Physics 413: Introduction to Electrodynamics

Instructor: Jim Imamura
444 Willamette Hall
imamura@uoregon.edu
541-346-5212

Office Hours: 10-noon Tu, noon-2pm Th

Grading:

Test 1	40	points
Test 2	40	points
HWs	40	points
Final	80	points
	200 points	

Exams: Test 1: Wednesday, 2013 Feb. 6
Test 1: Friday, 2013 March 8

Final: Friday, 2013 March 22, 10:15

Material: Griffiths, Intro to Electrodynamics

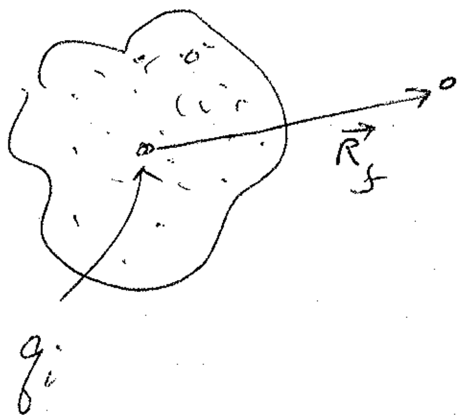
Chapters 5, 6, 7, 8, (9)

Chapter 5: Magnetostatics

Comment: Not "static", magnetic fields arise from charges in motion $\Rightarrow$ magnetostatics $\Rightarrow$ steady currents $\Rightarrow$ steady fields

Goals:

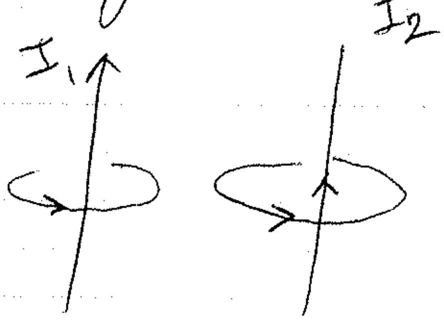
(i) Electrostatics



$$q \Rightarrow \vec{F}_q = \frac{q}{4\pi\epsilon_0} \sum_i \frac{q_i \vec{r}_i}{r_i^3}$$

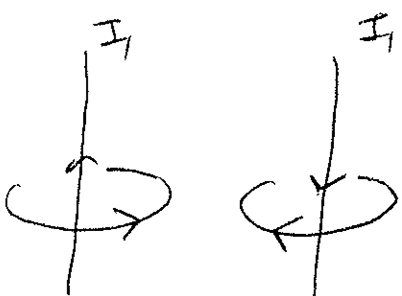
field at q ($\Rightarrow$ force on q)

(ii) Magnetostatics



$I_1 // I_2$

$\Rightarrow$ force is attractive, I_1, I_2 pulled together

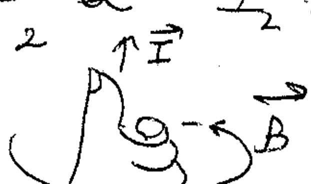


I_1 anti-// to I_2

$\Rightarrow$ force is repulsive, I_1, I_2 repel each other

follows from $\vec{F}_2 \propto I_2 \hat{n}_2 \times \vec{B}_1$

and

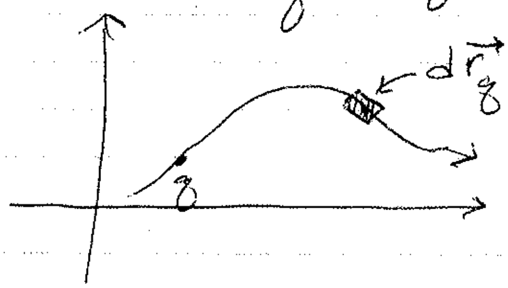


for a particle w/ charge q moving through an EM field,

$$\boxed{\vec{F} = \overbrace{q\vec{E}}^{\text{Coulomb force}} + \overbrace{q\left(\vec{v}\times\vec{B}\right)}^{\text{Lorentz force}}}$$

<u>Work:</u>

$$\vec{F}\cdot d\vec{r}_q = q\vec{E}\cdot d\vec{r}_q + \cancelto{0}{q\left(\vec{v}\times\vec{B}\right)\cdot d\vec{r}_q}$$



← region containing $\vec{E}$, $\vec{B}$

[unless stated otherwise, q does <u>not</u> affect external $\vec{E}$ & $\vec{B}$]

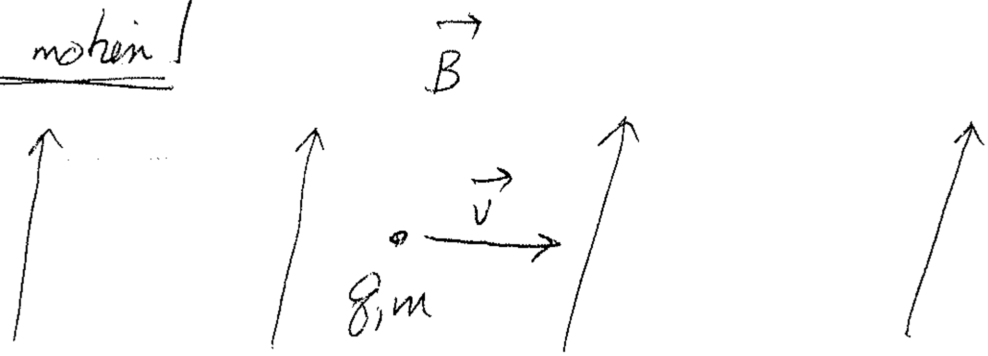
$$\Rightarrow \vec{F}\cdot d\vec{r}_q = q\left(\vec{E}\cdot d\vec{r}_q\right) + 0$$

and energy

$$\boxed{\text{Energy} = \frac{1}{2}mv^2 + qV}$$

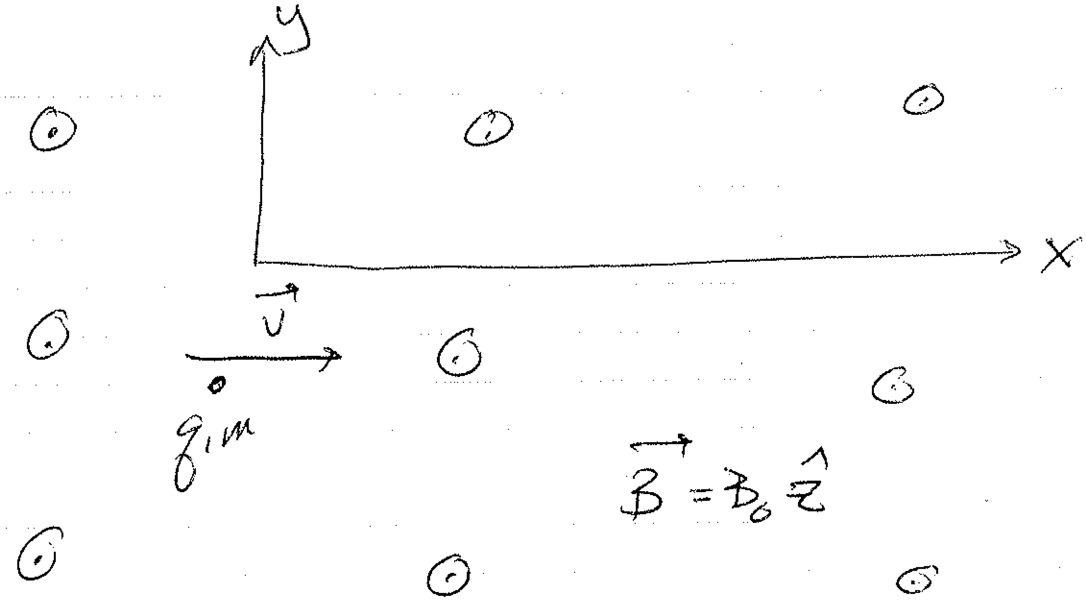
<u>note</u>: magnetic field does no work

