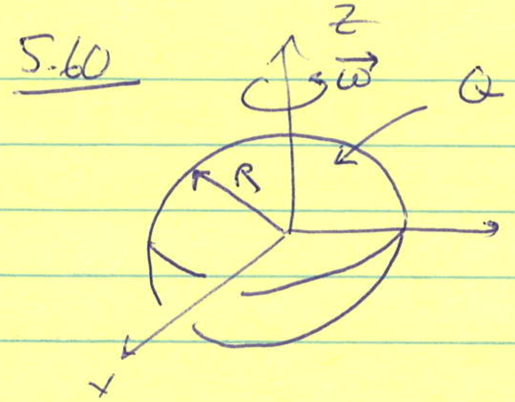
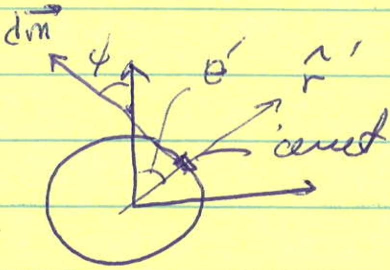
5.60



a) find $\vec{m}$ about z-axis

$$\vec{m} = \frac{1}{2}\int \vec{r}\,' \times \vec{J}(\vec{r}\,')\, d^3x'$$

$$= \frac{1}{2}\int r' \rho_0 \omega r' \sin\theta' (\hat{r}' \times \hat{\phi}')\, d^3x'$$



we want projection of $\hat{r}' \times \hat{\phi}'$ onto z. okay

(i) $\hat{r}' = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$, $\hat{\phi}' = (-\sin\phi', \cos\phi', 0)$

$$\Rightarrow (\hat{r}' \times \hat{\phi}') = (-\cos\theta'\cos\phi', -\cos\theta'\sin\phi', \sin\theta')$$

(ii) $\hat{z}' \cdot \hat{r}' = (0, 0, \sin\theta')$

$$\boxed{\Rightarrow \vec{m}_z = \frac{1}{2}\hat{z}\rho_0\omega \int r'^2 \sin^2\theta'\, r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi' = \frac{4\pi}{3}\rho_0 \omega \frac{R^5}{5}\hat{z}}$$

b) Find B_{ave} in sphere

Last week we found

$$\vec{B} = \frac{\mu_0 \omega \rho_0}{3}\left[\hat{r}\cos\theta\left(R^2 - \frac{3}{5}r^2\right) + \hat{\theta}\sin\theta\left(\frac{6r^2}{5} - R^2\right)\right]$$

rewrite this is in Cartesian coordinates

$$\vec{B} = \frac{\mu_0 R \rho_0}{3}\Big[(\cos\theta\sin\theta\cos\phi, \cos\theta\sin\theta\sin\phi, \cos^2\theta)\left(R^2 - \frac{3}{5}r^2\right)$$
$$+ (\cos\theta\sin\theta\cos\phi, \cos\theta\sin\theta\sin\phi, -\sin^2\theta)\left(\frac{6r^2}{5} - R^2\right)\Big]$$

$$= \frac{\mu_0 \Omega \rho_0}{3}\Big[\Big(\cos\theta\sin\theta\cos\phi\left\{\frac{3}{5}r^2\right\}, \cos\theta\sin\theta\sin\phi\left\{\frac{3}{5}r^2\right\},$$
$$\cos^2\theta\left(R^2 - \frac{3}{5}r^2\right) - \sin^2\theta\left(\frac{6r^2}{5} - R^2\right)\Big)\Big]$$

To find $\vec{B}_{ave}$, we will integrate the above over the sphere. Note:

$$d^3x = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

i) the integral over $d\phi$ $\Rightarrow$ x, y components will integrate to 0, leaving only the z-component.

ii) $$\int \vec{B}\, d^3x' = \hat{z}\,\frac{\mu_0 \Omega \rho_0}{3}\int \left\{\cos^2\theta\left(R^2 - \frac{3}{5}r^2\right) - \sin^2\theta\left(\frac{6r^2}{5} - R^2\right)\right\} d^3x'$$

$$= \hat{z}\,\frac{\mu_0 \Omega \rho_0}{3}\, 2\pi \left\{\int_0^\pi \int_0^R \left(\cos^2\theta\, \frac{3r^2}{5} - \frac{6r^2}{5} + R^2\right) r^2 dr \, \sin\theta\, d\theta\right\}$$

$$= \hat{z}\,\frac{2\pi \mu_0 \Omega \rho_0}{3}\int \left\{\cos^2\theta\, \frac{3r^2}{5} - \frac{6r^2}{5} + R^2\right\} r^2 dr \, \sin\theta\, d\theta$$

$$= \frac{\vec{m}}{2} \times 5 \mu_0 \frac{1}{R^5}\int \left\{\cos^2\theta\, \frac{3r^2}{5} - \frac{6r^2}{5} + R^2\right\} r^2 dr \, \sin\theta\, d\theta$$

