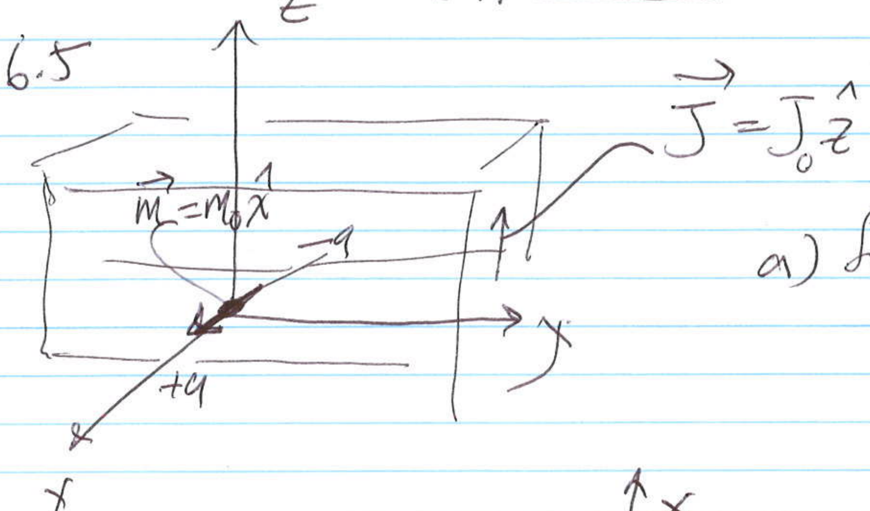
HW4

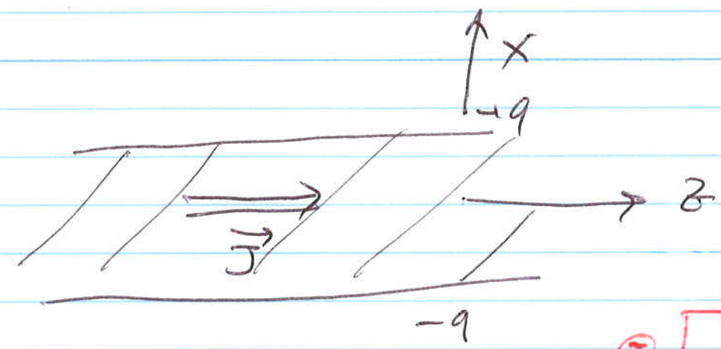
6.5



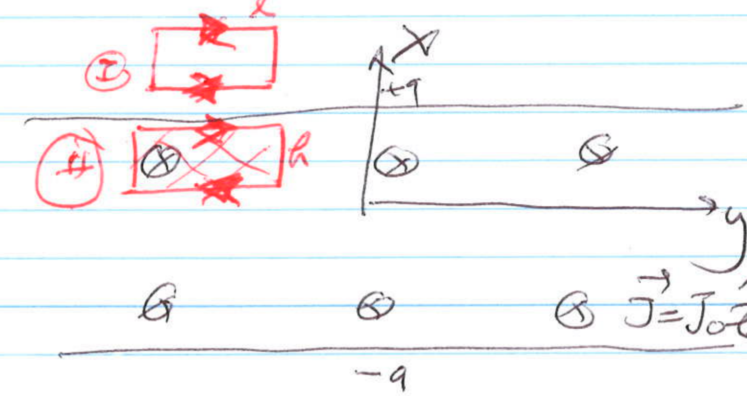
a) find the force on $\vec{m}$ using

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

(i)



we see that field will point in $\hat{y}$ direction for $x>0$ and in $-\hat{y}$ direction for $x<0$



(ii) Use Ampere's law

$$\int \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S}$$

$x > a$ I $\longrightarrow +B_y^{top} l - B_y^{bott} l = 0 \longrightarrow B_y = \text{constant}$

$0 < x < a$ II $\longrightarrow +B_y(x) l - B_y(0) l = \mu_0 J_0 l x$

$$\longrightarrow B_y(x) = \cancel{B_y(0)}^{=0,\ \text{by symmetry}} + \mu_0 J_0 x$$

and, below the $x=0$ plane same.