Cyclotron motion

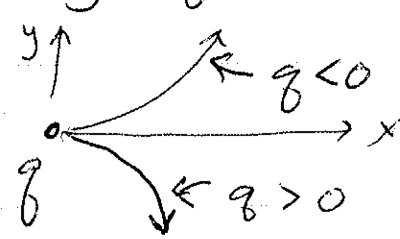


assume $\vec{E} = 0 \Rightarrow \vec{F} = q(\vec{v} \times \vec{B})$, Lorentz force.

okay (a)



$$\Rightarrow \vec{F} = q(\vec{v} \times \vec{B})$$
$$= q\Big(v_y B_z - \cancel{v_z B_y},\ \underbrace{\cancel{v_z B_x} - v_x B_z}_{F_y = -q v_x B_z},\ \underbrace{\cancel{v_x B_y} - \cancel{v_y B_x}}_{F_z = 0 \Rightarrow \text{motion is 2D}}\Big)$$



Solve:

$$\begin{cases} F_x = m\ddot{x} = q v_y B_z \\ F_y = m\ddot{y} = -q v_x B_z \\ F_z = m\ddot{z} = 0 \implies \dot{z} = V_z(0) \rightarrow \underbrace{z = V_z(0)t + z(0)} \end{cases}$$

subject to at $t=0$, $\vec{r}_0 = 0$, $\vec{V} = V_0\hat{x}$ initial conditions

drifts upward in z, if initial motion has a z-component
$\Rightarrow$ frame translation

Consider xy motion

$$\begin{cases} m\ddot{x} = q\dot{y}B_z \\ m\ddot{y} = -q\dot{x}B_z \end{cases}$$

from $\vec{B} = B_0\hat{z}$

(a)

(b)

divide by m and define the cyclotron frequency,

$$\boxed{\omega_c = \left| q\frac{B_0}{m} \right|}$$

Strategy

a) Integrate

$$\boxed{\begin{aligned} \ddot{x} = \dot{y}\omega_c &\implies \dot{x} = \cancelto{V_0}{\dot{x}_0} + \omega_c(y - \cancelto{0}{y(0)}) \\ \ddot{y} = -\dot{x}\omega_c &\implies \dot{y} = \cancelto{0}{\dot{y}_0} - \omega_c(x - \cancelto{0}{x(0)}) \end{aligned}}$$

b) replace $\dot{y}$ in (a)

$$\ddot{x} = \omega_c\left[-x\omega_c\right.$$

$$\Rightarrow \ddot{x} + \omega_c^2 x = 0$$

ω_c is real $\Rightarrow \boxed{x(t) = A\cos\omega_c t + B\sin\omega_c t}$

$$\Rightarrow \boxed{\dot{x}(t) = -\omega_c A \sin\omega_c t + \omega_c B\cos\omega_c t}$$

at $t=0 \Rightarrow x(0) = 0$, $\dot{x}(0) = V_0$

and so, $A = 0$, $B = \frac{V_0}{\omega_c}$

Sol^{n} subject to ICs is $\boxed{x(t) = \frac{V_0}{\omega_c}\sin\omega_c t}$

(c) replace $\dot{x}$ in $\dot{y}$ [(b)]

$$\ddot{y} = -\omega_c\left[V_0 + \omega_c y\right]$$

$$\ddot{y} + \omega_c^2 y = -\omega_c V_0$$

Can solve this by considering homogeneous sol^{n} and particular solution or ...

$$\ddot{y} + \omega_c^2\underbrace{\left(y + \frac{V_0}{\omega_0}\right)}_{u} = 0$$

$$\ddot{u} + \omega_c^2 u = 0 \Rightarrow u = C\cos\omega_c t + D\sin\omega_c t$$

$$y(t) = C\cos\omega_c t + D\sin\omega_c t - \frac{V_0}{\omega_c}$$

$$\boxed{\begin{aligned} y(t) &= C\cos\omega_c t + D\sin\omega_c t + (V_0/\omega_c) \\ \dot{y}(t) &= -\omega_c C\sin\omega_c t + \omega_c D\cos\omega_c t \end{aligned}}$$

ICs ($y(0)=\dot{y}(0)=0$) $\Rightarrow D=0$ & $C=\frac{V_0}{\omega_c}$

and y sol$^{\underline{n}}$ is

$$\boxed{y(t)=\frac{V_0}{\omega_c}\cos\omega_c t-\frac{V_0}{\omega_c}}$$

So, we have

$$\boxed{\begin{aligned} x(t) &= \frac{V_0}{\omega_c}\sin(\omega_c t) \\ y(t) &= \frac{V_0}{\omega_c}\cos(\omega_c t)-\frac{V_0}{\omega_c} \end{aligned}}$$

Perfectly fine, but, let us manipulate sol^{n}.

$$r_g^2 = x^2+\underbrace{\cancel{y^2}}_{\left(y(t)+\frac{V_0}{\omega_c}\right)^2} = \left(\frac{V_0}{\omega_c}\right)^2\left[\sin^2\omega_c t+\cos^2\omega_c t\right] = \left(\frac{V_0}{\omega_c}\right)^2$$

motion is circular w/ radius $\boxed{r_g = \text{gyro radius} = \frac{V_0}{\omega_c} = \frac{mV_\perp}{qB}}$

w/ frequency $\boxed{\omega_c = \frac{qB}{m}\text{, cyclotron frequency}}$

Typical #'s

Examples:

$$\omega_c = \frac{qB}{m} \qquad R = \frac{mV_\perp}{qB}$$

In most situations, electrons are much more mobile than ions and we look at cyclotron motion of e⁻'s.

$$\omega_{cyc} \simeq \frac{1.6\times10^{-19} B}{9.1\times10^{-31}\,kg} = 1.8\times10^{11} B\ s^{-1} \equiv \text{cyclotron frequency}$$

$$R = \frac{9.1\times10^{-31}\,kg\,V_\perp}{1.6\times10^{-19}\,C\cdot B} = 5.7\times10^{-12}\,\frac{V_\perp}{B}\ m \equiv \text{gyro radius}$$

