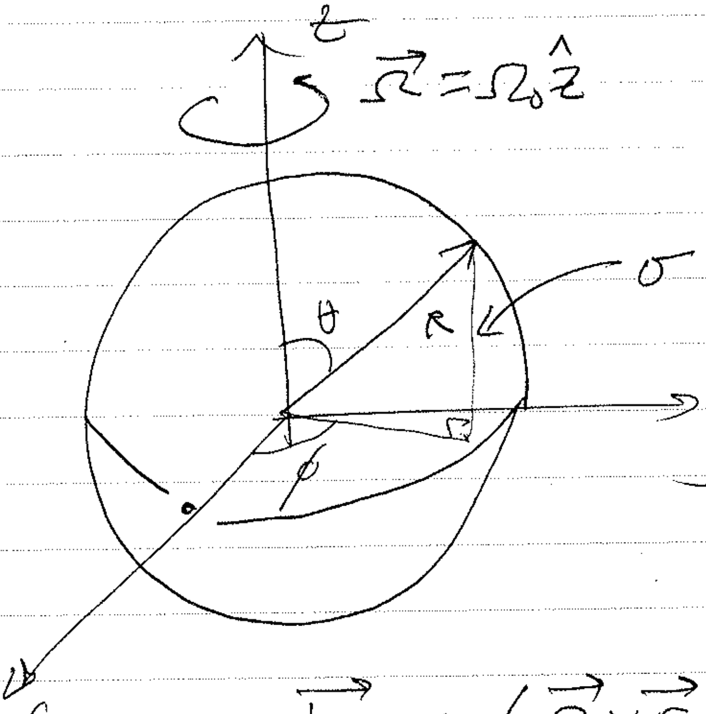
Applications: $\vec{A}$

find $\vec{A}$ for this shell of charge that spins about the z-axis



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\, ds}{\mathscr{r}}$$

(1) $\vec{K} = \sigma(\vec{\Omega}\times\vec{r}) = \sigma(0,0,\Omega_0)\times(R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$

$= \sigma\Omega_0 R(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$

(2) $\mathscr{r}^2 = r^2 + R^2 - 2rR\left[\cos\theta\cos\theta^P - \sin\theta\sin\theta^P\cos(\phi-\phi^P)\right]$

if we want the on-axis field, $\underbrace{\theta^P = 0, \phi^P = 0}_{\text{field point}}$

$\Rightarrow \boxed{\mathscr{r}^2 = r^2 + R^2 - 2rR\cos\theta}$ ← becomes independent of ϕ !

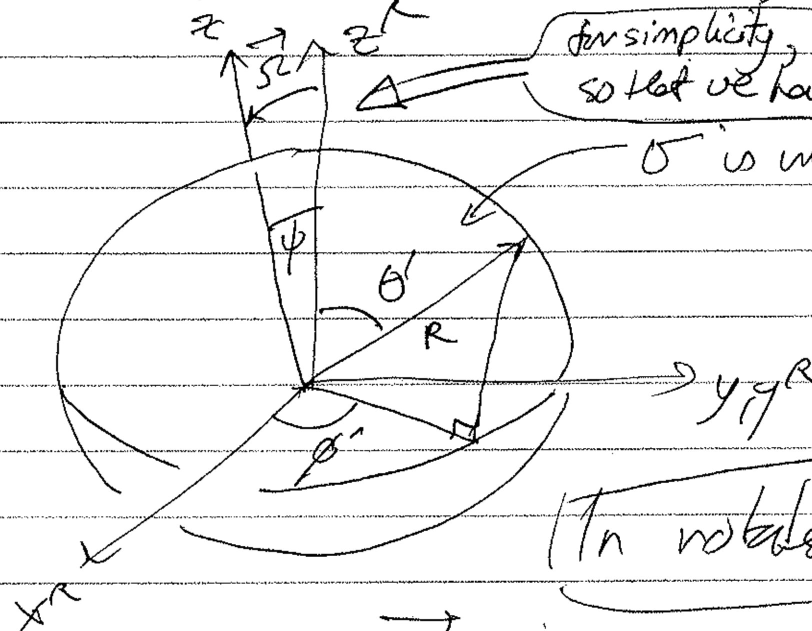
$$\Rightarrow \vec{A} = \frac{\mu_0\sigma\Omega_0 R}{4\pi}\int\frac{(-\sin\theta'\sin\phi', \sin\theta'\cos\phi', 0)}{\sqrt{r^2+R^2-2rR\cos\theta'}}\, s\, d s\, d\phi'$$

note: $\int \begin{pmatrix}\sin\phi'\\ \cos\phi'\end{pmatrix} d\phi' = 0 \Rightarrow \vec{A} = 0$ on-axis hmm, not so good ($\Rightarrow$ can't find $\vec{B}$ unless we know $\vec{A}$ off-axis anyway).

Rotate frame so that $\vec{\Omega}$ is off-axis and P is on-axis

for simplicity, let P sit in (x^R-z^R) plane so that we have simply rotated one $\measuredangle\,\psi$.

σ is uniform



In rotated frame, $\vec{\Omega} = \Omega_0(\sin\psi, 0, \cos\psi)$

$$\Rightarrow \vec{K} = \sigma\Omega_0(\sin\psi, 0, \cos\psi) \times (R\sin\theta'\cos\phi',\; R\sin\theta'\sin\phi',\; R\cos\theta')$$

$$= \sigma\Omega_0\left[-\cos\psi R\sin\theta'\sin\phi',\; \cos\psi R\sin\theta'\cos\phi' - \sin\psi R\cos\theta',\; \sin\psi R\sin\theta'\sin\phi'\right]$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\sigma\Omega_0 R\left(-\cos\psi\,\cancel{\sin\theta'\sin\phi'}^{\,0},\; \cos\psi\sin\theta'\cancel{\cos\phi'}^{\,0} - \sin\psi\cos\theta',\; \sin\psi\sin\theta'\cancel{\sin\phi'}^{\,0}\right)dS'}{\sqrt{r^2+R^2-2rR\cos\theta'}}$$

again, $\displaystyle\int_0^{2\pi} \begin{pmatrix} \sin\phi' \\ \cos\phi' \end{pmatrix} d\phi' = 0$

$$A_y = \frac{\mu_0 \Omega_0 \sigma R}{4\pi} \int \frac{-\sin\phi' \cos\theta' \, dS'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

after integrating over ϕ

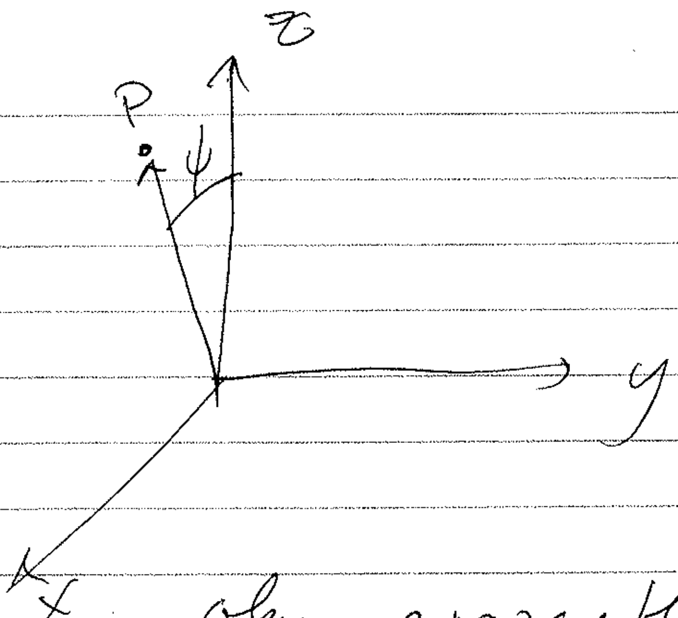
$$= -\frac{\mu_0 \Omega_0 \sigma R \sin\phi}{2} \int \frac{\cos\theta' (R^2 \sin\theta' d\theta')}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

$$= -\frac{\mu_0 \Omega_0 \sigma R^3 \sin\phi}{2} \int \frac{\cos\theta' \sin\theta' d\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

$$= \frac{\mu_0 \Omega_0 \sigma R^3 \sin\phi}{2} \int_{-1}^{1} \frac{u \, du}{\sqrt{r^2 + R^2 - 2Rru}}$$

$$= \frac{\mu_0 \Omega_0 \sigma R^3 \sin\phi}{2} \begin{cases} \dfrac{2r}{3R^2} & r < R \text{, inside} \\[2ex] \dfrac{2R}{3r^2} & r > R \text{, outside} \end{cases}$$

$$\boxed{A_y = \frac{\mu_0 \Omega_0 \sigma}{2} R^3 \begin{cases} \dfrac{2}{3R^2} r \sin\phi & r < R \\[2ex] \dfrac{2R}{3} \dfrac{\sin\phi}{r^2} & r > R \end{cases}}$$



In original frame where $\vec{\Omega} \parallel \hat{z}$ and $\vec{r}_*(\vec{P})$ is in xz plane

$$\vec{\Omega} \times \vec{r} = (0, 0, \Omega_0) \times (r\sin\psi, 0, r\cos\psi)$$

$$= (0, \Omega_0 r \sin\psi, 0)$$

$$= \Omega_0 r \sin\psi \, \hat{y}$$

okay, we now see that

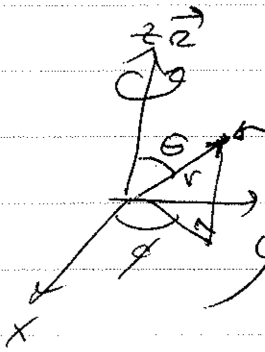
$$\vec{A} = \frac{\mu_0 \sigma R^3}{\cancel{2}} \frac{\cancel{2}}{3} \begin{cases} \dfrac{\vec{\Omega} \times \vec{r}}{R^2}, & r < R \\[2ex] \dfrac{R}{r^3} \vec{\Omega} \times \vec{r}, & r > R \end{cases}$$

$$\boxed{\vec{A} = \begin{cases} \dfrac{\mu_0 \sigma}{3} R (\vec{\Omega} \times \vec{r}) & r < R \\[2ex] \dfrac{\mu_0 \sigma}{3} R^4 \left(\dfrac{\vec{\Omega} \times \vec{r}}{r^3} \right) & r > R \end{cases}}$$

in general, off-axis calculations are harder and we resort to approximation methods; multipoles

(a) find $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{A} = \frac{\mu_0 \sigma}{3} \begin{cases} R(\vec{\Omega} \times \vec{r}) & r < R \\ R^4 \left(\dfrac{\vec{\Omega} \times \vec{r}}{r^3} \right) & r > R \end{cases}$$



$$= \frac{\mu_0 \sigma}{3} \begin{cases} R \Omega_0 r \, \hat{\phi} \sin\theta & r < R \\ \dfrac{R^4 \Omega_0}{r^2} \hat{\phi} \sin\theta & r > R \end{cases}$$

$$\vec{B} = \frac{1}{r\sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r}$$

$$+ \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} r A_\phi \right] \hat{\theta}$$

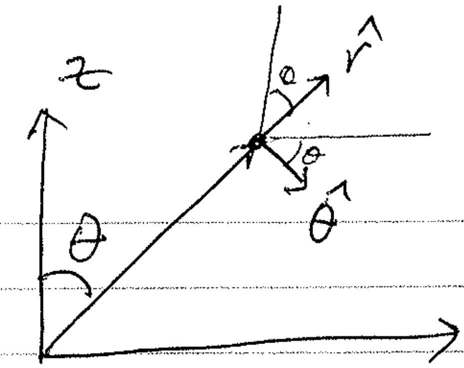
$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi}$$

$r < R$

$$= \frac{\mu_0 \sigma}{3} R \Omega_0 \overset{2\cos\theta}{\left[\frac{\cancel{\cos\theta}}{\sin\theta} \right]} \hat{r} - \frac{\mu_0 \sigma}{3} R \Omega_0 \sin\theta \, 2 \, \hat{\theta}$$

$$+ 0 \hat{\phi}$$

$$= \frac{\mu_0 \sigma}{3} \Omega_0 R \left(2\cos\theta \hat{r} - 2\sin\theta \hat{\theta} \right)$$



$$\Rightarrow \hat{z} = \hat{r}\cos\theta - \sin\theta.$$

$$\Rightarrow \vec{B} = \frac{2\mu_0 \sigma \Omega_0 R}{3}\hat{z}$$

$$\boxed{\vec{B} = \frac{2\mu_0 \sigma R}{3}\vec{\Omega}, \quad r < R}$$

$\underline{r > R}$

$$\vec{B} = \frac{\mu_0 \sigma \Omega_0 R^4}{r^3 \sin\theta}\left(2\sin\theta\cos\theta\right)\hat{r}$$

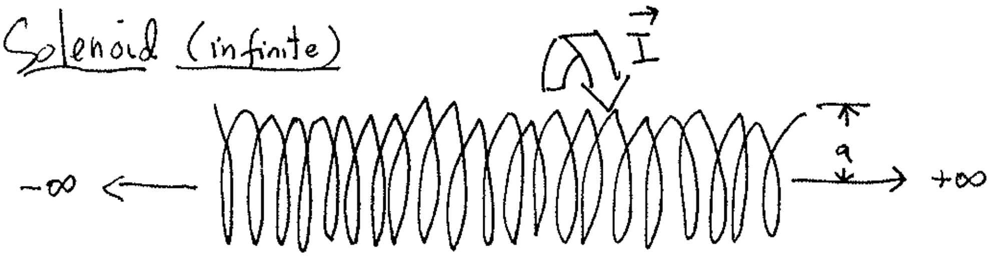
$$- \frac{\mu_0 \sigma R^4 \Omega_0 \sin\theta}{3r}\left(-\frac{1}{r^2}\right)\hat{\theta}$$

$$\boxed{\vec{B} = \frac{\mu_0 \sigma \Omega_0 R^4}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

"dipolar"

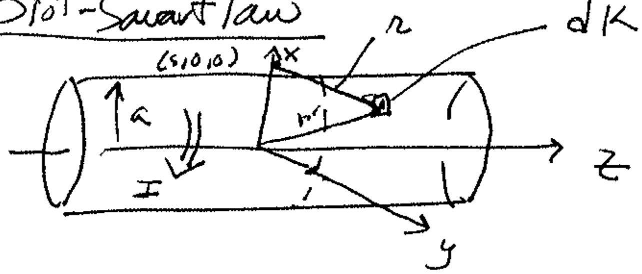
Applications : find $\vec{A}$

Solenoid (infinite)