$$\boxed{\vec{B}_y(x) = +\mu_0 J_0 x \hat{y}}$$

$$\rightarrow \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = \vec{\nabla}(m_0\hat{x}\cdot\mu_0 J_0 x\hat{y})$$

$$\boxed{\vec{F} = 0}$$

b) $$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = \vec{\nabla}(m_0\hat{y}\cdot\mu_0 J_0 x\hat{y})$$

$$\boxed{\vec{F} = m_0\mu_0 J_0\hat{x}}$$

c) $$\vec{F} = \vec{\nabla}(\vec{p}\cdot\vec{E}) = \vec{p}\times(\vec{\nabla}\times\vec{E}) + \vec{E}\times(\underbrace{\vec{\nabla}\times\vec{p}}_{0}) + (\vec{p}\cdot\vec{\nabla})\vec{E} + (\underbrace{\vec{E}\cdot\vec{\nabla})\vec{p}}_{0}$$

ID #4

$\vec{p}$ = constant vector

$$= \vec{p}\times(\underbrace{\vec{\nabla}\times\vec{E}}_{0}) + (\vec{p}\cdot\vec{\nabla})\vec{E}$$

electrostatics

$$= (\vec{p}\cdot\vec{\nabla})\vec{E} \checkmark$$

In general $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_f}$ and so, $\vec{\nabla}\times\vec{B}$ is generally $\neq 0$

and $\vec{\nabla}(\vec{m}\cdot\vec{B}) \neq (\vec{m}\cdot\vec{\nabla})\vec{B}$

in general

6.10

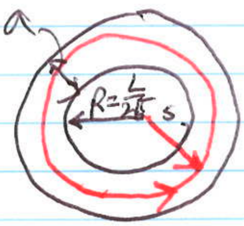
$\vec{M} = M_0 \hat{\phi}$

→ Current wraps around toroid as shown

a) from class notes we know that a torus w/ uniform cross-section has only an azimuthal field

Top

⇒



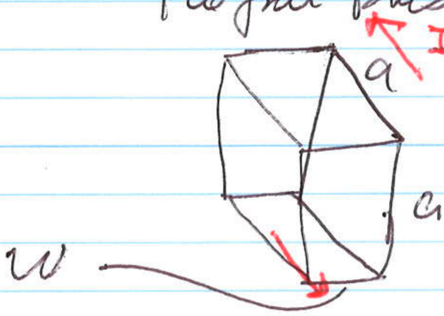
$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{s}$

(a) $R < s < a$

$B_\phi 2\pi s = \mu_0 M_0 2\pi R$

$\Rightarrow \boxed{\vec{B}_\phi = \mu_0 \frac{M_0 R}{s} \hat{\phi}}$

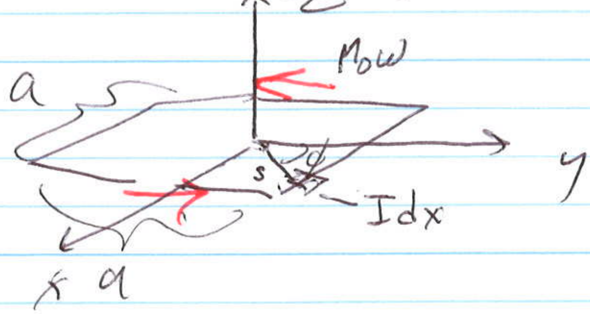
b) superpose a square loop w/ reversed current onto the full torus



, recall: $|\vec{K}_b| = M_0$, $I = M_0 w$

where w is the width of the gap

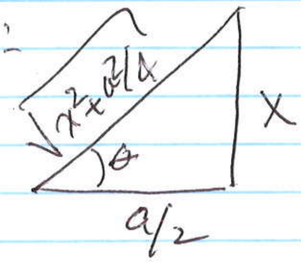
find the field of the loop below at its center. To do so, find the field due to one side at $(x, y, z) = (0,0,0)$



Only have z-component of field

$$dB_z = \frac{\mu_0}{4\pi}\frac{M_0 w\, dx \sin\left(\frac{\pi}{2}-\phi\right)}{\left(x^2+a^2/4\right)} \rightarrow B_z = \frac{\mu_0 T M_0 w}{2\pi}\int_{-\frac{a}{2}}^{0}\frac{\frac{a}{2}\,dx}{\left(x^2+\frac{a^2}{4}\right)^{3/2}}$$