N $*$'s (Pulsars)

$$B_* \simeq 10^{13}\,G = 10^{9}\,T,\quad V_\perp \simeq 7\times10^{7}\,\frac{m}{s}\quad (T \approx 10^8 K) \rightarrow 7\,keV$$

$$\Longrightarrow \begin{cases} \omega_{cyc} \simeq 1.8\times10^{20}\,s^{-1} \Longrightarrow \text{X-rays} \approx 100\,keV \\ R \simeq 4\times10^{-13}\,m \approx 400\,a_{nucleon} \end{cases}$$

comment: $\hbar\omega_{cyc} \gg kT_e \Rightarrow$ low energy gas and motion is quantized in $\perp$ dir^n

$$E_\perp = n\hbar\omega_{cyc}$$

Sun (flare)

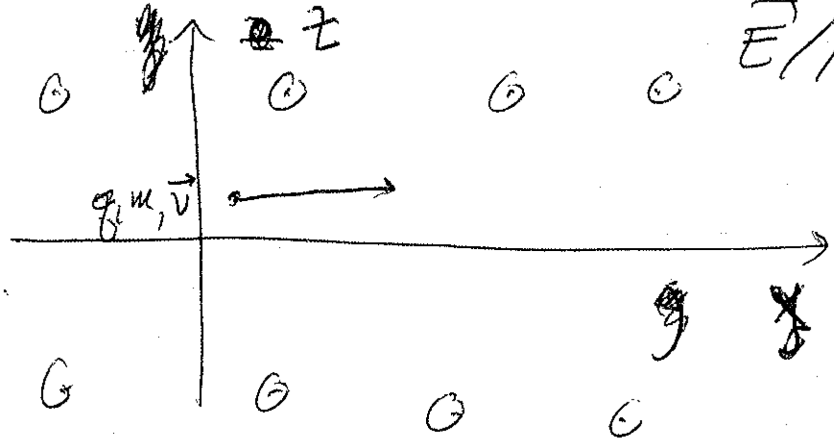
$$B_* \approx 10^{3}\,G \approx 0.1\,T,\quad V_\perp \approx 7\times10^{7}\,\frac{m}{s}$$

$$\Rightarrow \begin{cases} \omega_{cyc} \approx 1.8\times10^{10}\,s^{-1} \Longrightarrow \text{microwaves} \\ R \approx 4\times10^{-3}\,m \end{cases}$$

$\hbar\omega_{cyc} \ll kT \Rightarrow$ "classical" motion

Consider effect of $\vec{E}$

(a) $\vec{E} \parallel \vec{B}$; $\vec{E} = E_0 \hat{x}$, $\vec{B} = B_0 \hat{x}$

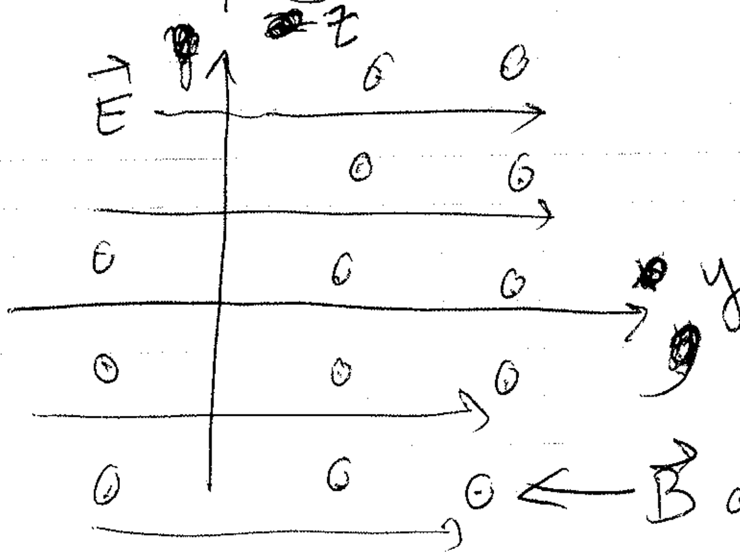


$$\vec{F} = \underbrace{q\vec{E}}_{\text{uniform acceleration in } \hat{x} \text{ direction}} + \underbrace{q(\vec{v}\times\vec{B})}_{\text{2D circular rotation in } y\text{–}z \text{ plane}}$$

$\Longrightarrow$ "Opening" helical motion

(b) Crossed $\vec{E}$ & $\vec{B}$

$\vec{B} = B_0 \hat{x}$, $\vec{E} = E_0 \hat{y}$



$\leftarrow \vec{B}$ out of paper

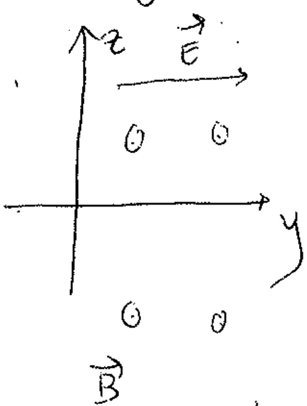
Lorentz force $\Longrightarrow$ circular motion projected onto yz plane

Coulomb force $\Longrightarrow$ frame translates to right (or left) but motion can be complicated drawn for an e^-