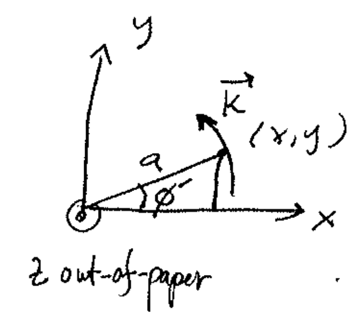
Radius a, current I, and coil density N. What is field inside and outside the solenoid? [2 layers, w/reverse tilt ⇒ cancel I_z]

Biot-Savart law



$$\Rightarrow d\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{\vec{K}\,dS \times \hat{r}}{r^2}\right]$$

write the vecters in Cartesion coordinates



$$\vec{K} = k\left(-\sin\phi', \cos\phi', 0\right)$$

$$\vec{r} = \left(x - a\cos\phi', -a\sin\phi', -z'\right)$$

$$\Rightarrow (\vec{K} \times \vec{r}) = k\left(-z'\cos\phi', -z'\sin\phi', a\sin^2\phi' - x\cos\phi' + a\cos^2\phi'\right)$$

$$= k\left(-z'\cos\phi', -z'\sin\phi', a - x\cos\phi'\right)$$

$$d\vec{B} = \frac{\mu_0 K}{4\pi}\frac{\left(-z'\cos\phi', -z'\sin\phi', a - x\cos\phi'\right)}{\left([x - a\cos\phi']^2 + a^2\sin^2\phi' + z'^2\right)^{3/2}}$$

$$= \frac{\mu_0 K}{4\pi}\frac{\left(-z'\cos\phi', -z'\sin\phi', a - x\cos\phi'\right)}{\left(x^2 + a^2 + z'^2 - 2ax\cos\phi'\right)^{3/2}}$$

Consider dB_x, dB_y

$$\begin{pmatrix} dB_x \\ dB_y \end{pmatrix} = \frac{\mu_0 K}{4\pi} \frac{-z'\begin{pmatrix}\cos\phi' \\ \sin\phi'\end{pmatrix}}{(x^2+a^2+z'^2-2ax\cos\phi')^{3/2}}$$

Integrate over z'

$$-\frac{\mu_0 K}{4\pi}\begin{pmatrix}\cos\phi' \\ \sin\phi'\end{pmatrix}\int_{-\infty}^{\infty}\frac{z'\,dz'\,a\,d\phi'}{(\underbrace{x^2+a^2-2ax\cos\phi'}_{A^2 \text{ (independent of } z')}+z'^2)^{3/2}}$$

let: $u = (A^2+z'^2) \rightarrow du = 2z'dz'$

$$\propto \int_{u_1}^{u_2}\frac{du}{2u^{3/2}} = -\frac{1}{u^{1/2}} = -\frac{1}{\sqrt{A^2+z'^2}}\Bigg|_{-\infty}^{\infty} = 0$$

$$\Rightarrow B_x = B_y = 0 \quad \text{(as expected)}$$

$$dB_z = \frac{\mu_0 K}{4\pi} \frac{(a - x\cos\phi')}{(x^2+a^2-2ax\cos\phi'+z'^2)^{3/2}} \Rightarrow B_z = \frac{\mu_0 K}{4\pi} \int \frac{(a-x\cos\phi')\,dz'\,a\,d\phi'}{\left[x^2+a^2-2ax\cos\phi'+z'^2\right]^{3/2}}$$

Hints:

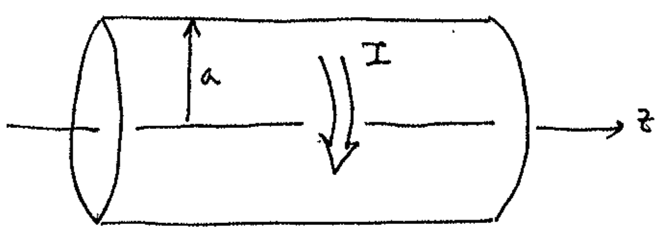
(i) z' integration: trig substitution

(ii) ϕ'-integration: $\displaystyle\int_0^{2\pi} \frac{\cos(mx)\,dx}{1-2b\cos x+b^2} = \begin{cases} \dfrac{2\pi b^m}{1-b^2} & b^2<1 \\ \dfrac{2\pi}{(b^2-1)b^m} & b^2>1 \end{cases}$

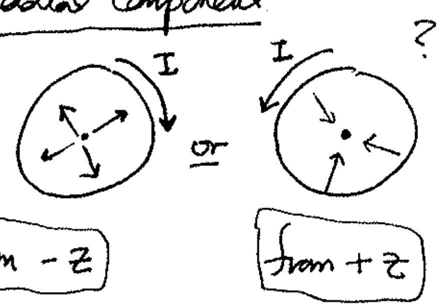
$$\Rightarrow B_z = \begin{cases} \mu_0 K = \mu_0 N I & x^2<a^2 \\ 0 & x^2>a^2 \end{cases}$$

Ampere's Law

a)

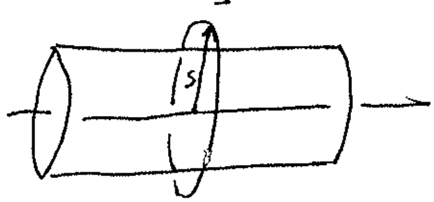


Radial Component



$? \Rightarrow$ No $\Rightarrow B_s = 0$

b)



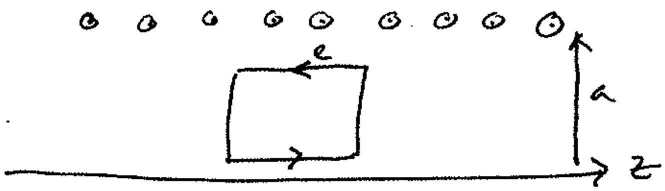
$$\oint \vec{B}\cdot d\vec{l} = \oint \mu_0 \vec{J}\cdot d\vec{S}$$

$\vec{J} \perp d\vec{S} \Rightarrow \vec{J}\cdot d\vec{S} = 0$

$$B_\phi\, 2\pi s = 0$$

$$\rightarrow B_\phi = 0$$

c)

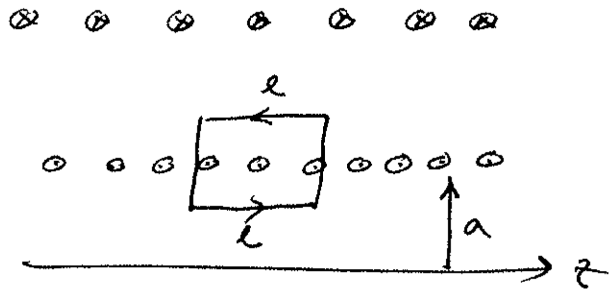


$$-B_z^2 \ell + B_z^1 \ell = 0$$

or, $B_z(s) - B_z(0) = 0$

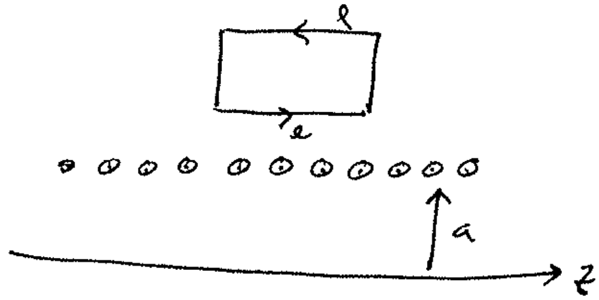
$\Rightarrow B_z(s) = \text{constant} = B_0$

for $s < a$



$$-B_z(s)\ell + \overbrace{B_z(s_0)}^{B_0}\ell = \mu_0 K \ell$$

$$B_z(s) = \underbrace{B_0}_{B_z(s<a)} - \mu_0 K \cancel{\ell} \quad s > a$$



$$-B_z(s)\ell + B_z(s_1)\ell = 0$$

$B_z(s) = \text{constant}, s > a$

as $s \to \infty \to B_z \to 0$

$\Rightarrow B_z(s) = 0$ for all $s > a$

d) $\Rightarrow B_z(s<a) - \mu_0 K = 0$

$$\begin{cases} B(s<a) = \mu_0 K = \mu_0 N I & s < a \\ = 0 & s > a \end{cases}$$

Vector Potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,d\tau}{r}$$, if $\vec{J}$ is <u>not</u> infinite ; So how can we find $\vec{A}$?

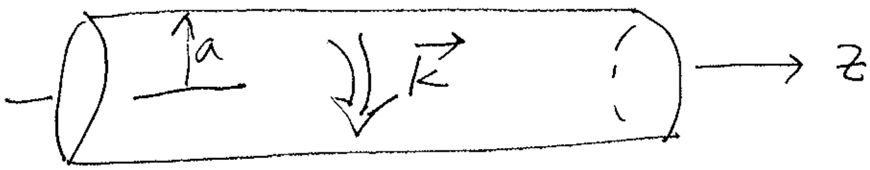
<u>why?</u>

$\vec{\nabla}\cdot\vec{A} = 0$

$\rightarrow \int(\vec{\nabla}\cdot\vec{A})\,d^3x = 0$

$\oint \vec{A}\cdot d\vec{S} = 0$

$\Rightarrow$ we must be able to find a surface which contains all of $\vec{A}$



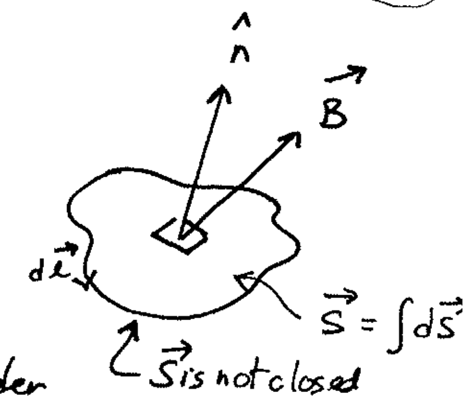
magnetic flux

$$\Rightarrow \vec{B} = \begin{cases} 0 & s > a \\ \mu_0 K\hat{z} & s < a \end{cases}$$

a) $\vec{B} = (\vec{\nabla}\times\vec{A})$

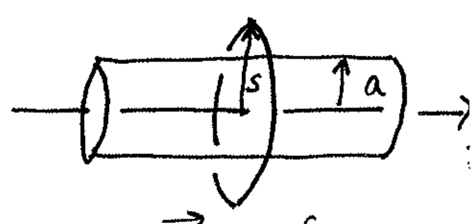
$$\Phi_B = \int \vec{B}\cdot d\vec{S} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

$$= \oint \vec{A}\cdot d\vec{\ell}$$ ← line integral around the perimeter



<u>Solenoid</u>

$$\begin{cases} B_z \cdot \pi a^2 = A_\phi\, 2\pi s & s > a \\ B_z\, \pi s^2 = A_\phi\, 2\pi s & s < a \end{cases}$$



$$\vec{B}_z = \mu_0 \begin{cases} NI & s< \\ 0 & s> \end{cases}$$

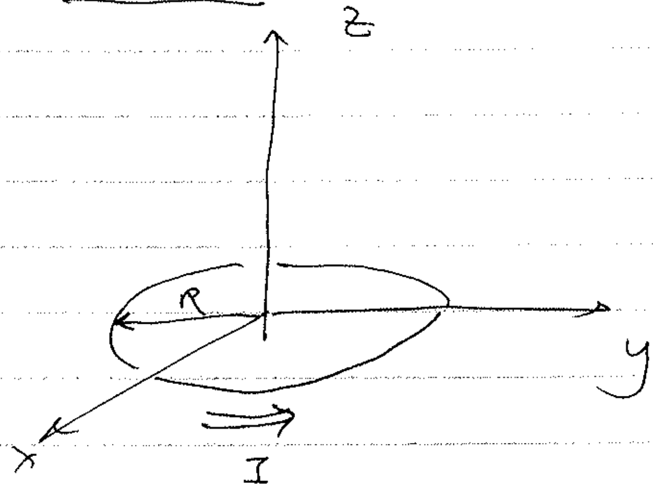
$$\Rightarrow A_\phi = \begin{cases} \mu_0 K \frac{a^2}{2s} & s > a \\ \mu_0 K \frac{s}{2} & s < a \end{cases}$$

<u>Vs</u>

$$B_z = \begin{cases} 0 & s > a \\ \mu_0 K & s < a \end{cases}$$

Approximation Schemes ($\rightarrow$ Multipole formulation for $\vec{A}$)

Wire loop

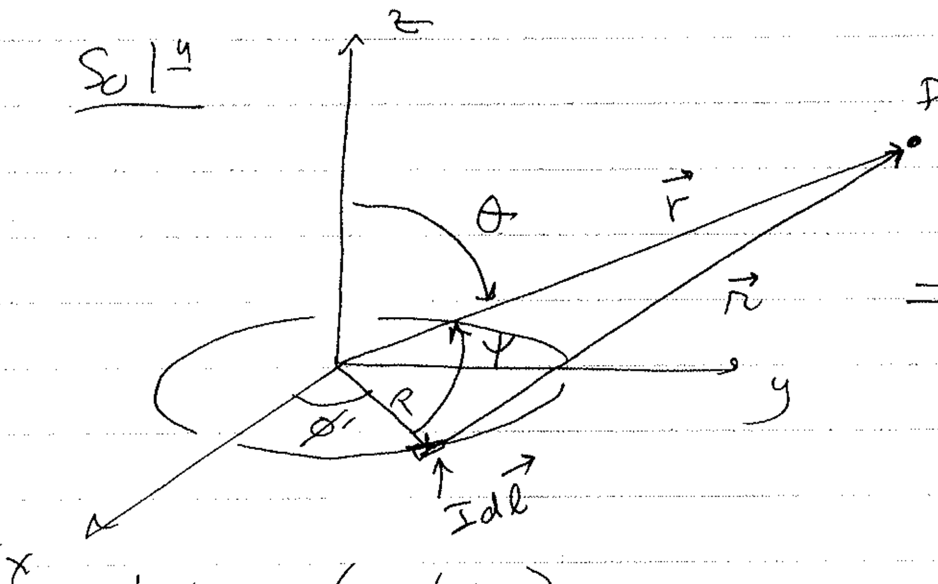


On-axis, the field is

$$\vec{B} = \hat{z}\,\frac{\mu_0 I}{2}\,\frac{R^2}{(R^2+z^2)^{3/2}}$$

Q: Can we find the field for arbitrary locations?

