Homework 7

Due: February 25, 2013

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32. 7.1

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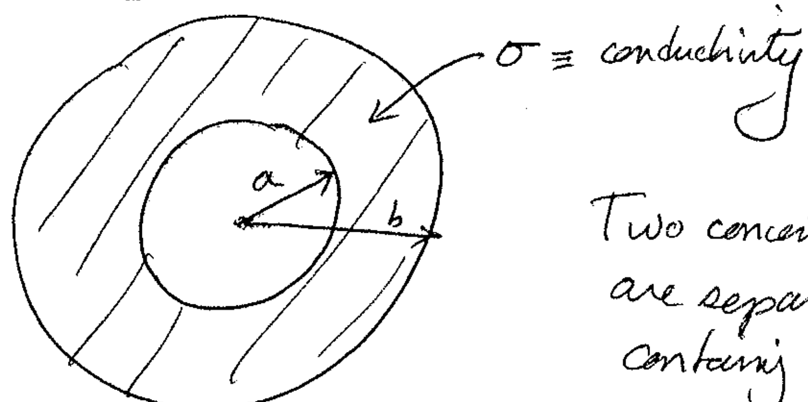
Prob 7.1



Two concentric conducting shells are separated by a space containing material w/ conductivity σ.

a) If the shells are maintained at $\Delta V = V_0$, find the current

(i) $\vec{J} = \sigma \vec{E} \longrightarrow \vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot \sigma \vec{E} = 0$, in steady state

$$\Rightarrow \vec{\nabla} \cdot \sigma \vec{E} = \sigma (\vec{\nabla} \cdot \vec{E}) = -\sigma (\nabla^2 V) = 0$$

(ii) for spherical symmetry, we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V\right) = 0$$

$$\rightarrow r^2 \frac{\partial}{\partial r} V = \text{Constant} = C_1$$

$$\rightarrow \boxed{V = C_0 - \frac{C_1}{r}}$$

(iii) apply BC's. let $V = V_0$ at $r = a$ and $V = 0$ at $r = b$

(1) at $r = a$, $V_0 = C_0 - \frac{C_1}{a}$

(2) at $r = b$, $0 = C_0 - \frac{C_1}{b} \longrightarrow C_0 = \frac{C_1}{b}$

$$\Longrightarrow V_0 = C_1\left(\frac{1}{b} - \frac{1}{a}\right) \longrightarrow C_1 = \frac{ab}{a-b} V_0$$

the potential is then

$$V(r) = \frac{a}{a-b} V_0 - \frac{ab}{a-b} \frac{V_0}{r}$$

$$\boxed{V(r) = \frac{a}{a-b} V_0 \left(1 - \frac{b}{r} \right)}$$

$$\Longrightarrow \vec{E}(r) = \frac{a V_0}{a-b} \left(+ \frac{b}{r^2} \right) \hat{r}$$

$$\rightarrow \boxed{\vec{J} = \frac{b a V_0 \sigma}{a-b} \left(\frac{\hat{r}}{r^2} \right)}$$

and so, $|\vec{J}| = \frac{I}{4\pi r^2} \hat{r} \rightarrow \boxed{I = 4\pi V_0 \sigma \left| \frac{ab}{a-b} \right|}$