$v_y^0 = 0$

v_x^0

recall that my coordinate system was



$\Rightarrow$ drift is $\downarrow$ in $-z$ direction

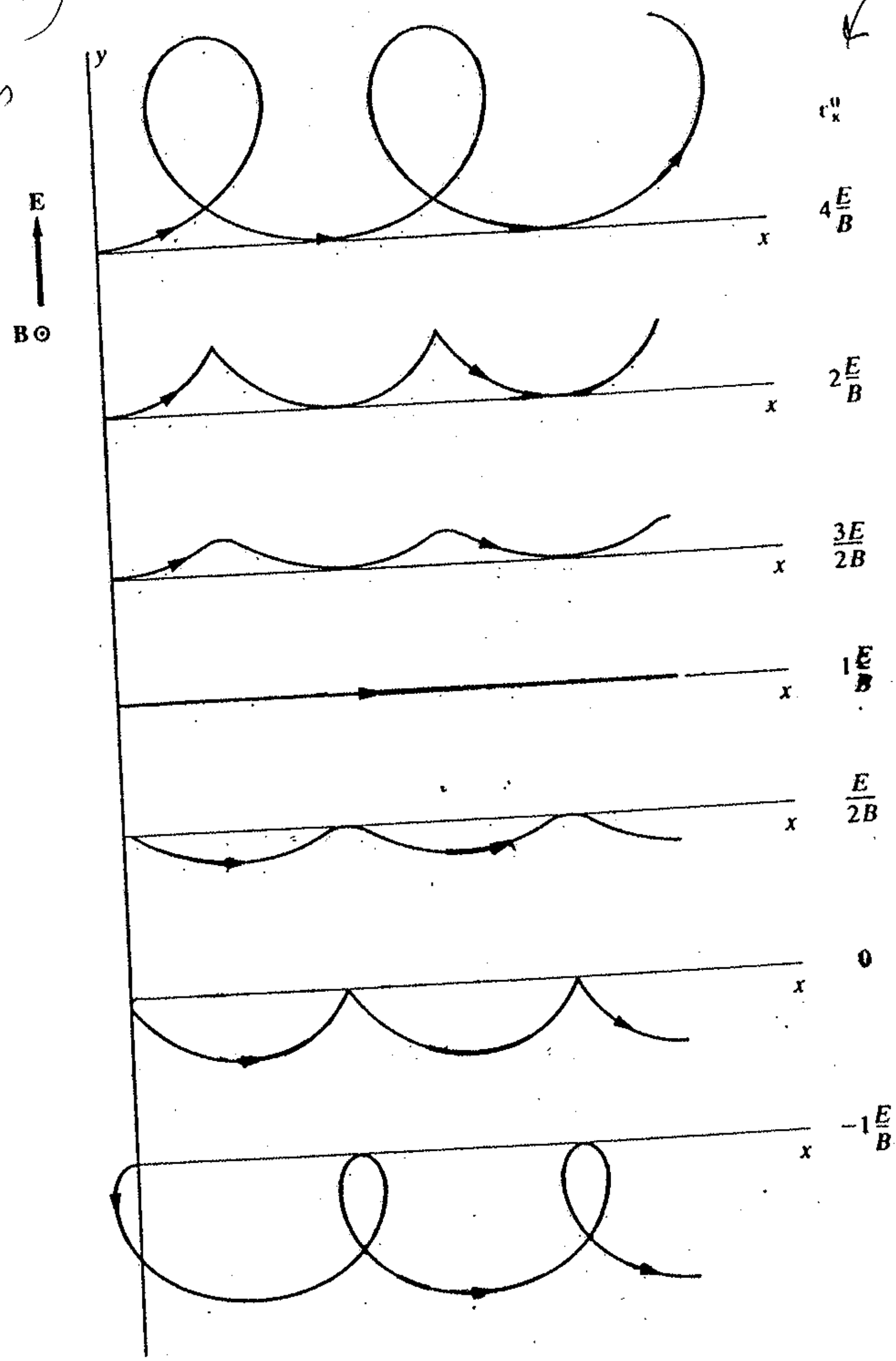


FIGURE 1-4 Trajectories in the xy plane of a charged particle in crossed E_y and B_z fields for various initial velocities along the x axis

Generalize from point charge q to current distributions

Consider magnetic forces on currents

$$\vec{F} = q\,(\vec{E} + \vec{v}\times\vec{B})$$

for a collection of charges

$$= \sum q_i\,(\vec{E}_m(\vec{r}_i) + \vec{v}_i \times \vec{B}_m(\vec{r}_i))$$

microscopic fields

Macroscopic fields are averages of microscopic fields. Cast $\vec{F}$ into a more reasonable form. For example,

$$\vec{E}_m(\vec{r}) = \vec{E}_M(\vec{r}) + \delta\vec{E}$$

$\delta\vec{E}$: fluctuations about $\vec{E}_M(\vec{r})$

$\vec{E}_M$: macroscopic field

and so, the force on a volume element ΔV is $\Delta\vec{F}$

$$\Delta\vec{F} = \int_{\Delta V}(\rho_M + \delta\rho)\left[\vec{E}_M + \delta\vec{E}\right]dV + \int_{\Delta V}(\vec{J}_M + \delta\vec{J})\times(\vec{B}_M + \delta\vec{B})\,dV$$

$$= \int\left[\rho_M\vec{E}_M + \rho_M\delta\vec{E} + \delta\rho\,\vec{E}_M + \delta\rho\,\delta\vec{E}\right]dV + \int\left[\vec{J}_M\times\vec{B}_M + \vec{J}_M\times\delta\vec{B} + \delta\vec{J}\times\vec{B}_M + \delta\vec{J}\times\delta\vec{B}\right]dV$$

all 1st order terms integrate (average) to 0.

$$= \underbrace{\int\left[\rho_M\vec{E}_M + \vec{J}_M\times\vec{B}_M\right]dV}_{\text{macroscopic fields}} + \underbrace{\int\left[\delta\rho\,\delta\vec{E} + \delta\vec{J}\times\delta\vec{B}\right]dV}_{\text{fluctuation forces}}$$

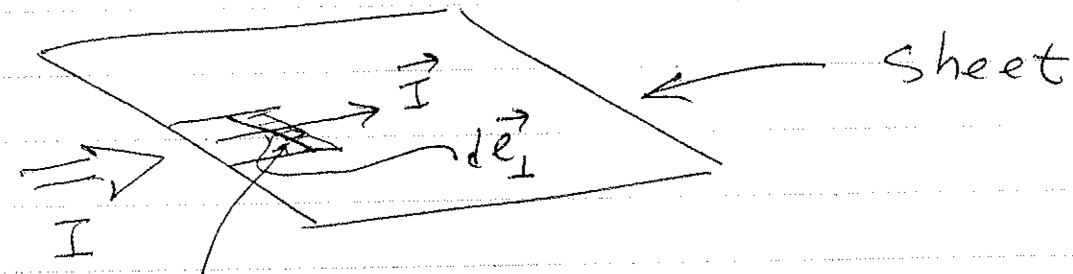
can be important

fluctuation forces (In a wire driven by a battery $\vec{F} = 0$) ("resistance" → Ohmic losses)

Current Distributions

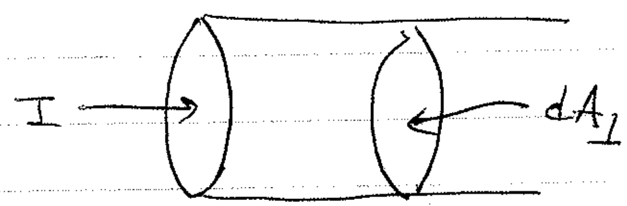
Define:

a)



$$\vec K = \frac{d\vec I}{d\vec l_\perp} \longrightarrow d\vec I = \vec K\, dl_\perp$$

b)

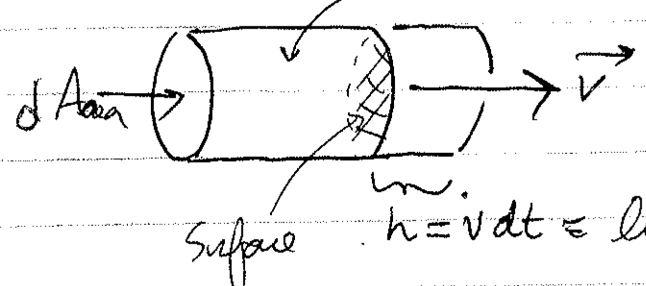


[current density is a current flux.]

$$\vec J = \frac{d\vec I}{dA_\perp} \longrightarrow d\vec I = \vec J\, dA_\perp$$

c) Suppose I have a volume of charge which moves?