Sol^{n}



$\Rightarrow$ we need $\cos\gamma$ to find $|\vec{\mathscr{r}}|$

look up (or derive)

$$\cos\gamma = \cos\theta\underset{\theta'=\frac{\pi}{2}}{\cos\theta'} + \sin\theta\underset{\theta'=\frac{\pi}{2}}{\sin\theta'}\underset{\text{choose }\phi=0}{\cos(\phi-\phi')}$$

$$= \sin\theta\cos(-\phi') = \sin\theta\cos\phi'$$

$$\Rightarrow \boxed{\begin{aligned}\mathscr{r}^2 &= r^2+R^2-2rR\cos\gamma\\ &= r^2+R^2-2rR\sin\theta\cos\phi'\end{aligned}}$$

$$\rightarrow \boxed{dA_\phi = \frac{\mu_0}{4\pi} \frac{I R \cos\phi' d\phi'}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos\phi'}}}$$

elliptic integral!

Q: How to proceed?

$r > R$

A:
$$\frac{1}{\underbrace{\sqrt{r^2+R^2-2rR(\sin\theta\cos\phi')}}} = \frac{1}{r}\frac{1}{\sqrt{1+\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)(\sin\theta\cos\phi')}}$$

well, we will express this term using the generating function

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l P_l(\sin\theta\cos\phi')$$

$$= \quad \vdots$$

$\rightarrow$ Multipole Formulation for $\vec{A}$

or, let's not immediately go to $r \gg R$

$$A_\phi = \frac{\mu_0 I R}{4\pi} \int \frac{\cos\phi' \, d\phi'}{\sqrt{r^2+R^2}} \left\{ 1 + \frac{rR}{r^2+R^2} \sin\theta \cos\phi' - \frac{3}{4} \frac{r^2 R^2 \sin^2\theta \cos^2\phi'}{(r^2+R^2)^2} + \frac{5}{8} \frac{r^3 R^3 \sin^3\theta \cos^3\phi'}{(r^2+R^2)^3} - \cdots \right\}$$

<u>note:</u> $\int_0^{2\pi} \cos\phi' \, d\phi' = \sin\phi' \Big|_0^{2\pi} = 0$

$$\int_0^{2\pi} \cos^2\phi' \, d\phi' = \pi$$

$$\int_0^{2\pi} \cos^3\phi' \, d\phi' = \int_0^{2\pi} (1 - \sin^2\phi') \, d(\sin\phi') = \sin\phi' - \frac{1}{3}\sin^3\phi' \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \cos^4\phi' \, d\phi' = \int_0^{2\pi} (1 - \sin^2\phi')(1 - \sin^2\phi') \, d\phi'$$

$$= \int_0^{2\pi} \left[1 - 2\sin^2\phi' + \sin^4\phi' \right] d\phi'$$

$$A_\phi = \frac{\mu_0 I R}{4\pi \sqrt{r^2+R^2}} \left[0 + \frac{\pi r R}{(r^2+R^2)} \sin\theta + 0 \right], \quad \text{let } r \gg R$$

$$\simeq \frac{\mu_0}{4\pi} \left[\frac{I \pi R^2}{r^2 \left(1 + \frac{R^2}{r^2}\right)^{3/2}} \sin\theta \right]$$

$$\simeq \frac{\mu_0}{4\pi} \left(I \pi R^2 \right) \frac{\sin\theta}{r^2} \left(1 - \frac{3}{2} \frac{R^2}{r^2} \right)$$

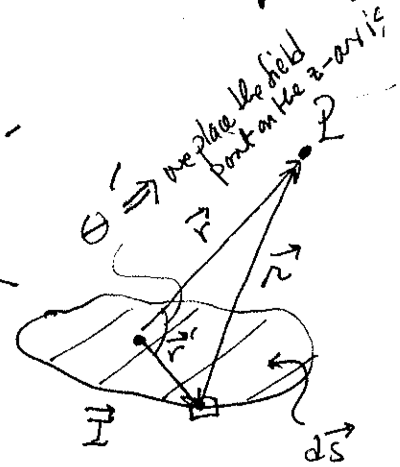
Multipole Formulation

More generally (as we alluded to earlier)

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\,d\tau}{\mathscr{r}} \quad \text{(return to this form later)}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\oint \frac{I\,d\vec{r}\,'}{\mathscr{r}} \quad \text{for a current loop} \left(\begin{array}{l}\text{for now, because}\\ \text{this is simpler}\end{array}\right)$$

$$= \frac{\mu_0}{4\pi}\oint \frac{I\,d\vec{r}\,'}{\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}} \quad ; \text{if } r > r'$$



$$= \frac{\mu_0}{4\pi}\frac{I}{r}\oint d\vec{r}\,' \frac{1}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'}}$$

generating function

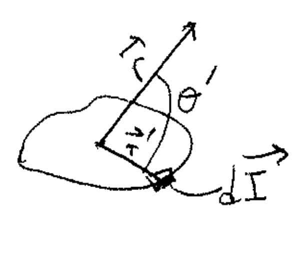
$$= \frac{\mu_0 I}{4\pi r}\oint d\vec{r}\,' \sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta')$$

$$= \frac{\mu_0 I}{4\pi r}\oint d\vec{r}\,'\Big[\underbrace{1}_{\substack{l=0\\ \rightarrow 0,\ \text{always!}}} + \underbrace{\left(\frac{r'}{r}\right)P_1(\cos\theta')}_{\text{dipole}} + \underbrace{\left(\frac{r'}{r}\right)^2 P_2(\cos\theta')}_{\text{quadrupole}} + \cdots\Big]$$

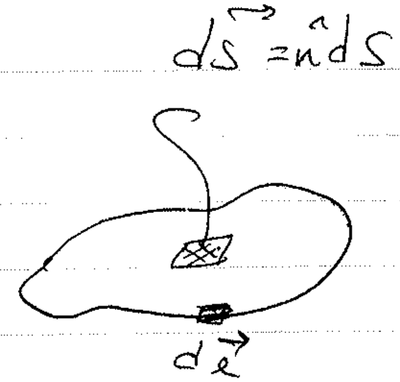
Consider

$$\vec{A}_d = \frac{\mu_0 I}{4\pi r^2}\oint r'\,d\vec{r}\,'\cos\theta' \quad \Longleftarrow \text{dipole term}$$

$$= \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}\,')\,d\vec{r}\,'$$



$$\vec{A}_d = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r}\,') \, d\vec{r}\,'$$



Equation (1.108)

$$\oint (\vec{c} \cdot \vec{r}) \, d\vec{l} = \left(\int_{\text{surface}} d\vec{S} \right) \times \vec{C}$$

now $\vec{c} \equiv \hat{r}, \ \vec{r} = \vec{r}\,'$

$$\Rightarrow \vec{A}_d = \frac{\mu_0}{4\pi r^2} I \left[\int_{\text{surface}} d\vec{S} \times \hat{r} \right]$$

$$\boxed{\vec{A}_d = -\frac{\mu_0}{4\pi r^2} \hat{r} \times \int_{\text{surface}} I \, d\vec{S}}$$

for a flat, wire loop $\boxed{\vec{m} = \text{moment} = \int I \, d\vec{S}}$

and $\boxed{\vec{A}_d = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}}$

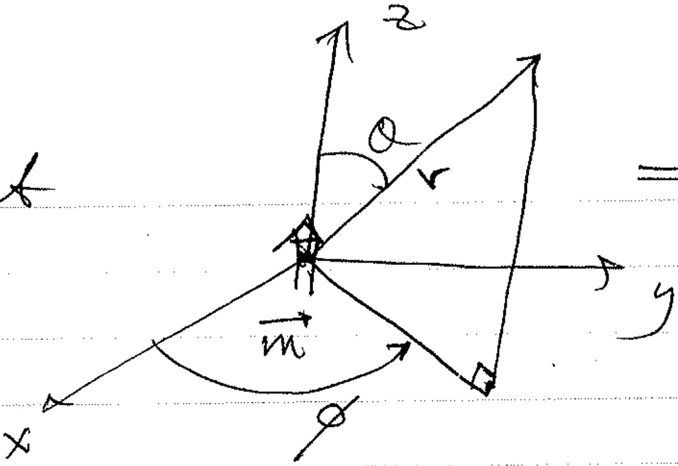
$$\Rightarrow \vec{B}_d = \vec{\nabla} \times \vec{A}_d$$

$$= \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} (\sin\theta \, A_\phi) - \frac{\partial A_\theta}{\partial\phi} \right] \hat{r}$$

$$+ \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta}$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial}{\partial\theta} A_r \right] \hat{\phi}$$

now set



$\Rightarrow \vec{m} \times \vec{r} = mr\sin\theta\,\hat{\phi}$

$$\Rightarrow \vec{B}_d = \frac{1}{r\sin\theta}\left(\frac{2\sin\theta\cos\theta\, m}{r^2}\frac{\mu_0}{4\pi}\right)\hat{r}$$

$$+ \frac{1}{r}\left(-\frac{m\sin\theta}{-r^2}\frac{\mu_0}{4\pi}\right)\hat{\theta}$$

$$+ (0)\,\hat{\phi}$$

$$\boxed{\vec{B}_d = \frac{\mu_0 m}{4\pi}\left(\frac{2\cos\theta}{r^3}, \frac{\sin\theta}{r^3}, 0\right)}$$

Derivation of Equation 1.108

Prob 1.60

$$\int(\vec{\nabla}\times\vec{T})\cdot d\vec{S} = \oint \vec{T}\cdot d\vec{l}$$, Stokes' theorem

Let $\vec{T} = \vec{c} f$

constant vector

scalar function

$$\int(\underbrace{\vec{\nabla}\times\vec{c}f}_{\text{ID \#5}})\cdot d\vec{S} = \oint \vec{c}f\cdot d\vec{l} = \vec{c}\cdot\oint f\,d\vec{l}$$

constant vector

$$f(\cancel{\vec{\nabla}\times\vec{c}}^{\,0}) - \vec{c}\times(\vec{\nabla}f)$$

$$-\int\underbrace{\left[\vec{c}\times(\vec{\nabla}f)\right]\cdot d\vec{S}} = \vec{c}\cdot\oint f\,d\vec{l}$$

$$-\int\underbrace{\vec{c}\cdot(\vec{\nabla}f\times d\vec{S})}_{\text{work it out by plugging in}}$$

work it out by plugging in $\vec{c} = (c_x, c_y, c_z)$

$$\vec{\nabla}f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)f$$

$$d\vec{S} = (dS_x, dS_y, dS_z)$$

$$\Rightarrow \vec{c}\cdot\left[\oint \vec{\nabla}f\times d\vec{S} + \oint f\,d\vec{l}\right] = 0$$

$$\Rightarrow \boxed{\oint \vec{\nabla}f\times d\vec{S} = -\oint f\,d\vec{l}}$$

(1.60e) $\oint \vec{\nabla} f \times d\vec{S} = -\oint f \, d\vec{l}$, f is a scalar function

Let $f = \vec{C} \cdot \vec{r}$ ← constant vector

$$\oint \left[\underbrace{\vec{\nabla}(\vec{C} \cdot \vec{r})} \times d\vec{S} \right] = -\oint (\vec{C} \cdot \vec{r}) \, d\vec{l}$$

Vector ID 4, $\vec{\nabla}(\vec{C} \cdot \vec{r}) = \vec{C} \times (\vec{\nabla} \times \vec{r}) + \vec{r} \times \cancel{(\vec{\nabla} \times \vec{C})}^{\,0}$

$$+ (\vec{C} \cdot \vec{\nabla})\vec{r} + \cancel{(\vec{r} \cdot \vec{\nabla})\vec{C}}^{\,0}$$

$$= \vec{C} \times (\underbrace{\vec{\nabla} \times \vec{r}}_{\to 0}) + (\vec{C} \cdot \vec{\nabla})\vec{r}$$

$$= (\vec{C} \cdot \vec{\nabla})\vec{r}$$

$$\Rightarrow \oint \underbrace{(\vec{C} \cdot \vec{\nabla})\vec{r}}_{\vec{C}} \times d\vec{S} = -\oint (\vec{C} \cdot \vec{r}) \, d\vec{l}$$

$$\vec{C} \times \oint d\vec{S} = -\oint (\vec{C} \cdot \vec{r}) \, d\vec{l}$$

$$\boxed{\left(\oint d\vec{S}\right) \times \vec{C} = \oint (\vec{C} \cdot \vec{r}) \, d\vec{l}}$$

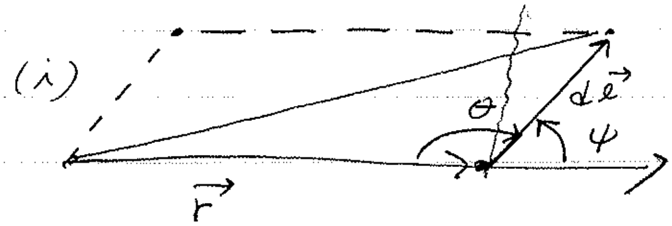
Eqn. (1.108)

Egiston 1.108

surface integral

note: $\oint (\vec{c}\cdot\vec{r})\, d\vec{\ell} = \left(\int_S d\vec{s}\right) \times \vec{c} \qquad (1.108)$

Surface area element. Consider

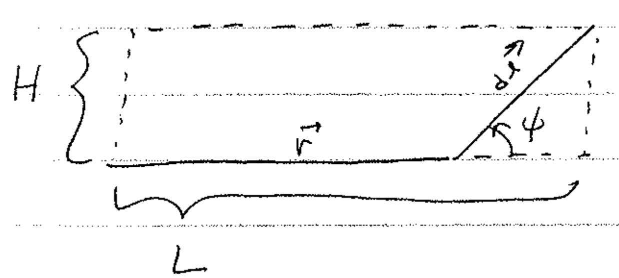


{ Consider a vector $\vec{r}$ and $d\vec{\ell}$ and take their cross-product }

$\psi + \theta = \pi$

$$\Longrightarrow |\vec{r} \times d\vec{\ell}| = \|\vec{r}\|\,\|d\vec{\ell}\|\, \underbrace{\sin\psi}_{\sin\psi = \sin(\pi-\theta) = \sin\theta}$$

a) Consider the area of the rectangle



{ $L * H$ = Area }

$$\text{Area} = \left[|\vec{r}| + |d\vec{\ell}|\cos\psi\right] * \left[|d\vec{\ell}|\sin\psi\right]$$

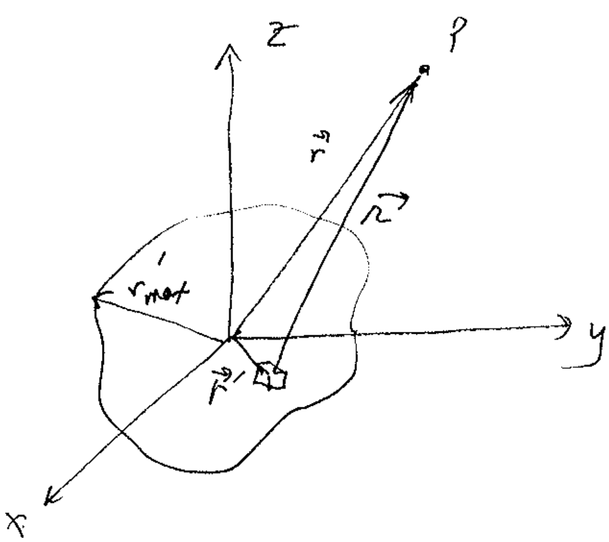
$$= \underbrace{|\vec{r}|\,|d\vec{\ell}|\sin\psi}_{\text{area of parallelogram}} + \underbrace{|d\vec{\ell}|^2 \sin\psi\cos\psi}_{2\times\text{area of triangles}}$$

Area of 1/2 the parallelogram is

$$\tfrac{1}{2}\|\vec{r}\|\,\|d\vec{\ell}\|\sin\psi$$

$\longrightarrow \frac{1}{2}\vec{r}\times d\vec{\ell}$ = area of region swept out by vector $\vec{r}$.