b) <u>What is the resistance, R ?</u>

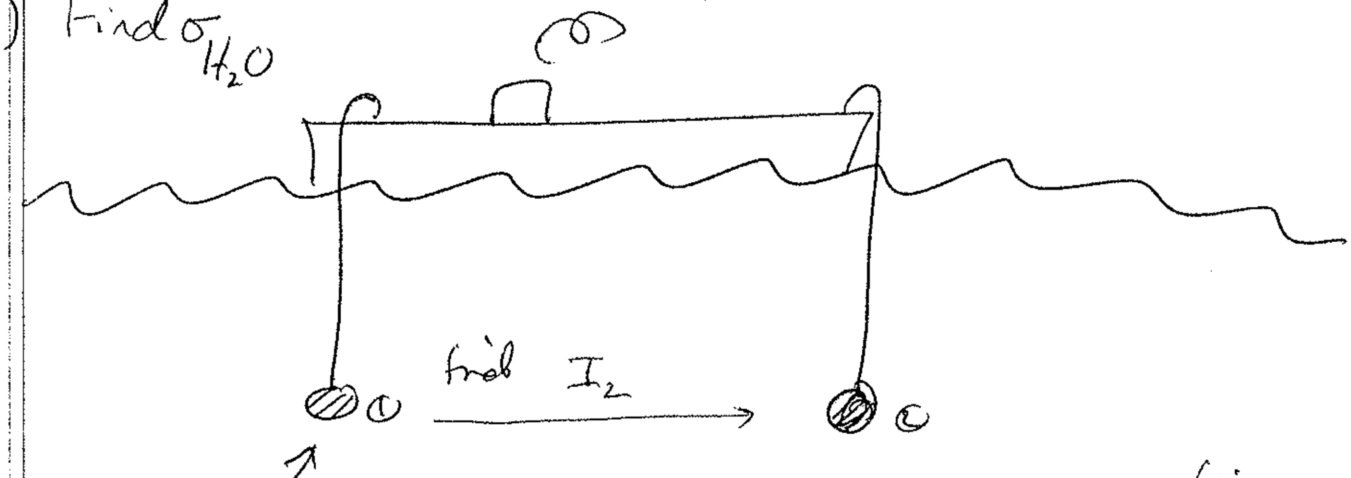
$$I = 4\pi \sigma V_0 \frac{ba}{b-a} \rightarrow V_0 = \underbrace{\left(\frac{b-a}{ba} \right) \frac{1}{4\pi\sigma}}_{R} I$$

$$\rightarrow \boxed{R = \frac{1}{4\pi\sigma} \left(\frac{b-a}{ba} \right)}$$

c) <u>let $b \rightarrow \infty$</u>

$$\rightarrow \boxed{I \rightarrow 4\pi\sigma V_0 \ \& \ R \rightarrow \frac{1}{4\pi\sigma}}$$

d) Find σ_{H_2O}



for sphere 1, the outer BC is at $\infty \rightarrow$ $I = 4\pi\sigma V_0$ flows off the sphere and escapes to ∞ w/ spherical symmetry

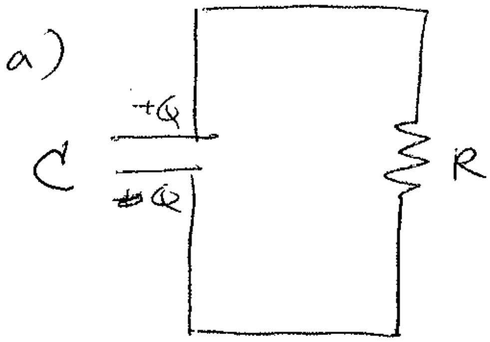
for sphere 2, the outer B.C. is at $\infty \rightarrow I_\infty = 4\pi\sigma V_0$ and flows toward sphere with spherical symmetry

$$\Longrightarrow \quad I_2 = 4\pi\sigma V_0 \left(\rightarrow R = 4\pi\sigma\right)$$

and so, $\boxed{\sigma = \left(I/4\pi V_0 \right)}$

Prob 7.2

a)



as chge flows off capacitor → I↑

$$V = \frac{Q}{C} = IR = -\frac{dQ}{dt}R$$

$$\rightarrow \frac{dQ}{dt} + \frac{Q}{RC} = 0$$

and $\boxed{Q(t) = Q_0 \exp\left(-\frac{t}{RC}\right)}$

b) $W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$ (2.55)

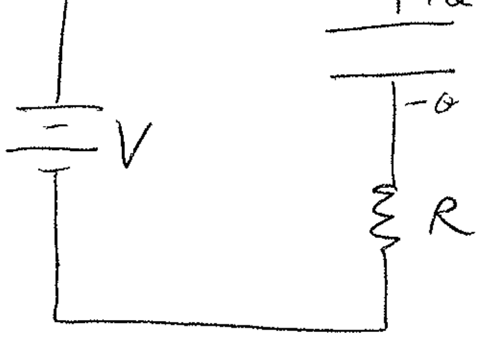
$$P = IV = I^2R = +\left(\frac{Q_0}{RC}\right)^2 \exp\left(-\frac{2t}{RC}\right)R$$