Use:



$$\Rightarrow \cos\theta = \frac{a}{2\sqrt{x^2+\frac{a^2}{4}}} \longrightarrow -\sin\theta\, d\theta = -\frac{a}{2}\frac{x\,dx}{\left(x^2+a^2/4\right)^{3/2}}$$

$$\sin\theta = \frac{x}{\sqrt{x^2+\frac{a^2}{4}}}$$

$$B_z = \frac{\mu_0 M_0 w}{2\pi}\int\left(\sqrt{x^2+a^2/4}\cos\theta\right)\frac{2}{a}\sin\theta\, d\theta\,\frac{1}{x}\Big) \cancel{d\theta}$$

$$= \frac{\mu_0 M_0 w}{2\pi}\int\frac{\cos\theta\, d\theta \sin\theta}{\sin\theta}\frac{2}{a}$$

$$= \frac{\mu_0 M_0 w}{2\pi}\frac{2}{a}\sin\theta\Big|_{\theta_1}^{\theta_2} = \frac{\mu_0 M_0 w}{\pi a}\left[\frac{x}{\sqrt{x^2+a^2/4}}\right]_{-a/2}^{0}$$

$$= \frac{\mu_0 M_0 w}{\pi a}\left[\frac{a/2}{\sqrt{a^2/2}}\right]$$

$$= \frac{\mu_0 M_0 w}{\pi a\sqrt{2}}$$

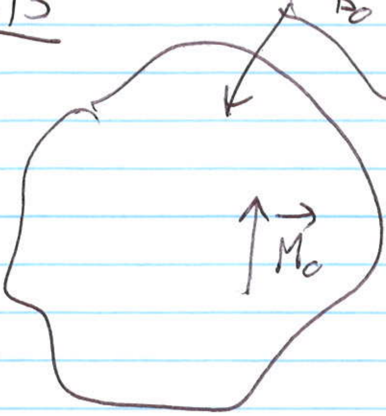
the field for all 4 sides is then,

$$B_z = \frac{\mu_0 M_0 w}{\pi a}2\sqrt{2}$$

$$\Rightarrow \boxed{\vec{B}_\phi = \mu_0 M_0\left[\frac{R}{R+\frac{a}{2}} - \frac{2\sqrt{2}\,w}{\pi a}\right]\hat{\phi}}$$

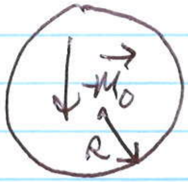
6.13

$\vec{B}_0$ & $\vec{H}_0 = \frac{\vec{B}_0}{\mu_0} - \vec{M}_0$



a) Hollow out a spherical cavity, what is $\vec{B}$ at the center of the cavity? Find $\vec{H}$ at the center of the cavity.

By superposition use a sphere w/ $\vec{M} = -\vec{M}_0$ and superpose.



(i) $\vec{J}_b = \vec{\nabla} \times (-\vec{M}_0) = 0$

$\vec{K}_b = -\vec{M}_0 \times \hat{r} = -M_0 \sin\theta \, \hat{\phi}$

from 5.11, we know that

(ii) $\vec{B} = \frac{2}{3}\mu_0 \underbrace{\sigma R \vec{\Omega}} = -\frac{2}{3}\mu_0 M_0 \hat{z}$ follows

$\vec{\Omega}$ ← spin frequency

for a uniformly spinning shell

(iii) only $\vec{K} = \sigma \vec{\Omega} \times \hat{r} r = \sigma \Omega R \sin\theta \, \hat{\phi}$ from which

$\Rightarrow$ Total field is $\boxed{\vec{B} = \vec{B}_0 - \frac{2}{3}\mu_0 M_0 \hat{z}}$, where $\vec{B}_0$ in $\hat{z}$ direction