Moment formulation for $\vec{J}(\vec{r})$, arbitrary currents



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')d\tau}{\mathscr{r}}$$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$\vec{\mathscr{r}}\cdot\vec{\mathscr{r}} = \mathscr{r}^2 = (\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')$

arbitrary coordinate system

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')d\tau}{\sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'}}$$

$r > r'_{max}$

$$\approx \frac{\mu_0}{4\pi r}\int \frac{\vec{J}(\vec{r}')d\tau}{\sqrt{1-2\frac{\vec{r}\cdot\vec{r}'}{r^2}+\frac{r'^2}{r^2}}}$$

$$\approx \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{r}')\, d\tau\left[1+\frac{\vec{r}\cdot\vec{r}'}{r^2} - \dots\right]$$

difficult because of $\vec{J}$; $\vec{J}$ is a vector.

To deal with this, let's define an auxiliary function

define $\left[\underbrace{f}_{\text{some arbitrary function}}\vec{J}\right]$ as an auxiliary function and then consider its properties

$$\int \vec{\nabla}_{r'}\cdot\left(f(\vec{r}')\vec{J}(\vec{r}')\right) d\tau \overset{\text{divergence theorem}}{=} \oint f(\vec{r}')\vec{J}(\vec{r}')\cdot d\vec{S} = 0$$

Surface encloses all current

okay $\Rightarrow \vec{\nabla}_{r'} \cdot \left[f(\vec{r}') \vec{J}(\vec{r}') \right] d^3x = 0$

$$\Rightarrow \int \left[f(\cancel{\vec{\nabla}_{r'} \cdot \vec{J}}) + (\vec{\nabla}_{r'} f \cdot \vec{J}) \right] d\tau = 0$$

from continuity eqn $\Rightarrow \boxed{\int \vec{J} \cdot \vec{\nabla}_{r'} f \, d\tau = 0}$

We're ready!

a) Suppose $f = x_i'$

$$\Rightarrow \int (J_i \, d\tau) = 0 \Rightarrow \boxed{\int \vec{J} \, d\tau = 0 \qquad (i)}$$

$\Rightarrow$ monopole term $\rightarrow 0$ (no magnetic charges)

b) Suppose $f = x_i' x_j'$

$$\Rightarrow \boxed{\int \left[J_i x_j' + J_j x_i' \right] d\tau = 0 \qquad (ii)}$$

or

$$\int \left[x_j' \hat{x}_i + x_i' \hat{x}_j \right] \cdot \vec{J} \, d\tau = 0$$

Now return to dipole term

$$\vec{A}_d = \frac{\mu_0}{4\pi r^3} \int \vec{J} (\vec{r} \cdot \vec{r}') \, d\tau$$

$(x_1 x_1' + x_2 x_2' + x_3 x_3')$

Consider i-th component of $\vec{A}_d$

$$A_i = \frac{\mu_0}{4\pi r^3} \sum_{j=1}^{3} x_j \int J_i x_j' \, d\tau$$

symmetric part $\rightarrow 0$ from (ii); anti-symmetric part

$$= \frac{\mu_0}{4\pi r^3} \sum_{j=1}^{3} x_j \int \left[\cancel{\frac{1}{2}(x_j' J_i + x_i' J_j)} + \frac{1}{2}(x_j' J_i - x_i' J_j) \right] d\tau$$

sums to 0

$$A_i = \frac{\mu_0}{4\pi r^3}\sum_{j=1}^{3}\int \frac{1}{2}\left(J_i x_j x_j' - x_j x_i' J_j\right) d\tau$$

<u>In vector form</u>

$$A_i = \frac{\mu_0}{4\pi r^3}\left\{\frac{1}{2}\int\left(J_i(\vec{r}\cdot\vec{r}') - r_i'(\vec{r}\cdot\vec{J})\right)d\tau\right\}$$

$$= \frac{\mu_0}{4\pi r^3}\,\frac{1}{2}\int\left[\vec{J}(\vec{r}\cdot\vec{r}') - \vec{r}'(\vec{r}\cdot\vec{J})\right]_i d\tau$$

$$B(A\cdot C) - C(A\cdot B)$$

$$= \frac{\mu_0}{4\pi r^3}\,\frac{1}{2}\int(\vec{r}\times\vec{J}\times\vec{r}')_i\, d\tau$$

go to vector form (generaliz

<u>define:</u> $\boxed{\vec{m} = \frac{1}{2}\int(\vec{r}'\times\vec{J})\,d\tau}$

$$\vec{A}_d = \frac{\mu_0}{4\pi r^3}\,\vec{r}\times(-\vec{m})$$

$$\boxed{\vec{A}_d = \frac{\mu_0}{4\pi r^3}(\vec{m}\times\vec{r})}$$, as before but with

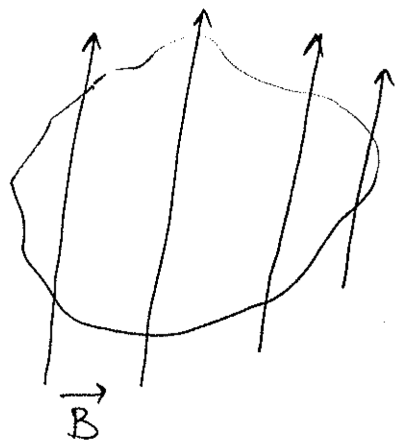
for a planar circuit,

$$\vec{m} = \frac{1}{2}\oint \vec{r}'\times I\,d\vec{r}'$$

$$\Rightarrow \boxed{\vec{m} = I\hat{n}A}$$

normal to circuit

Chapter 6: Magnetic Fields in Matter



what happens?

a) $\vec{B}\uparrow$ (para magnetism)

b) $B\downarrow$ (dia magnetism)

$\Delta\vec{B}$ goes away after the $\vec{B}$ is turned off.

note:(*) dia- << para-

(c) $\vec{B}=0$. Ferro magnetism; retains magnetism even after $\vec{B}$ is turned-off.

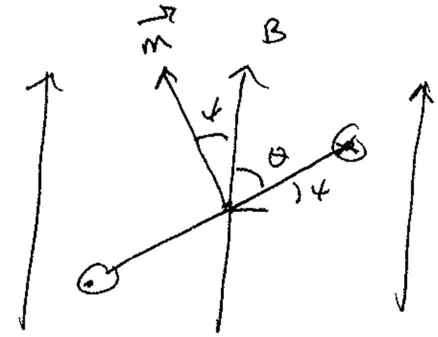
To study these effects, we first consider the origins of para, dia, and ferro-magnetism.

---

* for dielectrics recall that

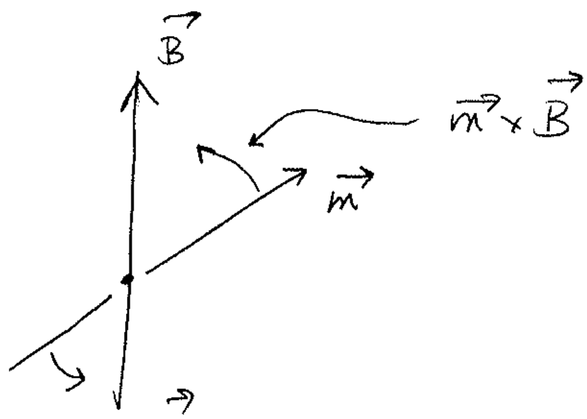
Induced Polarization (distort atoms or molecules) << Orientational Polarization (align molecules or materials)

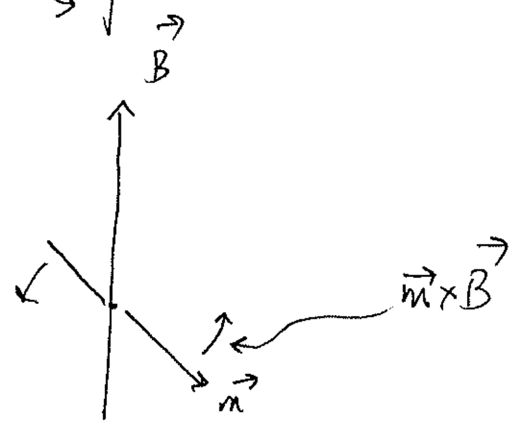
$$\vec{N} = \underbrace{I l^2}_{\substack{|\vec{m}| \\ = \\ IA}} B \sin\psi\,(-\hat{x})$$

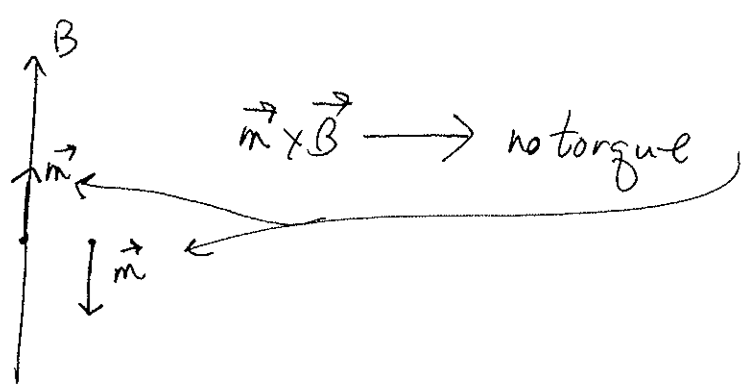


$$\Rightarrow \vec{N} = \vec{m} \times \vec{B}$$, and generalizes

Example,

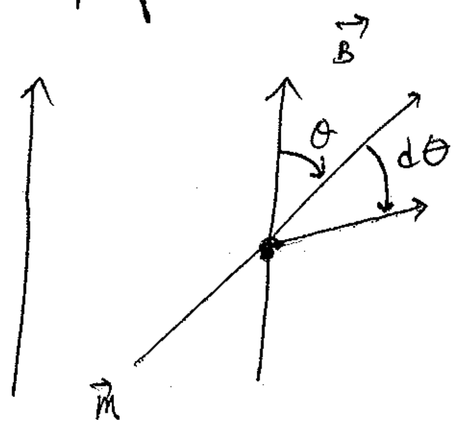






Energy

The preferred $\vec{N}=0$ state is determined by U_B



a) How much work is required to rotate the dipole by $d\theta$

b) $\vec{N} = \vec{m} \times \vec{B}$

$\vec{N}$ is in $-\hat{\theta}$-dir^{n}

$$\rightarrow dW = -\left[-mB\sin\theta\right] d\theta$$

$$W = -mB\cos\theta = -\vec{m}\cdot\vec{B}$$

$$\longrightarrow \boxed{U_B = -\vec{m}\cdot\vec{B}}$$

$\longrightarrow$ U_B is minimized when $\vec{m} \parallel \vec{B}$

an alternate way to view force

$$\Rightarrow \quad \vec{F}_B = -\vec{\nabla} U_B = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

But, $\vec{N}_e = \vec{p}\times\vec{E}$

$U_e = -\vec{p}\cdot\vec{E}$

but, $\vec{F}_e = (\vec{p}\cdot\vec{\nabla})\vec{E}$ (?)