$$\int_0^\infty P\,dt = \frac{Q_0^2}{C^2R}\int_0^\infty \exp\left(-\frac{2t}{RC}\right)dt$$

$$= -\frac{1}{2}\frac{Q_0^2}{C}\exp\left(-\frac{2t}{RC}\right)\Big|_0^\infty$$

$$\boxed{\int_0^\infty P\,dt = \frac{1}{2}\frac{Q_0^2}{C}}$$

c)



$$V = \frac{Q}{C} + IR$$

$$= \frac{Q}{C} + \frac{dQ}{dt}R$$

$$\frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{CR} = \frac{-1}{CR}\underbrace{\left(Q - VC\right)}_{\lambda}$$

$$\frac{d\lambda}{dt} = -\frac{\lambda}{CR} \longrightarrow \ln \lambda = -\frac{t}{CR}$$

$$Q - VC = \exp\left(\frac{-t}{CR}\right)\left[\cancelto{0}{Q_0} - CV\right]$$

$$\boxed{Q(t) = CV\left[1 - \exp\left(-\frac{t}{RC}\right)\right]}$$

$$\boxed{I(t) = \frac{V}{R}\exp\left(-\frac{t}{RC}\right)}$$

d) (i) $$\int_0^\infty P_V\,dt = \int_0^\infty V I\,dt = \int_0^\infty \frac{V^2}{R}\exp\left(-\frac{t}{RC}\right)dt$$

$$= CV^2$$

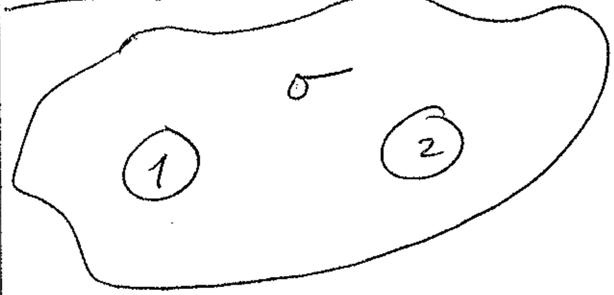
(ii) $$\int_0^\infty P_R\,dt = \int_0^\infty I^2 R\,dt = \int_0^\infty R\frac{V^2}{R^2}\exp\left(-\frac{2t}{RC}\right)dt$$

$$= \frac{1}{2}CV^2$$

(iii) $$Q(\infty) = CV \rightarrow W_C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

(iv) $$\boxed{\frac{W_C}{W_V} = \frac{1}{2}}$$

Prob 7.3



Show that $R = \frac{\epsilon_0}{\sigma C}$

a) Sol$^{\underline{n}}$

(i) $Q_1 = \int \rho \, d^3x = \epsilon_0 \oint \vec{E}_1 \cdot d\vec{S}$

(ii) $I_1 = \oint \vec{J}_1 \cdot d\vec{S} = \sigma \oint \vec{E}_1 \cdot d\vec{S}$

$$\Rightarrow \frac{Q_1}{\epsilon_0} = \frac{I_1}{\sigma} = \frac{CV_1}{\epsilon_0} = \frac{C I_1 R}{\epsilon_0}$$

$$\longrightarrow \boxed{R = \frac{\epsilon_0}{\sigma C}}$$

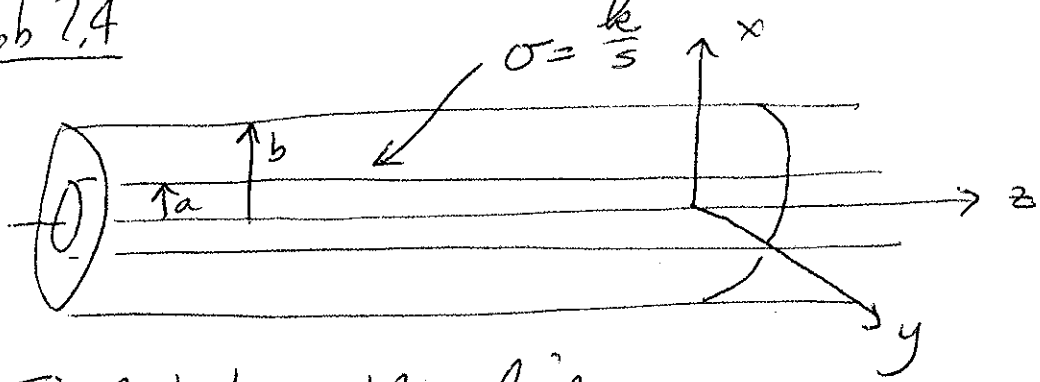
b) Show that $V(t) = V_0 \exp\left(-\frac{t}{\tau}\right)$

