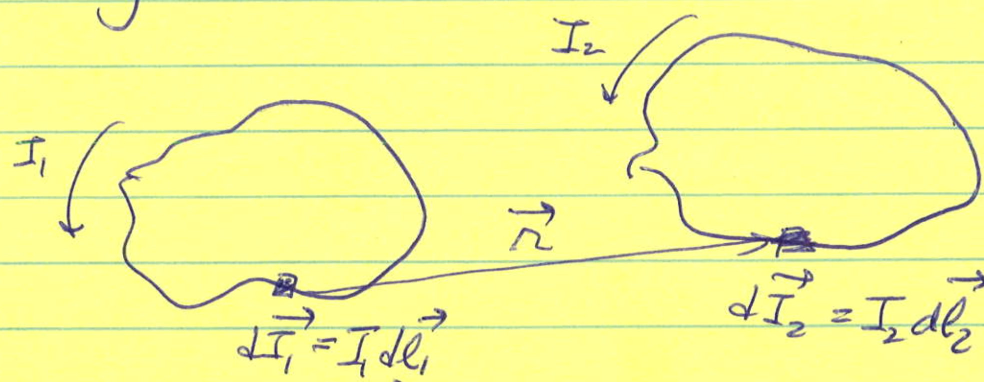
Prob 5.49

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\times\hat{r}}{r^2}\,, \quad \vec{F} = \int I(d\vec{l}\times\vec{B}) \text{ and}$$



Show that $\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\hat{r}}{r^2} d\vec{I}_1 \cdot d\vec{I}_2$

$$\vec{F}_{21} = \int I_2 d\vec{l}_2 \times \vec{B}_1$$

$$= \int I_2 d\vec{l}_2 \times \frac{\mu_0}{4\pi}\int \frac{I_1 d\vec{l}_1 \times \hat{r}}{r^2}$$

Integration variables are independent

$$\Rightarrow \vec{F}_{21} = \oint_1\oint_2 I_1 I_2\, d\vec{l}_2 \times \left[\frac{d\vec{l}_1 \times \hat{r}}{r^2}\right]\frac{\mu_0}{4\pi}$$

$$= \oint\oint \frac{\mu_0}{4\pi} I_1 I_2 \left\{\frac{d\vec{l}_1(d\vec{l}_2\cdot\hat{r}) - \hat{r}(d\vec{l}_1\cdot d\vec{l}_2)}{r^2}\right\}$$

~~look at~~ $\cancel{\oint \frac{\mu_0}{4\pi} I_1 I_2\, d\vec{l}_2 \cdot \left\{\oint \frac{\hat{r}\, d\vec{l}_1}{r^2}\right\}}$

~~→ integral returns to same spot after 1 loop → $\oint \to 0$.~~

look at $\oint\oint \frac{\mu_0}{4\pi} I_1 I_2 \left\{ \frac{d\vec{l}_1 (d\vec{l}_2 \cdot \hat{r})}{r^2} \right\}$

$$= \oint\oint \frac{\mu_0}{4\pi} I_1 I_2 \left[d\vec{l}_1 \left\{ d\vec{l}_2 \cdot \frac{\vec{l}_2 - \vec{l}_1}{|\vec{l}_2 - \vec{l}_1|^3} \right\} \right]$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint d\vec{l}_1 \left\{ d\vec{l}_2 \cdot \vec{\nabla}_2 \left[\frac{-1}{|\vec{l}_2 - \vec{l}_1|} \right] \right\}$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 d\vec{l}_1 \underbrace{\oint d\vec{l}_2 \cdot \vec{\nabla} \frac{-1}{|\vec{l}_2 - \vec{l}_1|}}_{\left. \frac{1}{|\vec{l}_2 - \vec{l}_1|} \right|_{\vec{l}_2}^{\vec{l}_2}}$$

goes around the loop

$$= 0$$

leaving

$$\boxed{\vec{F}_{21} = \oint_1\oint_2 \frac{\mu_0 I_1 I_2}{4\pi} (d\vec{l}_1 \cdot d\vec{l}_2) \frac{\hat{r}}{r^2}}$$