Comment

a) $$\vec{F}_B = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

$$= \vec{m}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times\cancel{(\vec{\nabla}\times\vec{m})}^{\,0,\text{ if } \vec{m} \text{ constant}}$$

$$+ (\vec{m}\cdot\vec{\nabla})\vec{B} + \cancel{(\vec{B}\cdot\vec{\nabla})\vec{m}}^{\,0,\text{ if } \vec{m} \text{ constant}}$$

$$\longrightarrow \boxed{\vec{F} = \vec{m}\times(\vec{\nabla}\times\vec{B}) + (\vec{m}\cdot\vec{\nabla})\vec{B}}$$

in general, $\vec{\nabla}\times\vec{B}\neq 0$

b) $$\vec{F}_{ES} = \vec{\nabla}(\vec{p}\cdot\vec{E}) = (\vec{p}\cdot\vec{\nabla})\vec{E},$$ because $\vec{\nabla}\times\vec{E} = 0$ (in electrostatics)

$\Longrightarrow$ $\vec{F}_B$ and $\vec{F}_{ES}$ are not "equivalent" forms unless

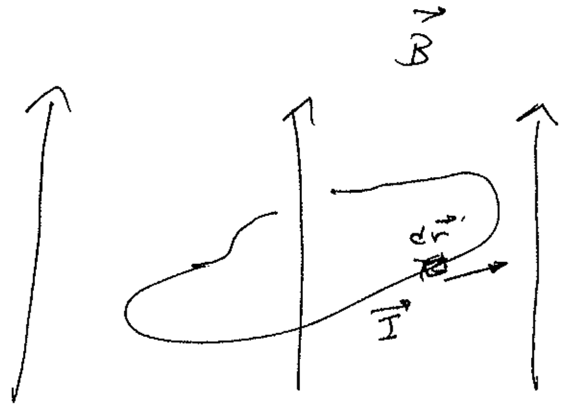
$$\vec{\nabla}\times\vec{B} = 0 \rightarrow \vec{J} = 0$$

Paramagnetism

Torque, Force, & Energy

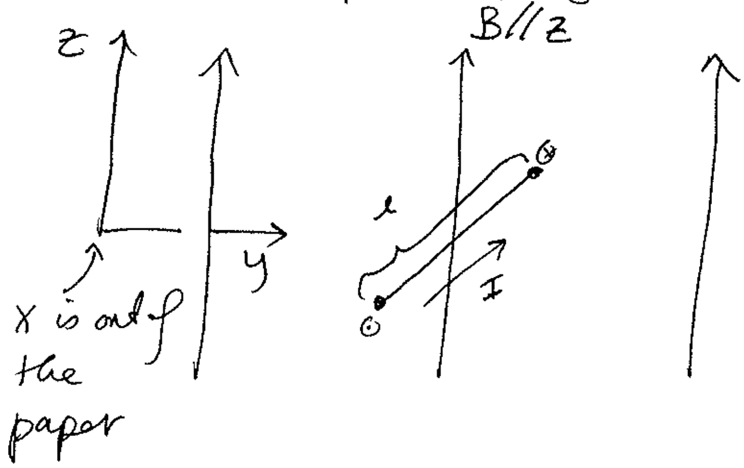
(i) $d\vec{F} = I\, d\vec{r}\,' \times \vec{B}$

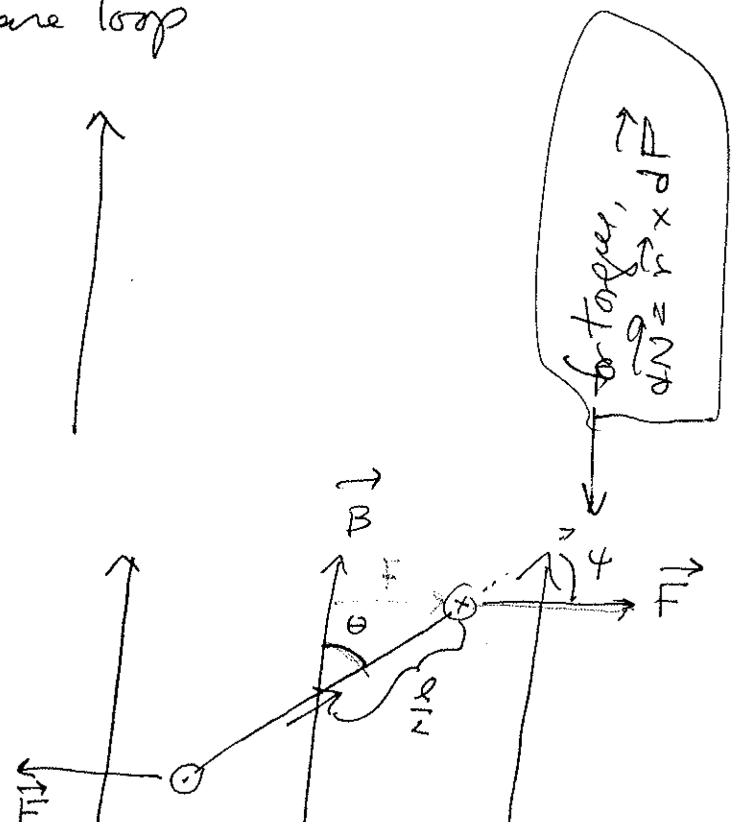
(ii) $d\vec{N} = \vec{r}\,' \times d\vec{F}$



Torque.

Consider a planar square loop





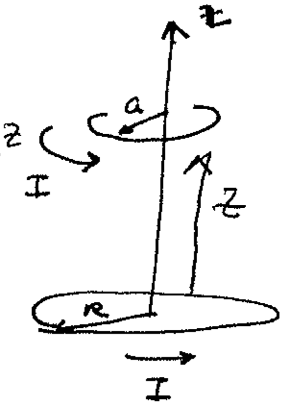
$$\Longrightarrow \vec{N} = \underbrace{\left(\frac{\ell}{2}\right)}_{|\vec{r}\,'|} \underbrace{\left[I \ell B \right]}_{|\vec{F}|} \sin\psi \, (-\hat{x}) \underbrace{\times 2}_{\text{other side of loop}}$$

← about x-axis

Forces on Dipoles

what is the force on $\vec{m}$

let:
$|a| << |R|, z$



approx. fields
$$\begin{cases} B_s \approx \dfrac{3\mu_0 I R^2}{4} \dfrac{sz}{(R^2+z^2)^{5/2}} \\ B_z \approx \dfrac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \left[1 + \dfrac{3s^2}{4} \dfrac{(R^2-4z^2)}{(R^2+z^2)^2} \right] \end{cases}$$
(see next page)

near z-axis

(i) $d\vec{F} = \vec{I}\,dr \times \vec{B}$ ← Lorentz Force law

(i) $\vec{I}dr \times B_z \hat{z} = I B_z \hat{s}\, dr$ → will integrate to 0 around loop

(ii) $\vec{I}dr \times B_s \hat{s} = I B_s \hat{z}\, dr$ → upward or downward force (we need B_s!)

$$\Rightarrow \boxed{\vec{F}_z = \oint \vec{I}dr \times B_s \hat{s} = 2\pi a I B_s \hat{z} = 2\frac{m}{a} B_s \hat{z}} \quad (A)$$

(ii) Note interaction Energy, $U_B = -\vec{m}\cdot\vec{B}$. interaction energy

Recall: $\vec{E} = -\vec{\nabla}V$ and $\vec{F} = q\vec{E} = -\vec{\nabla} qV$

$U_{Ed} = -\vec{p}\cdot\vec{E}$ and $\vec{F}_{Ed} = -\vec{\nabla}U_E = \vec{\nabla}(\vec{p}\cdot\vec{E})$

by analogy

$$U_B = -\vec{m}\cdot\vec{B} \Rightarrow \boxed{\vec{F}_D = \vec{\nabla}(\vec{m}\cdot\vec{B})}$$

$\vec{m} = m_0\hat{z}$, picks out B_z

Above, F_z seemed to depend on B_s; what's the deal?

'14

Well,

$F_z = \frac{\partial}{\partial z}(mB_z)$, the $\hat{s}$ component will average to 0

note: $\vec{\nabla}\cdot\vec{B} \longrightarrow m\frac{\partial B_z}{\partial z} = -\frac{m}{s}\frac{\partial}{\partial s}(sB_s)$

$$m\frac{\partial B_z}{\partial z} \approx -\frac{mB_s}{s} - m\frac{\partial B_s}{\partial s}$$

$$= -m\left[\frac{z}{(R^2+z^2)^{5/2}} + \frac{z}{(R^2+z^2)^{5/2}}\right]\frac{3\mu_0 IR^2}{4}$$

$$\boxed{F_z = -\frac{3\mu_0 IR^2 m}{2}\frac{z}{(R^2+z^2)^{5/2}}}$$

Same as (A) ?

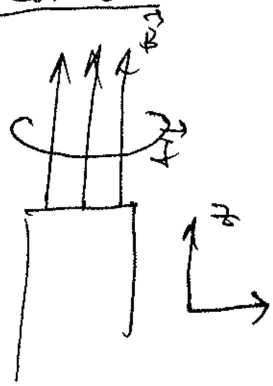
$= a$, radius of loop

from (A) $$F_z = -2\pi a I\left(\frac{3\mu_0 IR^2}{4}\frac{\overset{B_s}{s}z}{(R^2+z^2)^{5/2}}\right)$$

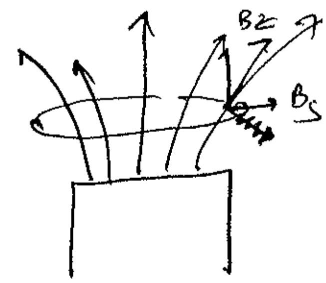
$$= -2\frac{m}{\cancel{\alpha}}\left(\frac{3\mu_0 IR^2}{4}\frac{\cancel{\alpha}z}{(R^2+z^2)^{5/2}}\right)$$

$$= -\frac{3\mu_0 IR^2 m}{2}\frac{z}{(R^2+z^2)^{5/2}}$$ ✓

Comment:

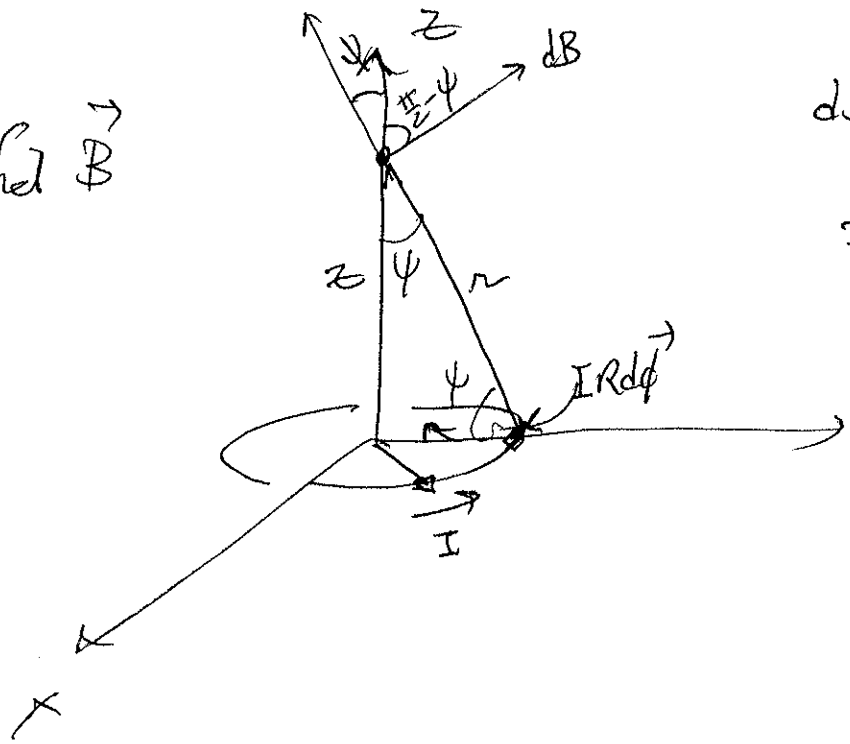
a) 

$\vec{F} = \vec{I}dl \times \vec{B}$ is radial if $\vec{B} = B_0 \hat{z}$

need 

need B_s to give z-component to force.

b) find $\vec{B}$



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{IR\,d\vec{\phi} \times \hat{r}}{(R^2+z^2)}$$

Find dB_z

$$dB_z = \frac{\mu_0 IR}{4\pi}\left(\frac{|d\vec{\phi}| \cdot z}{(R^2+z^2)^{3/2}}\right)$$

$$\boxed{B_z = \frac{\mu_0 IR}{2} \frac{R}{(R^2+z^2)^{3/2}}}$$

c) find $\vec{B}$ near z-axis. Use $\vec{\nabla}\cdot\vec{B} = 0$

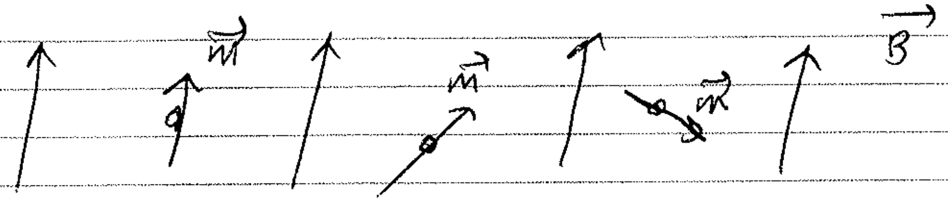
$$\frac{1}{s}\frac{\partial}{\partial s} sB_s + \cancel{\frac{1}{s}\frac{\partial}{\partial \phi} B_\phi}^{\,0} + \frac{\partial}{\partial z} B_z = 0$$

$$\Rightarrow \frac{1}{s}\frac{\partial}{\partial s}(sB_s) = -\frac{\partial}{\partial z} B_z = -\frac{\mu_0 IR^2}{2}\left(\frac{-\frac{3}{2}2z}{(R^2+z^2)^{5/2}}\right)$$

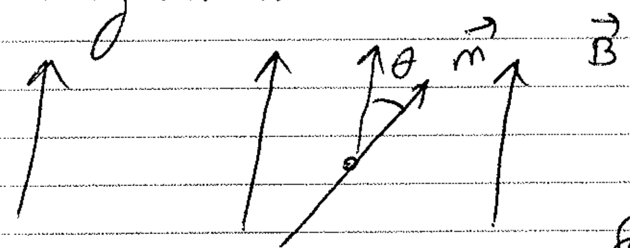
$$= \frac{3\mu_0 IR^2}{2}\left[\frac{z}{(R^2+z^2)^{5/2}}\right]$$

Paramagnetism (analogous to orientational polarization in electrostatics)

Imposed $\vec{B}$ aligns intrinsic spins (moments) of particles



As before, thermal motion (as measured by kT) opposes alignment.



