabla}\cdot\left[\cancelto{0}{\vec{\nabla}\times\vec{B}}\right] = \mu_0\left[\vec{\nabla}\cdot\vec{J} + \epsilon_0\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t}\right]$

$$0 = \mu_0\vec{\nabla}\cdot\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}$$

$$= \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} \text{ and charge is conserved}$$

but $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{E} = c_2\vec{J}_m - \frac{\partial\vec{B}}{\partial t}$ w/monopoles

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = c_2\vec{\nabla}\cdot J_m - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B}$$

$$= c_2\vec{\nabla}\cdot J_m - \frac{\partial}{\partial t}c_1\rho_m$$

$$= \frac{c_2}{c_1}\left[\vec{\nabla}\cdot J_m\right] - \frac{\partial\rho_m}{\partial t} = 0$$

this will be true if $\frac{c_2}{c_1} = -1$

(b) <u>force law</u>

$$\vec{F} = \left(\rho_e \vec{E} + \vec{J}_e \times \vec{B}\right) + \left(\rho_m \vec{B} + c_3 \vec{J}_m \times \vec{E}\right)$$

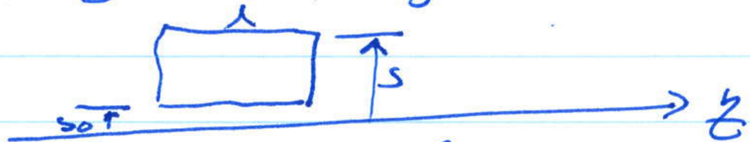
where c_3 is determined by units and charge conservation

Prob 5.26

(a) find $\vec{A}$ for an infinite line current.

(i) $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ if z and current I are aligned

(ii) $\int \vec{B}\cdot d\vec{S} = \int \vec{A}\cdot d\vec{\ell}$



(i) $\int \vec{B}\cdot d\vec{S} = \frac{\mu_0 I}{2\pi}\int_{s_0}^{s} dz\, ds\, \frac{1}{s} = \frac{\mu_0 I \ell}{2\pi}\ln\left(\frac{s}{s_0}\right)$

(ii) $\int \vec{A}\cdot d\vec{\ell} = A_z(s_0)\ell - A_z(s)\ell$

$$\Rightarrow \boxed{A_z(s) = A_z(s_0) - \frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{s_0}\right)}$$

(iv) find $\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s} s A_s + \frac{1}{s}\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z$

$\longrightarrow 0$, b/c $A_s = A_\phi = 0$ and A_z does not depend on z ✓

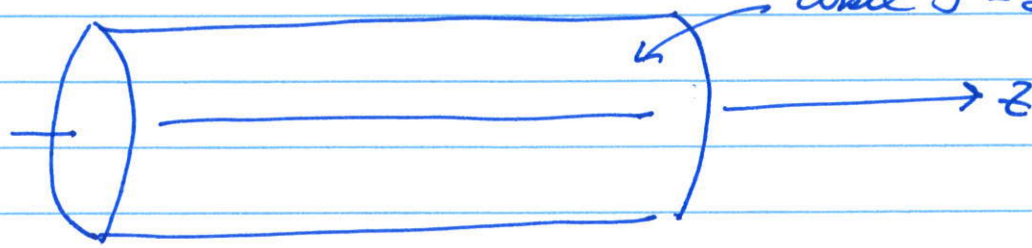
(v) find $\vec{\nabla}\times\vec{A} = \hat{s}\left[\cancel{\frac{1}{s}\frac{\partial A_z}{\partial \phi}}^{0}\right] + \left[-\frac{\partial A_z}{\partial s}\right]\hat{\phi}$