ρ = charge density



$h = v\,dt$ = length of cylinder which crosses surface

$$\rightarrow \left(\frac{\text{Charge which crosses surface}}{\text{second}}\right) = \frac{\rho\,(dA\,h)}{dt} = \frac{\rho\, dA\, v\, \cancel{dt}}{\cancel{dt}}$$

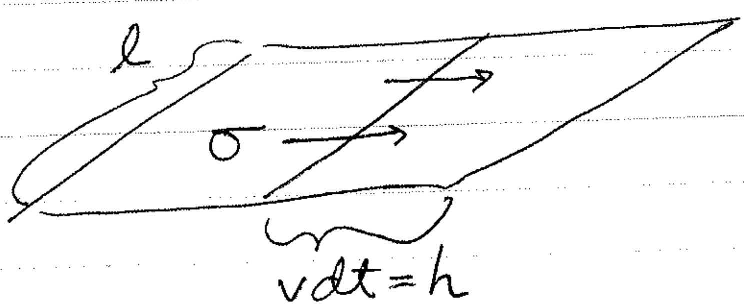
In terms of $\vec{J}$, what have we defined?

$$dI = J\,dA$$

$$\rightarrow J\,dA = \rho\,dA\,v \longrightarrow J = \rho v \quad \left[\frac{C}{l^2 \cdot t}\right]$$

In general, $\boxed{\vec{J} = \rho\vec{v}}$

d) Suppose I have a surface charge



$$I = \underbrace{\sigma l h}_{\text{charge}} \frac{1}{dt} = \frac{\sigma l v\,dt}{dt}$$

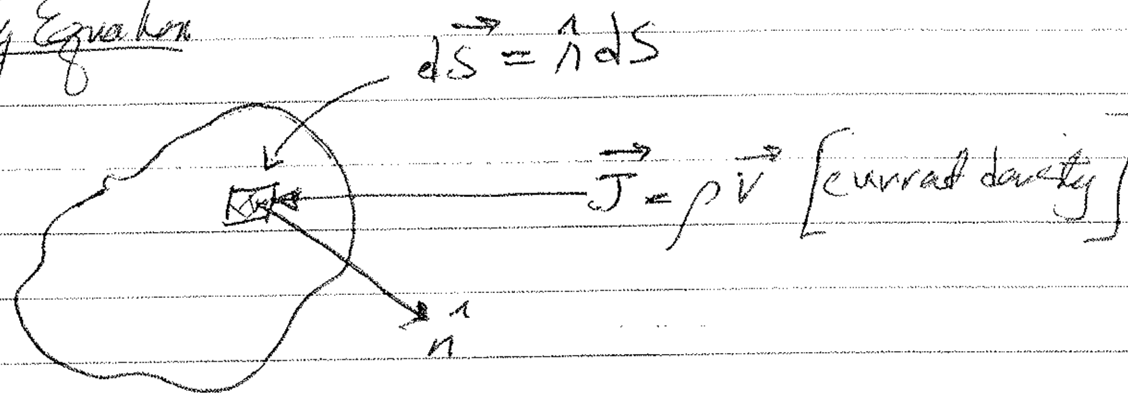
$$= \sigma l v$$

again, $K = \frac{I}{l} = \sigma v$

In general, $\vec{K} = \sigma\vec{v}$

Continuity Equation

(a)



We then have,

$$\oint \vec{J}\cdot d\vec{S} \equiv \text{Integral of flux over surface of volume} \quad [\text{charge}/\text{time}]$$

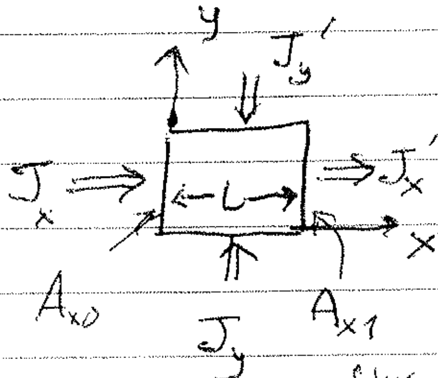
$$= -\frac{\partial}{\partial t}\int \rho\, d^3x \equiv \text{change in charge contained in volume per time}$$

$$\Rightarrow \int (\vec{\nabla}\cdot\vec{J})\, d^3x' = -\frac{\partial}{\partial t}\int \rho\, d^3x'$$

and

$$\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

(b)



Consider 2D flow for simplicity

$$(i)\ \Big[\overbrace{\rho_0 v_{x0}}^{\text{flux}} \times A_{x0} - \overbrace{\rho_1 v_{x1}}^{\text{flux}} \times A_{x1}\Big] \times \Delta t = \Delta Q$$

$$\Big[\rho_0 v_{x0} A_{x0} - \Big(\rho_0 v_{x0} A_{x0} + L\frac{\partial}{\partial x}\{\rho v_x A_x\}\Big|_{x0}\Big)\Big]\Delta t = \Delta Q$$

$$-L\frac{\partial}{\partial x}\left\{\rho v_x A_x\right\}\Delta t = \Delta Q$$

divide by the volume of the box, $(L A_x)$ Δ(density)

$$-\frac{1}{A_x}\frac{\partial}{\partial x}\left\{\rho v_x A_x\right\}\Delta t = \frac{\Delta Q}{\text{Volume}} = \left(\frac{\Delta Q}{L A_x}\right) = \Delta\rho$$

$$\Rightarrow \frac{\Delta\rho}{\Delta t} + \frac{\partial}{\partial x}\left(\rho v_x A_x\right)\frac{1}{A_x} = 0$$

$$\xrightarrow{\Delta t \to 0} \frac{\partial\rho}{\partial t} + \underbrace{\frac{1}{A_x}\frac{\partial}{\partial x}\left(\rho v_x A_x\right)}_{\vec{\nabla}\cdot\rho\vec{v}} = 0$$

$$\boxed{\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\rho\vec{v} = 0}$$

Continuity Equation

Lorentz force + Coulomb force

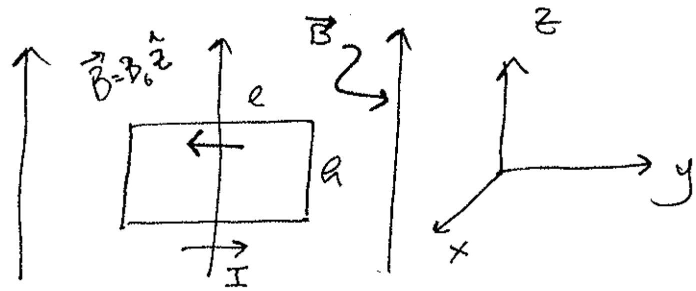
For purely macroscopic forces, we have

(i)	$\vec{F} = (q\vec{E} + q(\vec{V}\times\vec{B}))$	"point"
(ii)	$d\vec{F} = (\lambda dr\,\vec{E} + I d\vec{r}\times\vec{B})$	"line"
(iii)	$d\vec{F} = (\sigma\vec{E} + \vec{K}\times\vec{B})dS$	"surface"
(iv)	$d\vec{F} = (\rho\vec{E} + \vec{J}\times\vec{B})dV$	"volume"