(iii) $\vec{B}_{ave} = \dfrac{\int \vec{B}\, d^3x'}{\frac{4\pi}{3}R^3}$

$$= \hat{z}\, \frac{\mu_0 \Omega_0 \rho_0}{4\pi R^3} \underbrace{\int \left\{ \cos^2\theta \left(R^2 - \frac{3}{5} r^2\right) + \sin^2\theta \left(R^2 - \frac{6r^2}{5}\right) \right\} d^3x'}_{\text{do integral}}$$

$$= \iint \left\{ \cos^2\theta \left(\frac{R^5}{3} - \frac{3R^5}{25}\right) + \sin^2\theta \left(\frac{R^5}{3} - \frac{6R^5}{25}\right) \right\} \sin\theta\, d\theta\, d\phi$$

$$= 2\pi \int \left\{ \cos^2\theta \left(\frac{16R^5}{75}\right) + \sin^2\theta \left(\frac{7R^5}{75}\right) \right\} \sin\theta\, d\theta$$

$$= \frac{2\pi R^5}{75} \int \left\{ 16\cos^2\theta + 7\sin^2\theta \right\} d(-\cos\theta)$$

$$= \frac{40\pi R^5}{75}$$

$$\Rightarrow \vec{B}_{ave} = \frac{40\pi R^5}{75} \frac{\mu_0 \Omega_0 \rho_0}{4\pi R^3} \hat{z}$$

$$= \frac{2}{15} \mu_0 \Omega_0 \rho_0 R^2 \hat{z}$$

replace $\Omega_0 \rho_0 \hat{z}$ w/ $\vec{m}$

$$\Rightarrow \boxed{\vec{B}_{ave} = \frac{\mu_0}{4\pi} \frac{2\vec{m}}{R^3}}$$

(c) $\vec{m} = \frac{4\pi}{3}\rho_0 \Omega_0 \frac{R^5}{5}\hat{z}$

$$\Rightarrow \vec{A}_d = \frac{\mu_0}{\cancel{4\pi}} \frac{\cancel{4\pi}}{3} \frac{\rho_0 \Omega_0 R^5}{5} \frac{\hat{z}\times\hat{r}}{r^2}$$

$$\boxed{\vec{A}_d = \frac{\mu_0}{15} \frac{\rho_0 \Omega_0 R^5 \sin\theta}{r^2}\hat{\phi}}$$

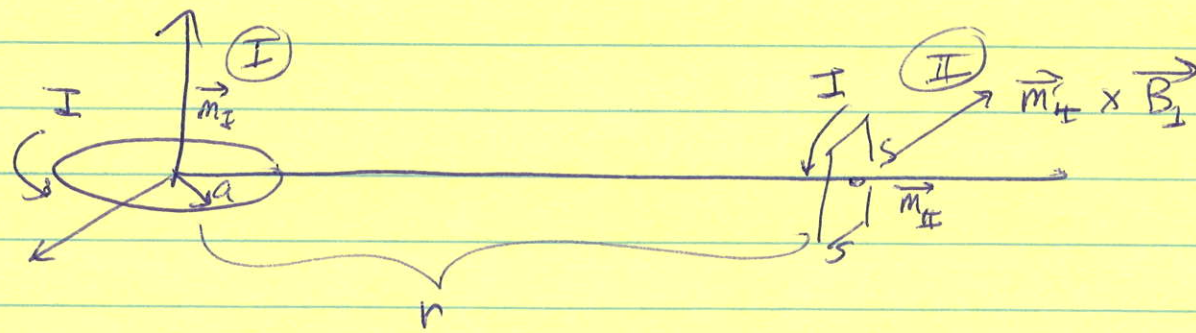
(d) from last week,

$$\boxed{\vec{A}_d = \frac{\mu_0 \Omega_0 \rho_0 R^5}{15}\left(\frac{\sin\theta}{r^2}\hat{\phi}\right)}$$