as charge flows off $1 \rightarrow Q\downarrow \rightarrow \dot{Q} < 0 \rightarrow I = -\dot{Q}$

(i) $CV = Q \longrightarrow C \frac{dV}{dt} = \frac{dQ}{dt} = -I = \frac{-V}{R}$

(ii) and so, $\frac{1}{V}\frac{dV}{dt} = \frac{-1}{RC} \longrightarrow \boxed{V = V_0 \exp\left(-\frac{t}{RC}\right)}$

Prob 7.4



Find R between the cylinders

find V: $\vec{\nabla}\cdot\vec{J} = 0$ in steady-state

$$\rightarrow \vec{\nabla}\cdot\sigma\vec{E} = \sigma(\vec{\nabla}\cdot\vec{E}) + (\vec{E}\cdot\vec{\nabla})\sigma = 0$$

$$-\sigma\nabla^2 V - \frac{\partial V}{\partial s}\frac{\partial \sigma}{\partial s} = 0$$

$$\nabla^2 V - \frac{1}{\cancel{\sigma}}\frac{\cancel{\sigma}}{s}\frac{\partial V}{\partial s} = 0$$

$$\frac{1}{s}\left[\frac{\partial}{\partial s} s \frac{\partial}{\partial s} V\right] - \frac{1}{s}\frac{\partial V}{\partial s} = 0$$

$$\frac{\partial V}{\partial s} + s\frac{\partial^2 V}{\partial s^2} - \frac{\partial V}{\partial s} = 0$$

$$\rightarrow \boxed{V = C_0 s + C_1}$$

Apply BC's: a) $s = a$, $V = V_0 \rightarrow V_0 = C_0 a + C_1$

b) $s = b$, $V = 0 \rightarrow 0 = C_0 b + C_1$

$$\Rightarrow V_0 - C_0 a = -C_0 b$$

$$\rightarrow C_0 = \frac{V_0}{a-b}, \quad C_1 = \frac{-V_0 b}{a-b}$$

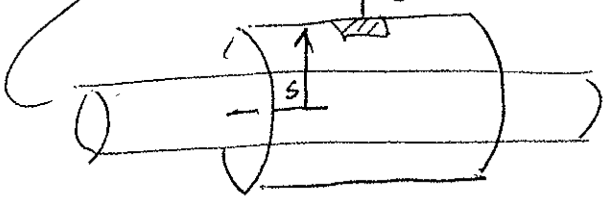
$$\Rightarrow \boxed{V(s) = \frac{V_0}{a-b}\left[s - b\right]}$$

$$\rightarrow \boxed{\vec{E}(s) = -\frac{V_0}{a-b}\hat{s}}$$

$$\rightarrow \boxed{\vec{J} = -\sigma\frac{V_0\hat{s}}{a-b}}$$

$$I = \int \frac{k}{s}\frac{V_0\, dS}{b-a} = \frac{kV_0}{b-a}\int \frac{1}{s}\, s\, d\phi\, dz$$