Examples,

A) Ignore gravity

loop sits in zy-plane



a) $\vec{F} = \oint I d\vec{r}\times\vec{B}$

$= Il\hat{x}B_0 + 0 + Il(-\hat{x})B_0 + 0$

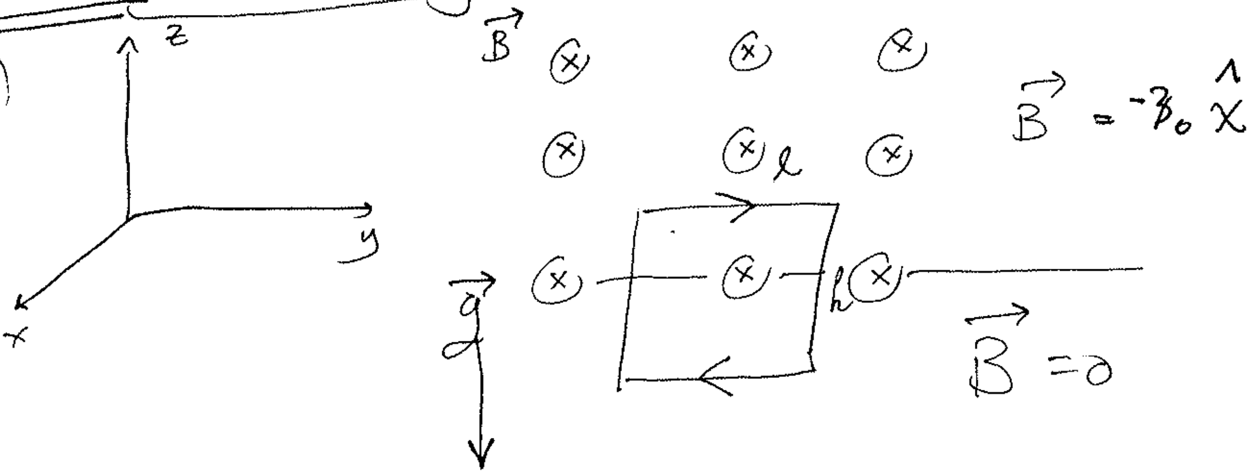
$= 0$

$\Longrightarrow$ no translational force, however,

b) $\vec{N} = \left[\frac{h}{2}Il + \frac{h}{2}Il\right]\hat{y}B_0 = Ihl\hat{y}B_0$

$\Longrightarrow$ rotate about y-axis so that loop sits in xy-plane

B)

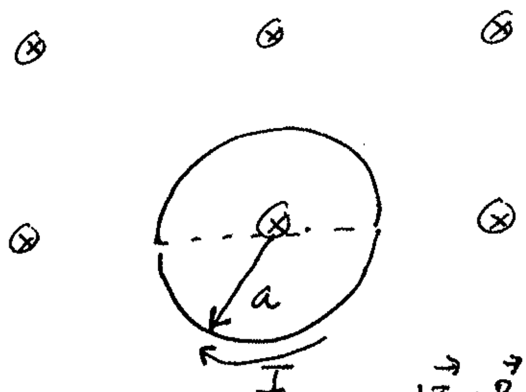


To make loop "float", I must be in CW direction

$$\vec{F}_B = IB\hat{z}l + \underbrace{IB\frac{h}{2}\hat{y} + 0 + IB\frac{h}{2}(-\hat{y})}_{\to 0}$$

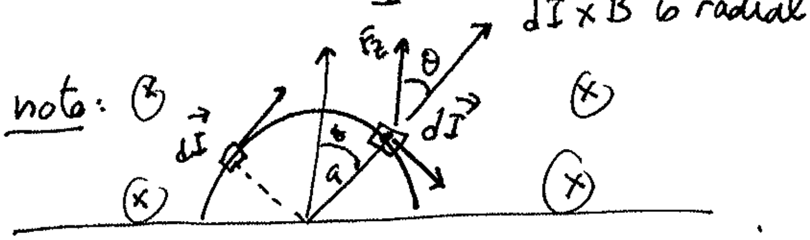
$$\boxed{\vec{F}_B + \vec{F}_g = IBl\hat{z} - mg\hat{z} = 0 \to I = \frac{mg}{Bl}}$$

c)



Q: what I is needed to make half-of-the circle "float" inside the field region?

note:



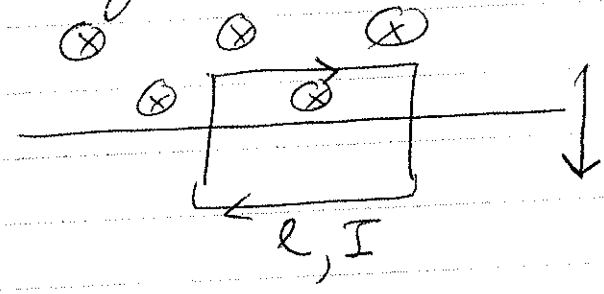
By symmetry we need only consider F_z.

$$dF_z = \left[I a\, d\theta\, B\right]\cos\theta$$

$$F_{z,\text{total}} = 2IaB\int_0^{\pi/2}\cos\theta\, d\theta = 2aIB$$

$$\Rightarrow \boxed{F_{z,\text{total}} + F_g = 0 \Rightarrow 2aIB - mg = 0 \Rightarrow I = \frac{mg}{2aB}}$$

Magnetic "Work" ← what is the meaning of this?

a)


$\downarrow g \Rightarrow$ I CW to support loop

a) $\vec{F}_B = \vec{I}\,d\vec{l} \times \vec{B} = I l B \hat{z}$

$\vec{F}_g = -Mg\,z$

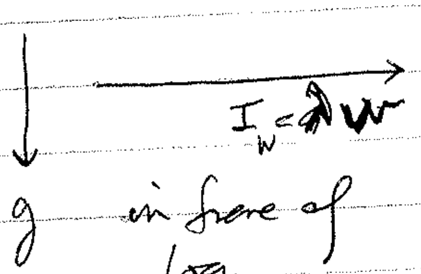
$$\boxed{I = \frac{Mg}{lB}}$$

b) If $I \uparrow \Rightarrow F_B \uparrow \Rightarrow$ loop rises

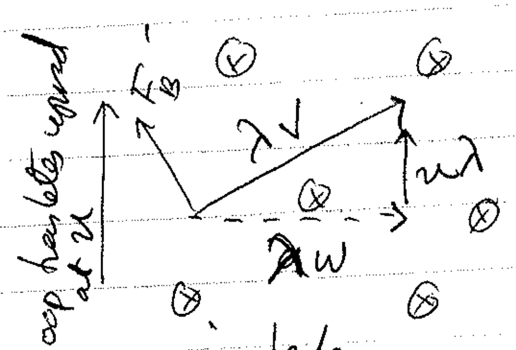
$$\text{Work} \equiv -\vec{F}_g \cdot d\vec{h} = \vec{F}_g \cdot d\vec{z} = Mgh$$

what if $-\vec{F}_B \cdot d\vec{h}$? $\rightarrow W = + I l B h$

Are these both right? Well, yes, but who did the work?