(e) Just did the tedious integral

6.1

find $\vec{N}$ for



when $a, s << r \Rightarrow$ treat as ideal dipoles

a) $\vec{B}_I = \frac{\mu_0}{4\pi}\frac{m_I}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

at $r >> a, s$ and $\theta = \frac{\pi}{2}$ (at second loop location)

$$\Rightarrow \boxed{\vec{B}_I = \frac{\mu_0}{4\pi}\frac{m_I}{r^3}(\hat{\theta})}$$

b) find torque $\vec{N} = \vec{m} \times \vec{B}$

$= \vec{m}_{II} \times \vec{B}_I$

$= m_{II} B_I$ into paper

$\Rightarrow$ square loop's $\vec{m}_{II}$ wants to point downward in $\hat{\theta}$ direction. It will stop when $\vec{m}_{II} // \vec{B}_I$

6-2

Show that $\vec{N} = \vec{m} \times \vec{B}$ for arbitrary current loops (can show for current distributions but is more involved)

Sol[n]

(i) $d\vec{F} = I\,d\vec{l} \times \vec{B}$

(ii) $d\vec{N} = \vec{r} \times (I\,d\vec{l} \times \vec{B})$

(a) Use Lagrange identity,

$$\Rightarrow \boxed{\vec{r} \times (I\,d\vec{l} \times \vec{B}) + I\,d\vec{l} \times (\vec{B} \times \vec{r}) + \vec{B} \times (\vec{r} \times I\,d\vec{l}) = 0}$$

suppress I

(b) look at $d\left[\vec{r} \times (\vec{r} \times \vec{B})\right]$

$$= d\vec{r} \times (\vec{r} \times \vec{B}) + \vec{r} \times (d\vec{r} \times \vec{B}) + \vec{r} \times (\cancelto{0}{\vec{r} \times d\vec{B}})$$

here $d\vec{r} = d\vec{l}$

$$\Rightarrow d\vec{l} \times (\vec{r} \times \vec{B}) = d\left[\vec{r} \times (\vec{r} \times \vec{B})\right] - \vec{r} \times (d\vec{l} \times \vec{B})$$

plug into Lagrange identity

$$0 = \vec{r} \times (d\vec{l} \times \vec{B}) + \left\{\vec{r} \times (d\vec{l} \times \vec{B}) - d\left[\vec{r} \times (\vec{r} \times \vec{B})\right]\right\} + \vec{B} \times (\vec{r} \times d\vec{l})$$

$$\Rightarrow 2\vec{r} \times (d\vec{l} \times \vec{B}) = d\left[\vec{r} \times (\vec{r} \times \vec{B})\right] - \vec{B} \times (\vec{r} \times d\vec{l})$$

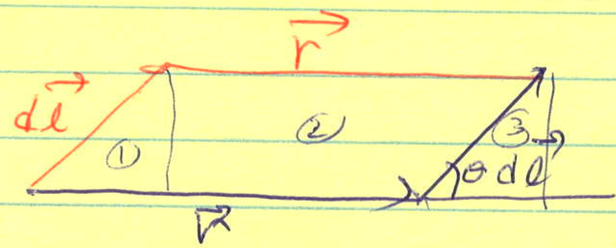
$$\Rightarrow d\vec{N} = \frac{I}{2}\left\{ d\left[\vec{r}\times(\vec{r}\times\vec{B})\right] - \vec{B}\times(\vec{r}\times d\vec{l})\right\}$$

$$\vec{N} = \frac{I}{2}\oint d\left[\cancelto{0}{\vec{r}\times(\vec{r}\times\vec{B})}\right] - \frac{I}{2}\oint \vec{B}\times(\vec{r}\times d\vec{l})$$

$$= \underbrace{\left[\frac{I}{2}\oint(\vec{r}\times d\vec{l})\right]}_{\text{what is } \frac{1}{2}\int \vec{r}\times d\vec{l}\ ?}\times\vec{B}$$

Geometrically, consider a parallelogram

(a)



Area of parallelogram is

$$A = \frac{1}{2}\left(\cancel{|d\vec{l}|\cos\theta\, |d\vec{l}|\sin\theta}\right) + \left(|\vec{r}| \cancel{- |d\vec{l}|\cos\theta}\right)|d\vec{l}|\sin\theta$$
$$+\frac{1}{2}\left(\cancel{|d\vec{l}|\cos\theta\, |d\vec{l}|\sin\theta}\right)$$
$$= |\vec{r}|\,|d\vec{l}|\sin\theta \Rightarrow \boxed{\vec{A} = \vec{r}\times d\vec{l}}$$

(b) $\Rightarrow$ area swept out by $\vec{r}$ is then $\frac{1}{2}\vec{A}$

$$\Longrightarrow \vec{N} = I\left(\frac{1}{2}\oint \vec{r}\times d\vec{l}\right)\times\vec{B} = I\int d\vec{S}\times\vec{B}$$
$$= \vec{m}\times\vec{B}$$