$\vec{J}\cdot d\vec{s} = J s\, d\phi\, dz$



$$\rightarrow I = \frac{kV_0}{b-a}2\pi dz$$

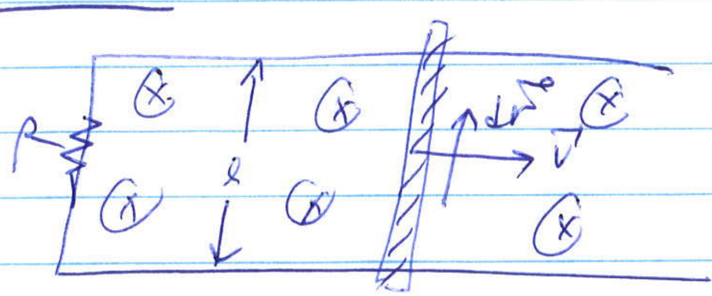
ad so

$$V = IR \longrightarrow R = \frac{V_0}{\left(\frac{kV_0}{b-a}\right)2\pi dz}$$

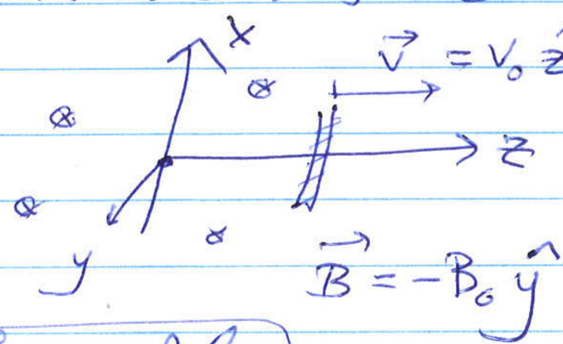
$$\boxed{R = \frac{(b-a)}{2\pi k\, dz}}$$

Prob 7.7

A metal bar, mass m, slides frictionlessly across the rails & a uniform $\vec{B}$ points into the page, fills the entire region



(a) Find I, in which direction does I flow?



$$\mathcal{E} = \oint \vec{v} \times \vec{B} \cdot d\vec{r}$$

$$= \int_0^\ell v_0 B_0 \, dx$$

$$= v_0 B_0 \ell \Longrightarrow \boxed{I = \frac{\mathcal{E}}{R} = \frac{v_0 B_0 \ell}{R}}$$

and flows upward through bar ($+\hat{x}$)
(flows CCW looking from y)

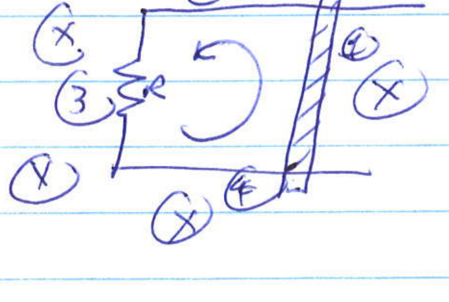
(b) Find v(t)

$$I(t) = \frac{v(t) B_0 \ell}{R}$$

around y-axis

from part (a) generalizing to arbitrary v(t).

find $\int I(t) \, d\vec{l} \times \vec{B} = ?$



(i) along legs ② + ④ ⇒ force ⊥ to wire

(ii) along leg ③ ⇒ force $+\hat{z}$ but this part of the circuit is fixed

(iii) along leg ① ⇒ force $-\hat{z}$ ← need this

$$f_z = m\ddot{z} = \hat{z}\cdot\int I(t)\,d\vec{l} \times \vec{B} = -\frac{B_0^2 l^2}{R}\underbrace{v(t)}_{\dot{z}}$$

(c) $$\Rightarrow \ddot{z} = -\frac{B_0^2 l^2}{mR}\dot{z}$$

$$\dot{z}\Big|_{v_0}^{v(t)} = -\frac{B_0^2 l^2}{mR} z\Big|_0^z$$

$$\Rightarrow \boxed{v(t) = v_0 - \frac{B_0^2 l^2}{mR} z}$$

* see next, next page

(d) Find the energy delivered to the resistor

$$P = IV = I^2 R$$

$$= \frac{B_0^2 l^2}{R^2} v^2 R$$

$$= \frac{B_0^2 l^2}{R} v^2$$

$$\int P\,dt = \frac{B_0^2 l^2}{R}\int v^2\,dt = \frac{B_0^2 l^2}{R}\int v^2\left(\frac{dt}{dz}\right)dz$$

$$= \frac{B_0^2 l^2}{R}\int v\,dz$$

$$= \frac{B_0^2 l^2}{R}\int_0^{z_{max}}\left[v_0 - \frac{B_0^2 l^2}{mR} z\right]dz$$

$$= \frac{B_0^2 l^2}{R}\left[z_{max} v_0 - \frac{B_0^2 l^2}{mR}\frac{z_{max}^2}{2}\right]$$