$$U_B = -\vec{m}\cdot\vec{B}$$

for left

$$U_B = -m_0 \cos\theta\, B_0$$

We find average θ by solving

$$\langle U_B \rangle = \langle -m_0 B_0 \cos\theta \rangle = \frac{\int U_B e^{-U_B/kT} d\Omega}{\int e^{-U_B/kT} d\Omega}$$

$$\Rightarrow \langle \cos\theta \rangle = \frac{kT}{m_0 B_0}\left\{ \frac{m_0 B_0}{kT} \coth\left(\frac{m_0 B_0}{kT}\right) - 1 \right\}$$

$$\simeq \left(\frac{m_0 B_0}{3kT}\right) \text{ if } (m_0 B_0 << kT)$$

weak $(B_0 m_0)$

So, for a gas w/ number density N_V, we have a total average magnetization of

$$\vec{M} \equiv \text{Magnetization} \equiv \left[\frac{\text{dipole moment}}{\text{volume}}\right] \approx \text{"}\vec{P}\text{"}$$

$$= N_V \langle \vec{m} \rangle$$

$$= N_V \langle m_0 \cos\theta \rangle \hat{e}_B \quad \leftarrow \text{direction of } \vec{B}$$

$$\simeq N_V m_0 \left[\frac{m_0 B_0}{3kT}\right] \hat{e}_B \,, \quad \frac{m_0 B_0}{kT} \ll 1$$

$$\boxed{\vec{M} = \frac{N_V m_0^2 \vec{B}}{3kT}}$$

the susceptibility χ_M is then defined as

$$\boxed{\vec{M} = \chi_M \vec{H}} \longleftarrow \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

as opposed to $\vec{P} = \epsilon_0 \chi_e \vec{E}$; Using the $\vec{H}$ field rather $\vec{E}$ is a quirk

$$\cancel{\Longrightarrow \chi_M \quad \vec{M} = \chi_M^{vac}\left[\vec{H} + \vec{M}\right]\mu_0}$$

$$\cancel{\vec{M}(1 - \mu_0 \chi_M^{vac}) = \chi_M^{vac} \vec{H}}$$

$$\cancel{\chi_M \vec{H} \left(\frac{\chi_M^{vac}}{1 - \mu_0 \chi_M^{vac}}\right) \simeq \chi_M \left(1 + \mu_0 \chi_M^{vac}\right) \vec{H}}$$

this is a little ahead, however, because

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \left(\text{recall: } \vec{D} = \epsilon_0 \vec{E} + \vec{P} \right)$$

$$\Rightarrow \vec{M} = \frac{N_V m_0^2}{3kT} \vec{B}$$

$$= \left(\frac{N_V m_0^2}{3kT} \right) \mu_0 \left(\vec{H} + \vec{M} \right)$$

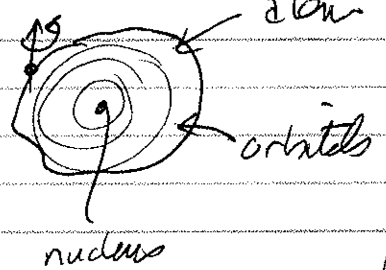
$$\vec{M} = \left(\frac{N_V m_0^2}{3kT} \right) \left(\frac{\mu_0}{1 - \frac{\mu_0 m_0^2 N_V}{3k}} \right) \vec{H}$$

$$\boxed{\vec{M} \simeq \frac{\mu_0 N_V m_0^2}{3kT} \vec{H}}$$

$\ll 1$

Q: What is the rough size expected for m_0?

in most circumstances, electrons are going to supply the magnetic moment, $\vec{m}$ for magnetization.



according to Pauli Exclusion Principle, can't have 2 fermions w/ identical "properties" (states) $\Rightarrow$ in deep orbitals, $S\uparrow$ + $S\downarrow$ are paired. Only unpaired valence e^- supply excess spin, $\vec{S}$, $\Rightarrow$ a moment

$\vec{S}$, $+q$ $\Rightarrow \vec{I} \Rightarrow \vec{m}$

$$\Rightarrow \chi_M \simeq \mu_0 \frac{N_v m^2}{3kT}\left(\frac{\overbrace{S}^{S \text{ state}}+1}{S}\right)$$

for particles,

$$\vec{m} = \underbrace{\overset{g\text{-factor}}{g}\left(\frac{Q}{2m}\right)}_{\text{gyromagnetic ratio}} \overset{\text{spin angular momentum}}{\vec{S}} \qquad \vec{S} \sim \hbar/2$$

Let's look at $\vec{m}$ for a second. Note: for an e^-,

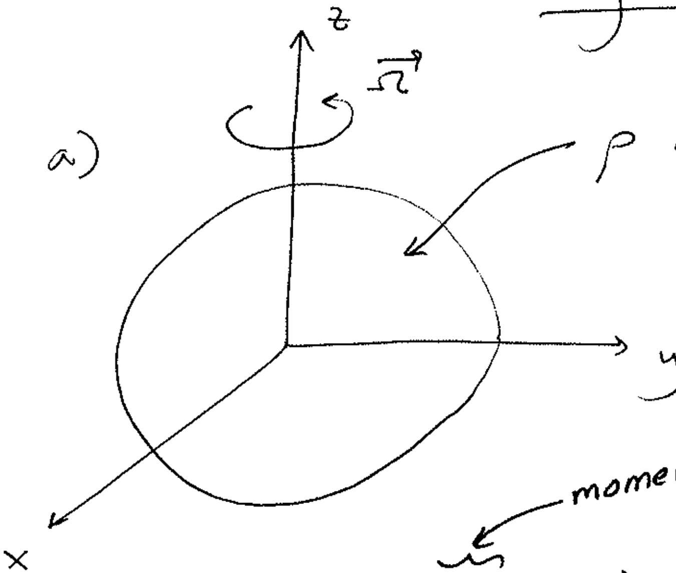
$$m_e = \frac{g_e}{2}\left|\frac{-e\hbar}{2m_e}\right|$$

$$= \frac{g_e}{2}\mu_{Bohr} \qquad \mu_{Bohr} = \text{Bohr magneton} = \frac{e\hbar}{2m_e}$$

$g_e \simeq 2$ (more on this later)

Gyromagnetic Ratios [looks at $\vec{m}/\vec{L}$ for an object]

a)



ρ and $\rho\left(\frac{M}{Q}\right) \equiv$ mass density $= \eta$

momentum element

ang. mom → (i) $d\vec{L} = \vec{r} \times d\vec{p} = \vec{r} \times \overbrace{\rho\left(\frac{M}{Q}\right)}^{\eta} \overbrace{\vec{\Omega} \times \vec{r}}^{\vec{v}}\, d\tau$

moment → (ii) $d\vec{m} = \frac{\vec{r} \times \vec{J} d\tau}{2} = \frac{1}{2} \vec{r} \times \rho \vec{\Omega} \times \vec{r}\, d\tau$

g-factor if $\frac{Q}{M}$ not constant

$$\frac{\vec{m}}{\vec{L}} = \frac{\int d\vec{m}}{\int d\vec{L}} = \left(\frac{Q}{2M}\right) \Longrightarrow \vec{m} = \left(\frac{Q}{2M}\right) \vec{L} = g\left(\frac{Q}{2M}\right) \vec{L}$$

comments?

gyromagnetic ratio

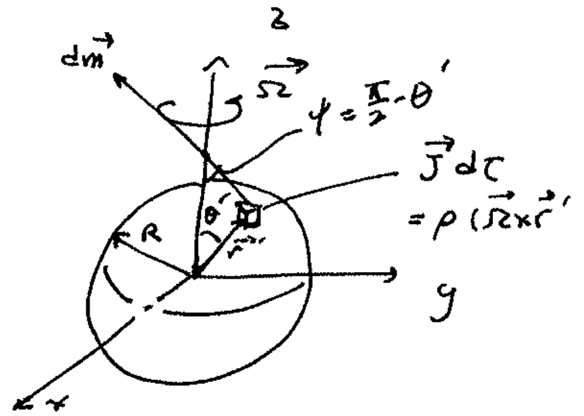
g-factor

b) $Q/M \neq$ constant

moment compared to angular momentum

b) $\boxed{\frac{Q}{M} \neq \text{constant case}}$

a) Moment of a charged Spinning sphere



$$\vec{m} = \frac{1}{2} \int \vec{r}' \times \vec{J}(\vec{r}') \, d\tau$$

$$M_z = \frac{1}{2} \int \underbrace{\rho \Omega r'^2 \sin\theta' d\tau}_{|d\vec{m}|} \; \underbrace{\cos\left(\tfrac{\pi}{2} - \theta'\right)}_{z \text{ projection}}$$

$$= \frac{1}{2} \rho \Omega \int \sin^2\theta' \sin\theta' r'^4 dr' d\theta' d\phi'$$

$$= \pi \rho \Omega \left(\frac{R^5}{5}\right) \int \sin^2\theta' d(-\cos\theta')$$

$$= \frac{2\pi}{15} \rho \Omega R^5 \left[-\cos\theta' + \frac{\cos^3\theta'}{3} \right]_0^\pi$$

$$\boxed{\vec{m} = \frac{4\pi}{15} \rho \Omega R^5 \hat{z}}$$

b) Angular momentum

$$\vec{L} = \int \vec{r}' \times \vec{p} \, d\tau$$

$$L_z = \int \left| \vec{r}' \times \rho \frac{M}{Q} \vec{v}' d\tau \right| \cos\left(\frac{\pi}{2} - \theta'\right)$$

$$= \int \rho \frac{M}{Q} r' \Omega r' \sin\theta' d\tau \sin\theta'$$

$$= \rho \frac{M}{Q} \Omega \int r'^4 dr' \sin^3\theta' d\theta' d\phi'$$

$$= \frac{2\pi}{5} \rho \frac{M}{Q} \Omega R^5 \int \sin^3\theta' d\theta'$$

$$\boxed{\vec{L} = \frac{8\pi}{15} \rho \frac{M}{Q} \Omega R^5 \hat{z}}$$

$$\Longrightarrow \boxed{\vec{m} = \frac{Q}{2M} \vec{L}} \Rightarrow \boxed{\text{gyromagnetic ratio} = \frac{Q}{2M}}$$

c) Suppose the charge is concentrated on the surface, but ρ_M is uniform

(i) $\sigma = Q_T / 4\pi R^2$

$$\vec{m}_z = \frac{1}{2}\int (\vec{r}\,' \times \sigma \vec{v}\,') \underbrace{\cos\left(\tfrac{\pi}{2} - \theta'\right)}_{\text{projection onto } z} \underbrace{dS}_{\text{area element}} \hat{z}$$

$$m_z = \frac{1}{2}\int (R\sigma\Omega R\sin\theta')\sin\theta' R^2 \sin\theta' d\theta' d\phi'$$

$$= \pi\sigma\Omega R^4 \int \sin^3\theta' d\theta'$$

$$= \frac{4\pi}{3}\left(\frac{Q_T}{4\pi R^2}\right)\Omega R^4$$

$$\boxed{\vec{m}_z = \hat{z}\,\frac{Q_T}{3}\Omega R^2 = \hat{z}\left(\frac{\Omega R^2}{3}\right)\left(\rho\frac{4\pi}{3}R^3\right) = \frac{4\pi}{9}\rho\Omega R^5 \hat{z}}$$

$$\Rightarrow \vec{m} = \frac{5Q}{6M}\vec{L} = \frac{5}{3}\frac{Q}{2M}\vec{L}$$

$\boxed{\text{gyromagnetic ratio} = \frac{5}{3}\frac{Q}{2M}}$

$g_{sh} = \frac{5}{3} g_{sphere}$

$\vec{m}/\vec{L}$ depends on $(Q/2M)$ ratio for a given $\vec{J}\,d\tau$ (current element)

c) <u>intrinsic $\vec{m}$ (due to spin $\vec{S}$)</u>, <u>e^-'s, p's, n's</u>

these intrinsic spins lead to <u>paramagnetism</u>.

$$\vec{m}_e = g_e \left|\left(-\frac{e}{2m_e}\right)\right| \vec{S}$$ ← spin angular momentum

g-factor

$|\vec{S}| = \hbar/2$ and

$$|\vec{m}_e| = \frac{g_e}{2}\left|\left(-\frac{e\hbar}{2m_e}\right)\right| = \left(\frac{g}{2}\right)\mu_{Bohr}$$ ← Bohr magneton

and

$$|\vec{m}_p| = \frac{g_p}{2}\left|\left(\frac{e\hbar}{2m_p}\right)\right| = \left(\frac{g_p}{2}\right)\mu_p$$ ← nuclear magneton

and $\frac{\mu_{Bohr}}{\mu_p} = \frac{m_p}{m_e} \approx 2{,}000 \Rightarrow$ nuclear spin effects are small compared to e^- spin effects

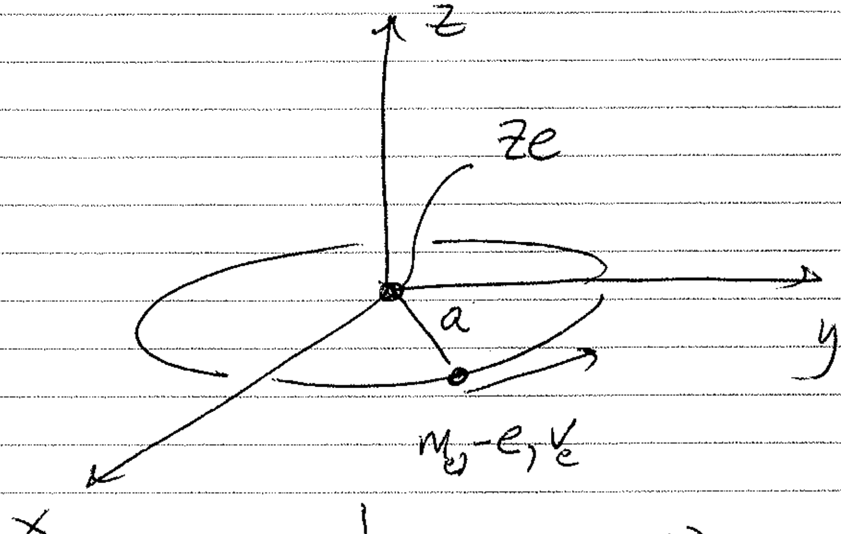
g	particle
5.591895	proton
−1.91	neutron
2.0023193043617	electron

<u>Comments</u>: Rel QM $\longrightarrow g_e = 2.0000$

QED $\longrightarrow$ little extra (above 2). Expt'ly verified confirming QED!

Diamagnetism (Induced $\vec{M}$)

Consider an "atom"



$\vec{m} = I\pi a^2(-\hat{z})$

the electron "orbits" about the nucleus in CCW direction as viewed from above.

$\Rightarrow \vec{I}$ is in CW direction

$\Rightarrow \vec{m} = \frac{1}{2} I \int \vec{r}' \times d\vec{r}'$

$= I\pi a^2(-\hat{z})$

$I = e\left(\frac{v_e}{2\pi a}\right) \delta(\vec{r}-\vec{r}')$ ← $\frac{1}{\text{period}}$

a) If a uniform $\vec{B} = B_0\hat{z}$ is applied

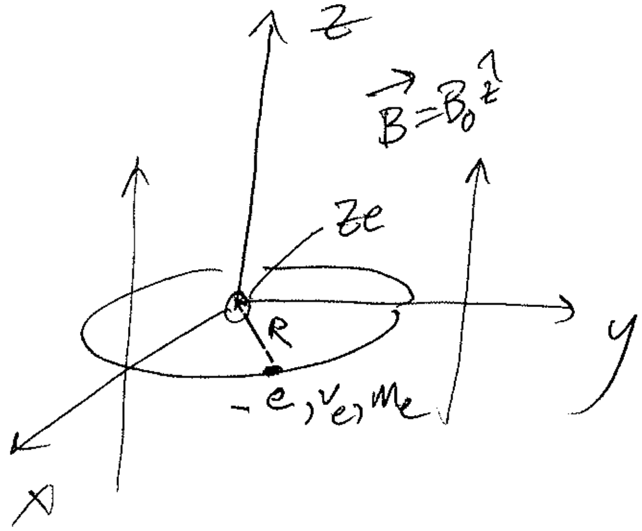
$\Rightarrow q\vec{v}\times\vec{B} \Rightarrow$ an inward radial force

$\Rightarrow v_e \uparrow$ to remain in orbit

$\Rightarrow \vec{m}$ gets larger in downward direction $\Rightarrow$ an increase in downward $\vec{B}$

$\Delta\vec{B}$ opposes the added $\vec{B}$.