$= -\left[-\frac{\mu_0 I}{2\pi}\frac{1}{s}\right]\hat{\phi}$

$= \frac{\mu_0 I}{2\pi s}\hat{\phi}$ ✓

find $\vec{A}$ inside the wire

assume $\vec{J} = J_0 \hat{z}$



Inside wire $\vec{B} = \dfrac{\mu_0 J_0 \pi s^2}{2\pi s}\hat{\phi} = \mu_0 \dfrac{J_0 s}{2}\hat{\phi}$

from Ampere's law

find $\vec{A}$

$$\int \vec{B}\cdot d\vec{S} = \int \vec{A}\cdot d\vec{l}$$

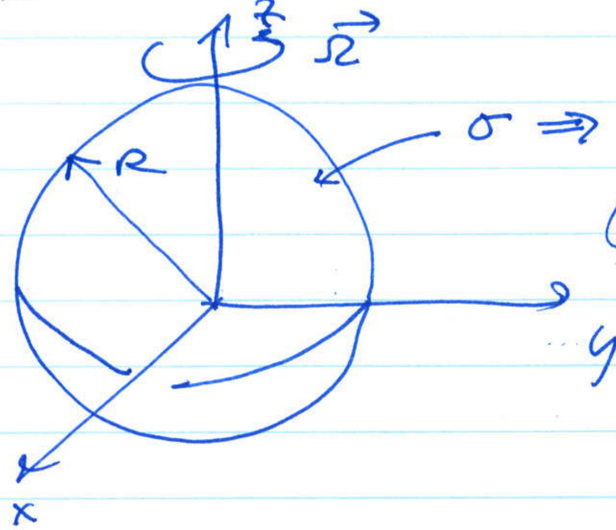
$$\int \mu_0 \frac{J_0}{2} s\,ds\,dz = A_z(0)l - A_z(s)l$$

$$\mu_0 \frac{J_0 s^2}{4}\Big|_0^s l = A_z(0)l - A_z(s)l$$

$$\boxed{A_z(s) = -\mu_0 \frac{J_0}{4} s^2 \hat{z}}$$

5.30

find the field inside a uniformly sphere of charge



$\sigma \Rightarrow \vec{K} = \sigma \vec{\Omega} \times \vec{r} \;\Big|$ at $r = R$

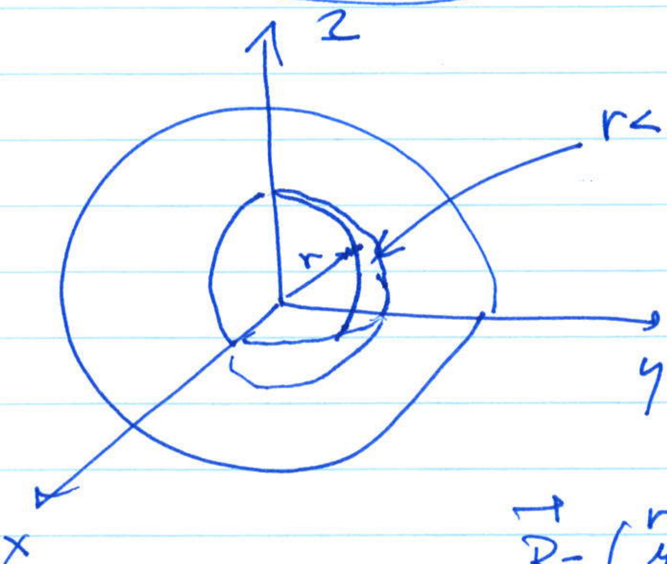
(a) Ex. 5.11

$$\Rightarrow \underbrace{\vec{B} = \frac{2}{3}\mu_0 \sigma \vec{\Omega} R}_{\text{inside a shell}}$$

(b) $\vec{B} = \vec{\nabla} \times \vec{A}$ outside shell

$$= \frac{\mu_0 R^4 \Omega \sigma}{3 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

note: $\boxed{\sigma = \dfrac{Q}{4\pi R^2}}$



$r < R$. Need to solve 2 integrations, one for $r' < r$ and one for $r' > r$

$\cancel{\vec{B} = \int_0^r \mu_0 r' \Omega \rho}$

$$\vec{B} = \int_0^r \frac{\mu_0 r'^4 \Omega \rho_0\, dr'}{3 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$$+ \int_r^R \frac{2}{3}\mu_0 \rho_0\, dr' \,\Omega_0 \hat{z} r'$$