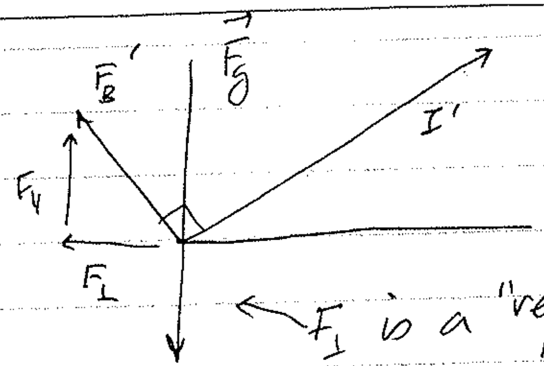


g in frame of loop



in lab frame

F_B' is *not* // (or anti-//) to $\vec{g}$



$F_\perp$ is a "retarding" force which opposes I'
$\Rightarrow$ a "battery" needs to push the charges

Energy?

(i) $F_\perp = \int I d\vec{r} \times \vec{B} = \int \lambda u B \, dr_\perp$

$= \lambda u B l$

$$\boxed{F_\perp = I\left(\frac{u}{w}\right) B l}$$

we define I as
$I = \lambda w$

(ii) $W = \int I\left(\frac{u}{w}\right) B l \times dl'$

distance charge travels in time dt along the $\perp$ direction

$dl' = w\,dt$

$= \int I\left(\frac{u}{w}\right) B l \times w\,dt$

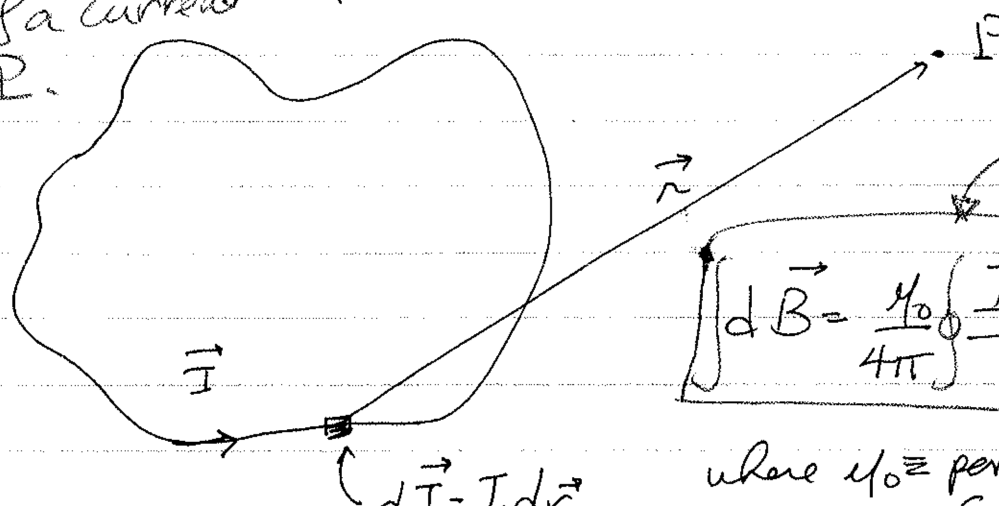
$= IBl \underbrace{u\int dt}_{h}$

$$\boxed{W = IBlh}$$ ✓

Field Produced by a Current

Biot-Savart law

field of a current loop at some point, P.



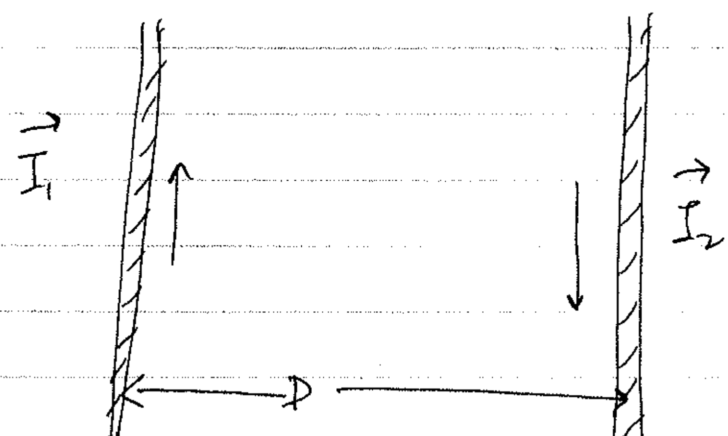
Biot-Savart law

$$\boxed{\int d\vec{B} = \frac{\mu_0}{4\pi}\oint \frac{I\,d\vec{r}\times\hat{r}}{r^2}}$$

where $\mu_0 \equiv$ permeability of free space
$= 4\pi \times 10^{-7}\ \frac{N}{A^2}$

$$\rightarrow\ \vec{B} = \frac{\mu_0}{4\pi}\begin{cases} \oint \frac{I\,d\vec{r}\times\hat{r}}{r^2} & \text{"line"} \\ \int \frac{\vec{K}\times\hat{r}}{r^2}\,dS & \text{sheet} \\ \int \frac{\vec{J}\times\hat{r}}{r^2}\,d\tau & \text{volume} \end{cases}$$

Example, Find force on $\vec{I_2}$ (Return to case of 2 wires)

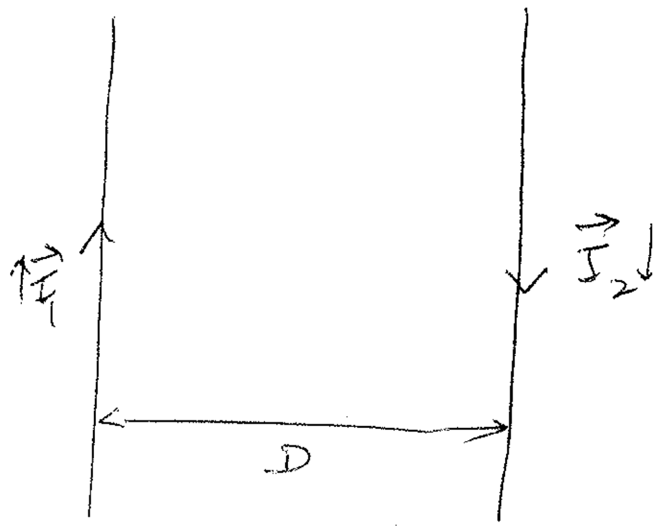


Sol^{n}

First,

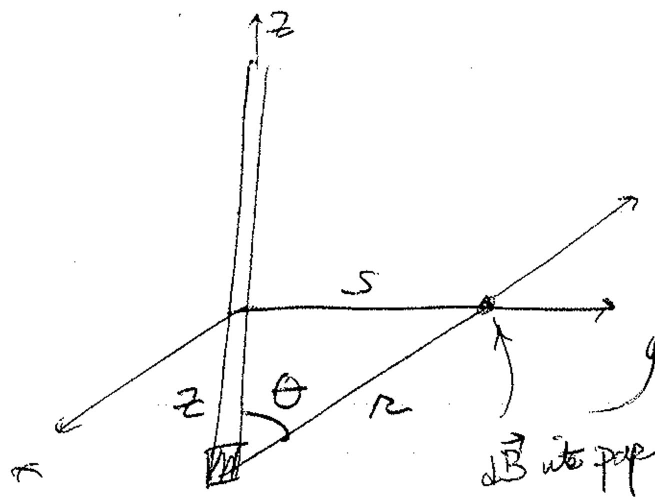
$$\vec{F_2} = \int I_2\,d\vec{r}\times\vec{B_1}$$