τ_{max} occurs when $v(t_{max}) = 0$

$$\Rightarrow \tau_{max} = \frac{m v_0 R}{B_0^2 \ell^2}$$

and
$$\int P\,dt = \frac{B_0^2 \ell^2}{R}\left\{\frac{m v_0^2 R}{B_0^2 \ell^2} - \frac{1}{2}\frac{B_0^2 \ell^2}{mR}\frac{m^2 v_0^2 R^2}{B_0^4 \ell^4}\right\}$$

$$= \left\{m v_0^2 - \frac{1}{2} m v_0^2\right\}$$

$$\boxed{\int P\,dt = \frac{1}{2} m v_0^2} \checkmark$$

Comment: I find

$$v(t) = V_0 - \frac{B_0^2 l^2}{mR} z$$

$$\frac{dz}{dt} = V_0 - \frac{B_0^2 l^2}{mR} z$$

$$\int \frac{dz}{V_0 - \frac{B_0^2 l^2}{mR} z} = \int dt$$

$$-\frac{mR}{B_0^2 l^2} \ln\left(V_0 - \frac{B_0^2 l^2}{mR} z\right)\Big|_0^z = t - 0$$

$$\Rightarrow \ln\left[\frac{V_0 - B_0^2 l^2/mR \, z}{V_0}\right] = -\frac{B_0^2 l^2 t}{mR}$$

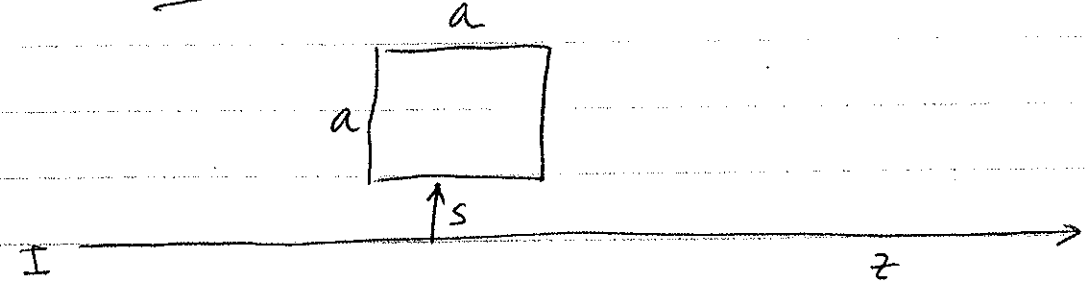
$$1 - \frac{B_0^2 l^2}{V_0 mR} z = e^{-B_0^2 l^2 t/mR}$$

$$z = \frac{mRV_0}{B_0^2 l^2}\left\{1 - e^{-B_0^2 l^2 t/mR}\right\}$$

$$\dot{z} = \frac{mRV_0}{B_0^2 l^2}\left\{+\frac{B_0^2 l^2}{mR} e^{-B_0^2 l^2 t/mR}\right\}$$

$$\boxed{\dot{z} = V_0 e^{-B_0^2 l^2 t/mR}}$$

Prob 7.8



a) Find $\oint \vec{B}\cdot d\vec{S}$ through the loop.

(i) Find $\vec{B}$ using Ampere's law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \longrightarrow \vec{B}_\phi = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

(ii) $$\Phi_B = \int_s^{s+a}\int_0^a \frac{\mu_0 I}{2\pi s}\, ds\, dz = \frac{\mu_0 I}{2\pi} a \ln\left(\frac{s+a}{s}\right)$$

b) If the loop is moved in the s-direction w/ speed v, what is the emf?