$\Rightarrow$ diamagnetism opposes added $\vec{B}$. Let's estimate effect.



a) Case I, $\vec{B} = 0$

$$\underbrace{-\frac{Ze^2}{4\pi\epsilon_0 R^2}}_{\text{es force}} + \underbrace{m_e\frac{v_e^2}{R}}_{\text{centrifugal fce}} = 0$$

$$\Rightarrow v_e^2 = \frac{Ze^2}{4\pi\epsilon_0 m_e R}$$

$$\boxed{\omega_e^2 = \frac{Ze^2}{4\pi\epsilon_0 m_e R^3}}$$

b) Case II, $\vec{B}$ turned on

$$-\frac{Ze^2}{4\pi\epsilon_0 R^2} + m_e\frac{v_e^2}{R} \pm e\,\vec{v}_e \times \vec{B} = 0$$

hold R fixed!

if $\vec{v}_e // \vec{B} \Rightarrow v_e \uparrow$ to maintain eq.
$\vec{v}_e$ anti-// $\vec{B} \Rightarrow v_e \downarrow$ to maintain eq.

$$-\left[\omega_e^2 (m_e R)\right] + m_e \omega^2 R \pm e\omega R B = 0$$

$- \Rightarrow \vec{v} // \vec{B}$
$+ \Rightarrow \vec{v}$ anti-// $\vec{B}$

$$\Rightarrow \boxed{\omega^2 \pm \frac{eB}{m_e}\omega - \omega_e^2 = 0}$$

$\underline{\text{let}}$: $\omega = \omega_e + \Delta\omega \leftarrow |\Delta\omega| \ll \omega_e$

$$\left(\cancel{\omega_e^2} + 2\Delta\omega\,\omega_e\right) \pm \underbrace{\frac{eB}{m_e}}_{\omega_c}(\omega_e + \Delta\omega) - \cancel{\omega_e^2} \simeq 0$$

$$\Delta\omega\left[2\omega_e \pm \omega_c\right] \pm \omega_e\omega_c \simeq 0$$

$$\Delta\omega \simeq \pm\frac{\omega_e\omega_c}{2\omega_e \pm \omega_c}$$

$$= \pm\frac{1}{2}\omega_c\left(1 \mp \frac{\omega_c}{2\omega_e}\right)$$

now, if $\vec{B}$ is a perturbation,

$$\boxed{\Delta\omega = \pm\frac{1}{2}\omega_c}$$

okay, if $\vec{B}\uparrow$ and $\vec{m}\downarrow$ originally

$\Rightarrow$ V_e increases $\Rightarrow$ $\vec{m}\downarrow$ gets larger

if $\vec{B}\downarrow$ and $\vec{m}\downarrow$ originally

$\Rightarrow$ V_e decreases $\Rightarrow$ $\vec{m}\downarrow$ gets smaller

$\Rightarrow$ $\vec{B} \parallel \vec{m} \Longrightarrow \vec{m}$ weakens

$\vec{B}$ anti-$\parallel$ $\vec{m} \Rightarrow \vec{m}$ gets larger

let's estimate $\Delta\vec{m}$,

$$\vec{m} = I\vec{A} = -\hat{z}\underbrace{\left(\frac{e}{P}\right)\pi a^2} = -\frac{e}{2}\omega a^2\hat{z}$$

$$\boxed{\Delta\vec{m} = -\frac{e}{2}a^2(\Delta\omega)\hat{z}}$$

$$|\Delta\vec{m}| = \frac{ea^2}{2}\left|\frac{1}{2}\omega_c\hat{z}\right|$$

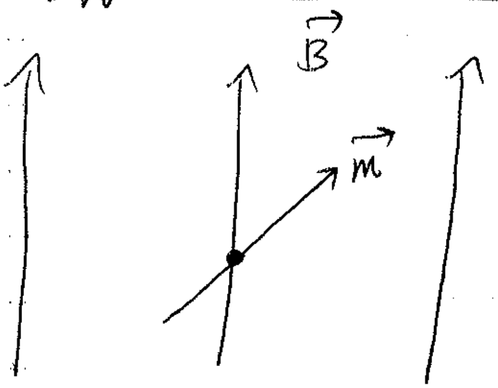
$$= \frac{ea^2}{4}\left(\frac{eB_0}{m_e}\right)$$

So:

$$\left|\frac{\Delta m_{para}}{\Delta m_{dia}}\right| \approx \frac{\frac{e\hbar}{2m_e}}{\frac{e^2a^2B_0}{4m_e}} = \frac{2\hbar}{ea^2B_0} \approx \frac{5\times10^5}{B} \gg 1$$

Larmor Precession

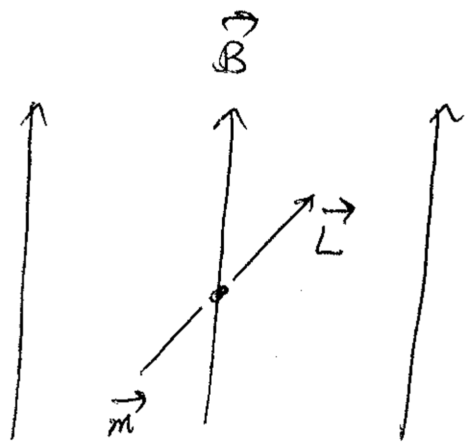
Suppose there is a $\vec{B}$-field



a) place a "wire loop" (moment $\vec{m}$) into the field.

b) what happens?

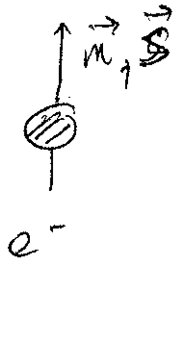
$\vec{N} = \vec{m} \times \vec{B} \longrightarrow \vec{m}$ & $\vec{B}$ will align



c) what if the object also has angular momentum $\vec{L} \parallel \vec{m}$?

$\vec{N} = \vec{m} \times \vec{B}$ still gives a torque, but we must now also consider $\vec{N} = \frac{d\vec{L}}{dt} \rightarrow$ something happens to $\vec{L}$

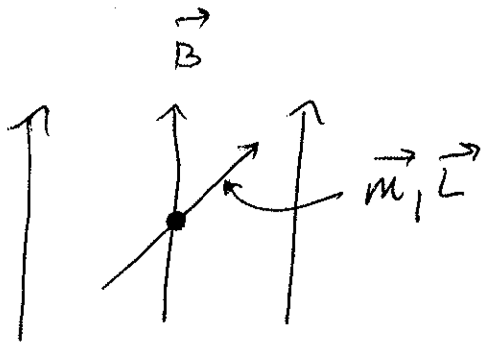
We'll consider this problem for an atom (classically). notice that atoms contain $\vec{L}$ (orbital), $\vec{S}$ (spin), and $\vec{I}$ (nuclear) angular momentum.



we need to do this problem and so we need to determine $\vec{m}, \vec{S}$ for particles.

Return to Larmor Precession

Eqn-of-Motion



$$\vec{m} = g\left(\frac{Q}{2M}\right)\vec{L}$$

$$\rightarrow \vec{N} = \frac{d\vec{L}}{dt} = \vec{m} \times \vec{B}$$

$$= g\left(\frac{Q}{2M}\right)\vec{L} \times \vec{B}$$

$$= \vec{L} \times \left(g\frac{Q\vec{B}}{2M}\right)$$

$$= \vec{L} \times (\vec{\omega}_L)$$

Larmor frequency

$$\longrightarrow \boxed{\frac{d\vec{L}}{dt} = -\vec{\omega}_L \times \vec{L}}$$

recall cyclotron motion where,

$$\frac{d\vec{u}}{dt} = -\vec{\omega}_B \times \vec{u} \Longrightarrow$$ circular motion about $\vec{B}$ at frequency ω_B

$$\frac{d\vec{L}}{dt} = -\vec{\omega}_L \times \vec{L} \Longrightarrow$$ circular motion about $\vec{B}$ at frequency ω_L

$\Rightarrow$ angular momentum vector $\vec{L}$ precesses about $\vec{B}$

"Larmor Precession"

Comments:

(a) Magnitude of $\vec{L} \Longrightarrow$ conserved (?)

$$\vec{L}\cdot\frac{d\vec{L}}{dt} = \begin{cases} \frac{d}{dt}\left(\frac{1}{2}L^2\right) \\ \vec{L}\cdot\left(\vec{\omega}_L\times\vec{L}\right) = 0 \end{cases}$$

$$\Longrightarrow \frac{d}{dt}\left(\frac{1}{2}L^2\right) = 0 \longrightarrow \|\vec{L}\| \text{ is conserved}$$

(b) Are all components conserved?

project $\vec{L}$ onto the $\vec{B}$-field axis (say z-axis)

$$\longrightarrow \frac{\vec{\omega}_L\cdot\vec{L}}{\|\vec{\omega}_L\|} = ? \quad ; \text{take } \frac{d}{dt}\left[\frac{\vec{\omega}_L\cdot\vec{L}}{\|\vec{\omega}_L\|}\right] = \frac{\vec{\omega}_L}{\|\vec{\omega}_L\|}\cdot\left(\frac{d\vec{L}}{dt}\right)$$

Consider $\vec{\omega}_L\cdot\left(\frac{d\vec{L}}{dt}\right) = \vec{\omega}_L\cdot\left(-\omega_L\times\vec{L}\right) = 0$

$\rightarrow \vec{\omega}_L\cdot\vec{L} =$ constant (projection of $\vec{L}$ on $\vec{B}$ does not change w/time)

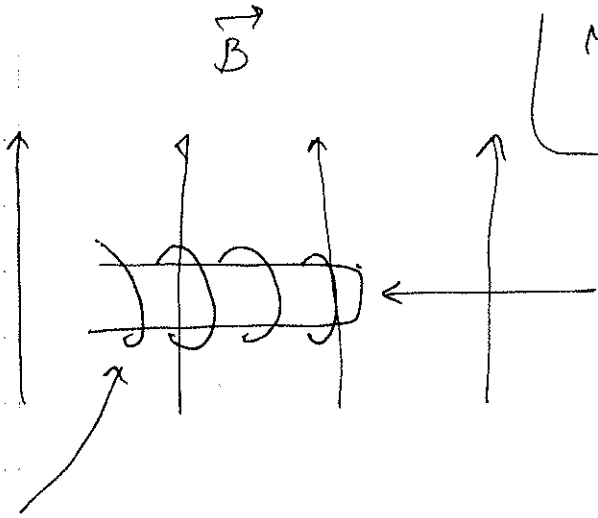
$\Longrightarrow \vec{L}$ components $\perp$ to $\vec{\omega}_L$ change $\Rightarrow$ precession

(c) Zeeman Effect

from Homework

recall: $\Delta\omega_z = \pm\frac{1}{2}\omega_B = \pm\frac{1}{2}\left(\frac{QB}{M}\right) = \pm\,\omega_L$!

MRI machine



hydrogens

$\Rightarrow \oplus \underbrace{\uparrow \downarrow}_{\Delta I}$

$$\Rightarrow \Delta\omega = \pm \omega_L = \pm g_p \frac{QB}{2M} = \pm g_p \mu_p B$$

and $\omega_L = \cancel{1.46 \times 10^{5} B_{1kG} \ R/s}$

Use 5–10kG

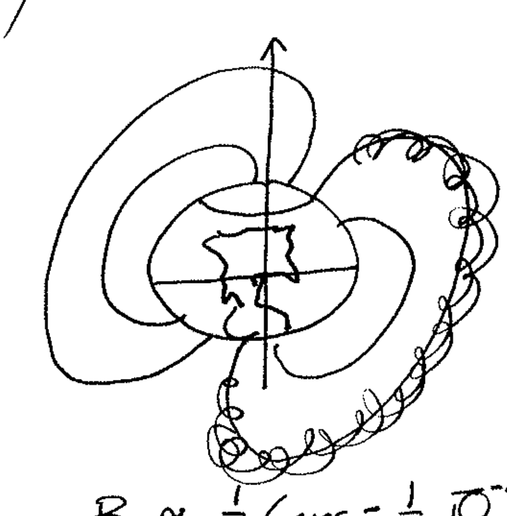
$$\approx 7 \times 10^{7} \ \frac{R}{s} \ B_{5kG}$$

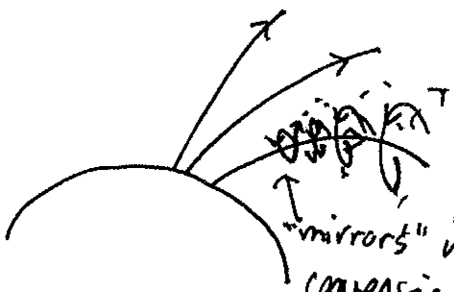
$$\boxed{\nu_L \approx 10 \text{ MHz } B_{5kG}}$$

RF signal

Magnetic Mirrors (Magnetic Bottles)

In ⊕'s and ♃'s atmosphere, see trapped "radiation" (high energy particles). Apparently bouncing around in the ⊕'s and ♃'s magnetosphere