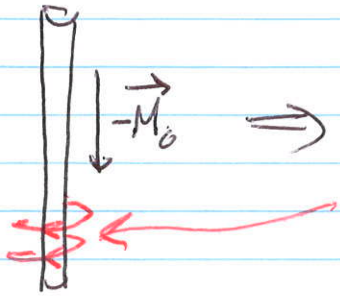
and $\cancel{\vec{H} = \frac{\vec{B}_0}{\mu_0} - \frac{2}{3} M_0 \hat{z}}$

oops $\vec{H} = \frac{\vec{B}}{\mu_0}$ in cavity

$= \left(\frac{\vec{B}_0}{\mu_0} - \frac{2}{3}\vec{M}_0\right) = \left(\frac{1}{\mu_0}\left[\mu_0 \vec{H}_0 + \mu_0 \vec{M}_0\right]\right) - \frac{2}{3}\vec{M}_0$

$$\rightarrow \boxed{\vec{H} = \vec{H}_0 + \frac{1}{3}\vec{M}_0}$$

b) a long needle



$\Rightarrow$ surface current w/ magnitude M_0 that wraps around needle

The field at the center of a finite solenoid is

$$\vec{B} = \frac{\mu_0 M_0 L}{2\sqrt{R^2+\frac{L^2}{4}}}\,\hat{axis}$$

where R, L are the radius and length of solenoid. If $L >> R$

$$\rightarrow \vec{B} = \frac{\mu_0 M_0}{2}\frac{L\,\hat{axis}}{\frac{L}{2}\sqrt{1+\frac{2R^2}{L^2}}} \simeq \frac{\mu_0 M_0 \hat{axis}}{4}\left(1-\frac{R^2}{L^2}\right)$$

$$\simeq \mu_0 M_0\left(1-\frac{R^2}{L^2}\right)(\hat{axis})$$

Superpose fields

$$\boxed{\vec{B} = \vec{B}_0 - \mu_0 M_0\left(1-\frac{R^2}{L^2}\right)\hat{z} \quad \& \quad \vec{H} = \frac{\vec{B}_0}{\mu_0} - M_0\left(1-\frac{R^2}{L^2}\right)\hat{z}}$$

$$\vec{H} = \frac{1}{\mu_0}\left[\mu_0\vec{H}_0 + \mu_0\vec{M}_0\right] - M_0\left(1-\frac{R^2}{L^2}\right)\hat{z}$$

$$= \vec{H}_0 + \frac{R^2}{L^2}\vec{M}_0 \simeq \vec{H}_0$$

c) consider a wafer. In this case, $R \gg L$ and

$$\vec{B} = \frac{\mu_0 M_0}{2} \frac{L}{R} \frac{(\hat{axis})}{\sqrt{1+\frac{L^2}{4R^2}}} \simeq \frac{\mu_0 M_0 L}{2R}(\hat{axis})$$

Superpose fields

$$\vec{B} = B_0\hat{z} - \frac{\mu_0 M_0 L}{2R}\hat{z}$$

&

$$\vec{H} = \frac{\vec{B_0}}{\mu_0} - \frac{M_0}{2}\frac{L}{R}\hat{z}$$

$$\vec{H} = \frac{1}{\mu_0}\left(\mu_0\vec{H_0} + \mu_0\vec{M_0}\right) - \frac{M_0}{2}\frac{L}{R}$$

$$= \vec{H_0} + \vec{M_0}\left(1 - \frac{L}{R}\right)$$

6.15 $\vec{J}_f = 0 \Rightarrow \vec{\nabla} \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla} W$

ad

recall $\vec{\nabla} \cdot \vec{B} = 0 = \vec{\nabla} \cdot \left[\mu_0 \vec{H} + \mu_0 \vec{M}\right] = 0$

$$\Rightarrow \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$$

ad so, $\vec{\nabla} \cdot \left[-\vec{\nabla} W\right] = -\vec{\nabla} \cdot \vec{M} \rightarrow \boxed{\nabla^2 W = \vec{\nabla} \cdot \vec{M}}$

a) find the field of a uniformly magnetized sphere.

(i) $\vec{J}_b = \vec{\nabla} \cdot \vec{M} = 0$, if $\vec{M} = M_0 \hat{z}$

(ii) $\vec{K}_b = \vec{M} \times \hat{r} = M_0 \sin\theta \hat{\phi}$

ad we see that $\nabla^2 W = 0$ holds everywhere except when $r = R$.

for $r \neq R$ $\quad W = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

b/c field is axisymmetric

b) <u>BCs</u>:

(i) at $r \rightarrow \infty$, $W \rightarrow 0$

(ii) at $r = 0$, W must be well-behaved

(iii) Inner and outer solutions must match at $r = R$