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 I a}{2\pi}\left[\frac{1}{s+a} - \frac{1}{s}\right]\dot{s} \leftarrow \dot{s} = v$$

$$\mathcal{E} = -\frac{\mu_0 I a}{2\pi}\left[\frac{-av}{s(s+a)}\right]$$

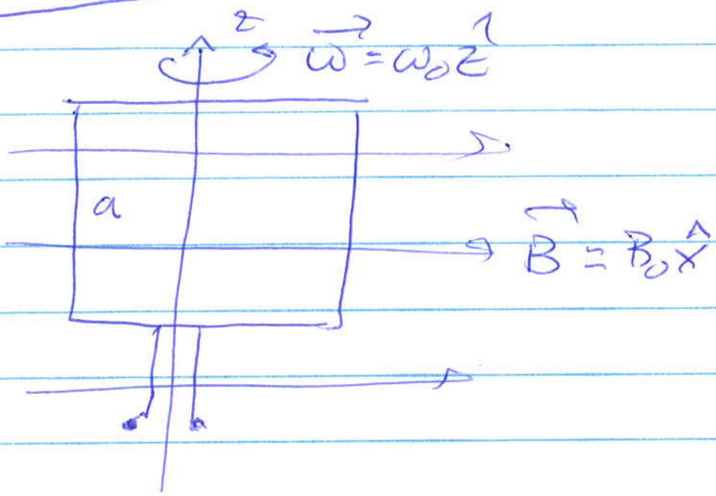
$$\boxed{\mathcal{E} = \frac{\mu_0 I a^2 v}{2\pi s(s+a)}} \rightarrow I \text{ in CCW}$$

sense to maintain Φ_B

c) If the loop is moved horizontally, Φ_B doesn't change and so,

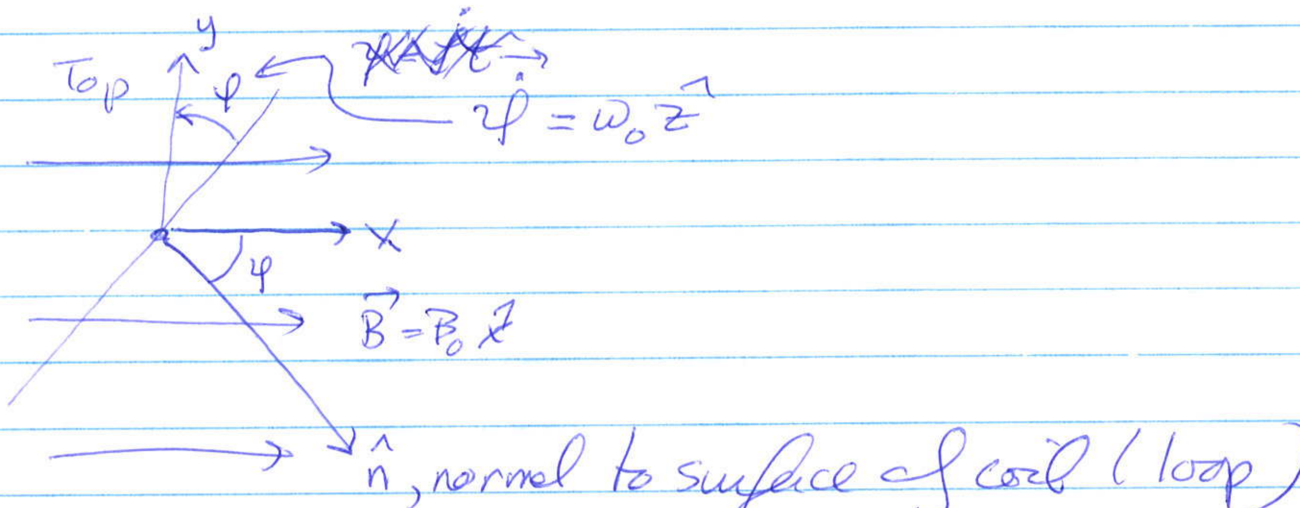
$$\boxed{\mathcal{E} = 0}$$

Prob 7.10



find $\mathcal{E}(t)$

a) find $\Phi_B(t) = \ ?$



$$\Phi_B = \oint \vec{B} \cdot d\vec{S} = \int B_0 \cos\varphi \, dS = B_0 \cos\varphi \, a^2$$

b) find $-\dfrac{d\Phi_B}{dt} = + B_0 a^2 (\sin\varphi)\, \dot{\varphi} = + B_0 a^2 \omega_0 \sin\varphi$