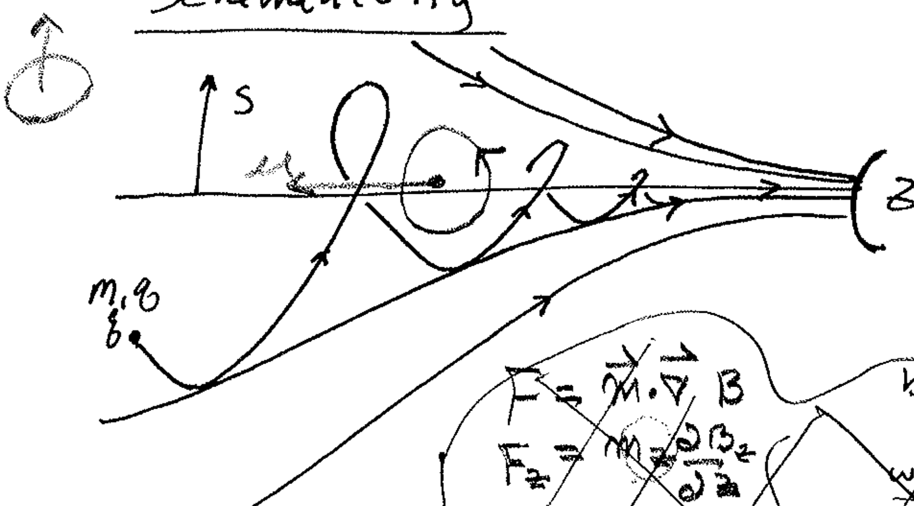


$B_\oplus \simeq \frac{1}{2}$ Gauss $= \frac{1}{2} 10^{-4}$ T

$$\Rightarrow \begin{cases} m_\oplus \simeq 8 \times 10^{22} \text{ A-m}^2 \\ m_♃ \simeq 1.4 \times 10^{27} \text{ A-m}^2 \end{cases}$$

~ dipole field for ⊕ & ♃

Schematically



a) As m, q moves into higher B_z-field

$\Rightarrow R_{gyro} = \frac{m v_\perp}{q B_z} \downarrow$

$F = \vec{m} \cdot \vec{\nabla} B$

$F_z = m_z \frac{\partial B_z}{\partial z}$

b) what about $v_\parallel$?

well, since $\vec{\nabla} \cdot \vec{B} = 0$

consider particles ~ on-axis (guiding center)

$$\frac{1}{s}\frac{\partial}{\partial s}(s B_s) + \frac{\partial}{\partial z} B_z = 0$$

$$\Rightarrow B_s \approx -\frac{s}{2}\left(\frac{\partial}{\partial z} B_z\right)\bigg|_{s=0}$$

(near axis) and so $B_z' =$ constant

$\frac{\partial}{\partial z} B_z > 0$ (converging field)

B_s is inward

Interestingly, as we will show, the magnetic moment arising from the cyclotron motion,

$$m = \frac{q V_\perp}{2\pi R_{gyro}} \times \pi R_{gyro}^2 = \frac{q}{2} V_\perp R_{gyro}$$

does not change (a lot) under certain conditions.

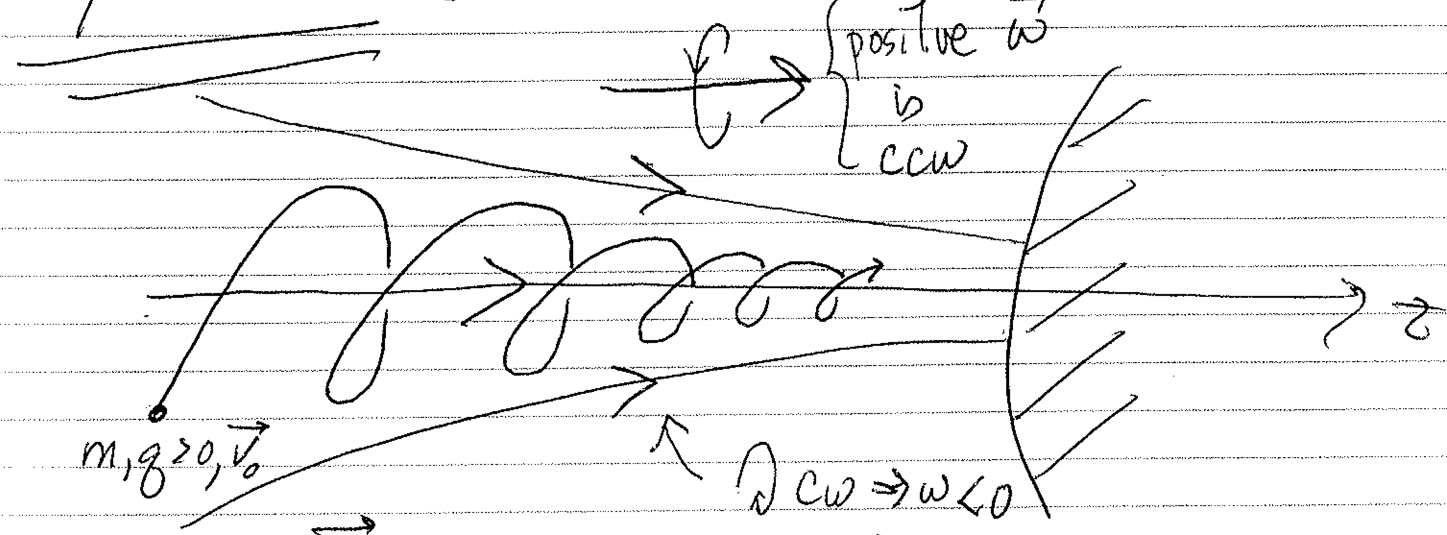
$\Rightarrow R_{gyro} \downarrow$ forces $V_\perp \uparrow$

Because a magnetic field does no work,

$K.E. = \frac{m}{2}(V_\perp^2 + V_\parallel^2)$ is fixed

$\Rightarrow$ as $V_\perp \uparrow \Rightarrow V_\parallel \downarrow$

and the particle can mirror. Let's work the problem out.



a) $\vec{B}$ converges as the $\oplus$ is approached $\Rightarrow \vec{B} = B_s \hat{s} + B_z \hat{z}$

b) q spirals as shown w/ local

$$R_{gyro} = \frac{m V_\perp}{q B_z}$$ and an ~~in~~ vertical motion, V_z

c) the B_s component $\Longrightarrow$ there is an "retarding" force in the z-direction

$$\left(q\,\vec{V}_\perp \times \vec{B_s}\right)$$

$$\Longrightarrow f_z = q\,v_\perp B_s$$

d) As before, let's get B_s from $\vec{\nabla}\cdot\vec{B} = 0$

$$\frac{1}{s}\frac{\partial}{\partial s}(sB_s) + \frac{\partial}{s\,\partial\phi}B_\phi + \frac{\partial}{\partial z}B_z = 0$$

$$\Longrightarrow \boxed{B_s \simeq -\frac{s}{2}\frac{\partial B_z}{\partial z}}$$

if $\frac{\partial B_z}{\partial z} > 0 \Longrightarrow$ force will become

$$f_z = \frac{q\,v_\perp s}{2}\frac{\partial B_z}{\partial z}$$

for the orbit of q, s is the gyro radius $\Longrightarrow s = \frac{v_\perp}{\omega_{cyc}}$

$$= \frac{q\,v_\perp}{2}\left(\frac{v_\perp m_{eq}}{qB_z}\right)\frac{\partial B_z}{\partial z} \quad \& \quad v_\perp = \omega_{cyc} R_{gyro}$$

$$= \frac{q}{2}\underbrace{\omega_{cyc}}_{-\frac{2\pi}{P}} R_{gyro}^2 \frac{\partial B_z}{\partial z}$$

current

$$= -\frac{q\pi R_{gyro}^2}{P}\frac{\partial B_z}{\partial z}$$

$$\boxed{f_z = -m_z\frac{\partial B_z}{\partial z}}$$

for a strengthening field in B_z $\Longrightarrow f_z < 0$, retarding

Recall: $\vec{f} = \vec{\nabla}(\vec{m}\cdot\vec{B})$

$\Rightarrow$ m_z is constant here. why? we assumed B_z changed slowly enough that R_{gyro} (circular motion) was a reasonable approximation for the motion; in this case, $\vec{m_z}$ is an adiabatic invariant.

Q: How do we define the mirror?

A: a) Kinetic energy

$$KE = \frac{m_q}{2}\left(v_\perp^2 + v_\parallel^2\right) = \text{constant}$$

(we ignored v_s^2; small in the adiabatic case)

b) $m_z = -\frac{1}{2} m_q \frac{v_\perp^2}{B_z} \equiv \text{constant}$

$$\Rightarrow KE = \frac{m_q}{2}\left(\frac{2B_z m_z}{m_q}\right) + \frac{m_q}{2} v_\parallel^2$$

$v_\parallel^2 = 0$ when $KE = m_z B_z = \frac{m_q}{2}\left(v_{\perp,0}^2 + v_{\parallel,0}^2\right)$

particles reflect if above is satisfied

$\boxed{\text{Okay, let us find the condition for reflection.}}$

the moment is:

$$m_z = q\frac{v_{\perp,0}}{2\pi R_{gyro}}\pi R_{gyro}^2$$

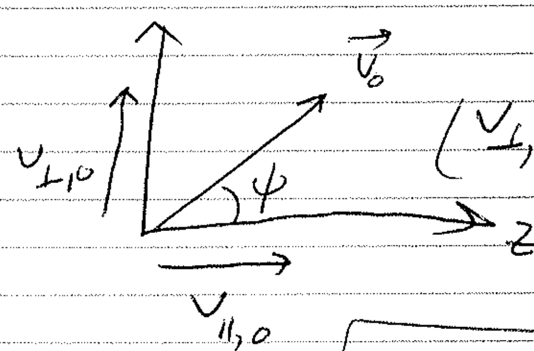
$$= \frac{q v_{\perp 0}}{2}\left(\frac{m_q v_{\perp,0}}{q B_{z,0}}\right)$$

$$= \frac{1}{2} m_q v_{\perp,0}^2 \frac{1}{B_{z,0}}$$

find the energy (KE)

$$\Rightarrow \frac{1}{2} m_q v_{\perp 0}^2 \frac{B_z}{B_{z0}} = \frac{m_q}{2}\left(v_{\perp 0}^2 + v_{\parallel,0}^2\right)$$

$$\frac{B_z}{B_{z,0}} = \left(1+\left(\frac{v_{\parallel,0}}{v_{\perp,0}}\right)^2\right) = \frac{1}{\sin^2\psi}$$



$(v_{\perp,0}/v_{\parallel,0}) = \tan\psi$, ψ = pitch ∡

$$\boxed{B_z = B_{z,0}/\sin^2\psi}$$

if $\psi = 0 \Rightarrow$ no mirroring (leakout)

if B_z small $\Rightarrow$ no leaking

Adiabatic Invariance

Assume $R_{gyro} \left(= \frac{mV_\perp}{qB_z}\right)$ didn't change over 1 orbit
($\Rightarrow$ circular orbit approximation).

Q: why would R_{gyro} change?

a) if $\frac{\partial B_z}{\partial z} \neq 0 \Rightarrow$ there will be a B_S and so,

$f_z \approx q V_\perp B_S \Rightarrow q$ will drift in z-direction

b) if q moves in $z \Rightarrow B_z$ changes (because $\frac{\partial B_z}{\partial z} \neq 0$)

c) if B_z changes $\Rightarrow R_{gyro}$ change

(i) $$R_{gyro} = \frac{mV_\perp}{qB_z} \rightarrow R_{gyro,0} + \Delta R_{gyro} = \underbrace{\frac{mV_\perp}{qB_{z,0}}}_{R_{gyro\,0}} + \Delta z \left[\underbrace{-\frac{mV_\perp}{qB_z^2}}_{-\frac{R_{gyro}}{B_{z,0}}} \frac{\partial B_z}{\partial z} \right]_0$$

$$\Rightarrow \frac{\Delta R_{gyro}}{R_{gyro}} \simeq \Delta z \left(\frac{1}{B_z} \frac{\partial B_z}{\partial z} \right)$$

(ii) $$f_z \simeq q V_\perp B_S \rightarrow \Delta z \simeq \frac{1}{2m} (q V_\perp B_S) t^2 \qquad t = \underbrace{\left(\frac{2\pi}{\omega_{cyc}} \right)}_{1 \text{ orbit}}$$

plug Δz into (i) and find

$$\left| \frac{1}{B_z} \frac{\partial B_z}{\partial z} \right| << \frac{2}{R_{gyro}}$$

$\Longrightarrow$ R_{gyro} will not change appreciably over 1 orbit

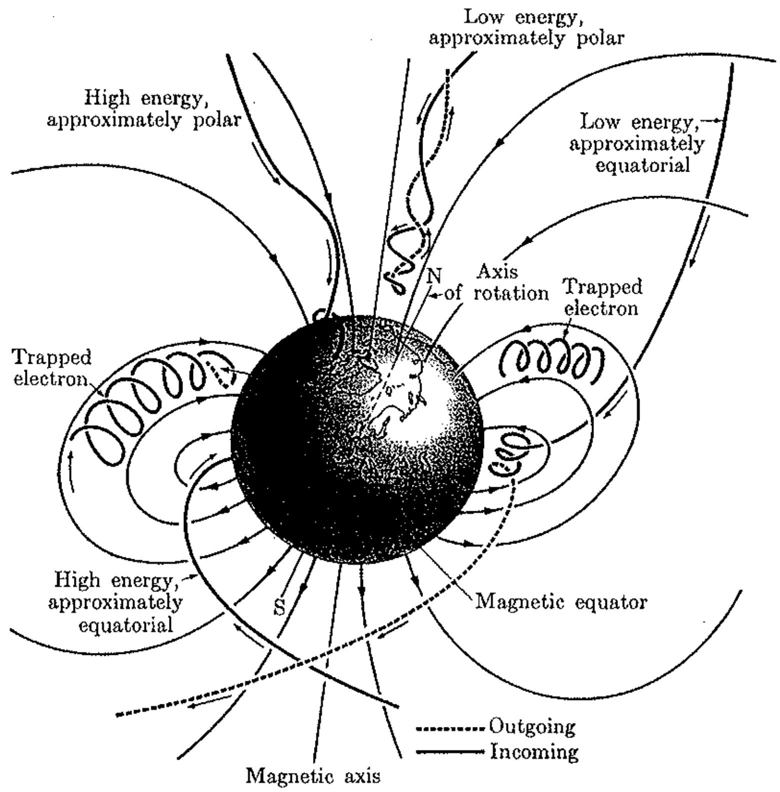


Fig. 15–11. Motion of charged cosmic-ray particles in the earth's magnetic field.

■ **Figure 9.6** The Van Allen radiation belts, showing a cross section through the Earth and the inner and outer belts. The *vertical line* represents Earth's magnetic axis, and the *horizontal line* represents Earth's magnetic equator. Both lines are marked in units of Earth radii. (From Zeilik, Michael and Elske v.P. Smith. *Introductory Astronomy and Astrophysics*, 2nd ed. Saunders College Publishing, Philadelphia, 1987.)