$\rightarrow$ we need $\vec{B_1}$



find the force on wire 2 as a result of $\vec{B}$ from wire 1

look at 1 wire

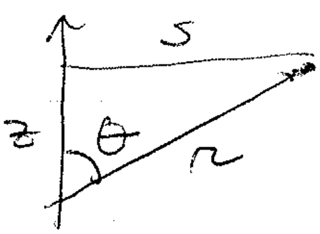


$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{z}\times\vec{r}}{r^3}$$

$d\vec{B}$ into paper w/ magnitude $\frac{I r}{r^2}\overbrace{\sin\theta}^{\text{from curl}}$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi}\hat{\phi}\int\frac{dz\,\sin\theta}{(z^2+s^2)}$$

$\boxed{\text{relate } \theta \text{ to } z}$



$$\cos\theta = \frac{z}{r} \quad \& \quad \sin\theta = \frac{s}{r}$$

$$-\sin\theta\,d\theta = \left[\frac{dz}{r} - \frac{\frac{1}{2}2z^2dz}{(z^2+s^2)^{3/2}}\right]$$

$$\cancel{= \left[\frac{\sin\theta}{s} - 2\cos\theta\frac{\sin\theta}{s}\right]dz}$$

$$-\sin\theta\, d\theta = \left[\frac{\sin\theta}{s} - \cos^2\theta\, \frac{\sin\theta}{s}\right] dz$$

$$= \frac{\sin\theta}{s} dz \left[1 - \cos^2\theta\right]$$

$$-\sin\theta\, d\theta = \frac{\sin\theta}{s} dz \left[\sin^2\theta\right]$$

$$\Longrightarrow dz = -\frac{s}{\sin^2\theta} d\theta$$

and the integral becomes

$$\vec{B} = \frac{\mu_0 I}{4\pi}\hat{\phi}\int_0^\pi \underbrace{\frac{s\,d\theta}{\sin^2\theta}}_{dz} \sin\theta\, \underbrace{\frac{\sin^2\theta}{s^2}}_{\frac{1}{r^2}}$$

$$= -\frac{\mu_0 I}{4\pi}\hat{\phi}\int_0^\pi \frac{1}{s}\sin\theta\, d\theta$$

$$= -\frac{\mu_0 I}{4\pi s}\hat{\phi}\left[\cos\theta\Big|_0^\pi\right]$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$$

the force is then

$$d\vec{F}_2 = (-) I_2 \, d\vec{z} \times \vec{B}_1 = - I_2 \, d\vec{z} \times \left(\frac{\mu_0 I}{2\pi D} \hat{\phi} \right)$$

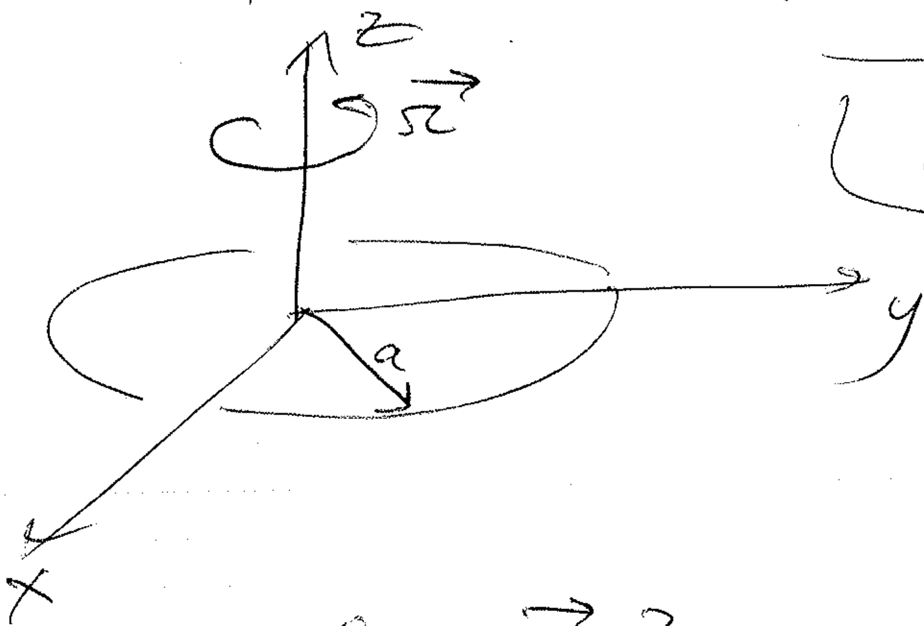
current is $\downarrow$

$$= \underbrace{I_2 \frac{\mu_0 I_1}{2\pi D}}_{\text{repulsive}} \hat{s} \, dz$$

$$\vec{F}_2 = \frac{\mu_0 I_1 I_2}{2\pi D} \hat{s} \int dz$$

$$\boxed{\frac{\vec{F}_2}{\Delta z} = \frac{\mu_0 I_1 I_2}{2\pi D} \hat{s} \quad ; \text{ for a piece with length, } \Delta z}$$

Consider "phonograph" record w/ $\sigma = \frac{Q}{\pi a^2}$



find $\vec{B}$ on the z-axis

Q: what is $\vec{K}$?

A: $\vec{K} = \sigma \vec{v}$

Physics 411 (Analytical Mechanics)

Q: what is $\vec{v}$?

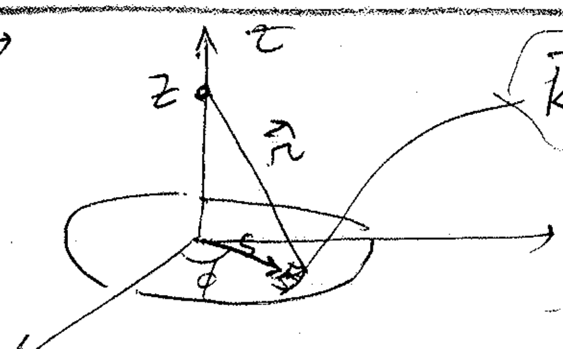
A: $\vec{v} = \cancel{\frac{d}{dt}(\hat{i}x)} + \frac{d}{dt}\left(\hat{i}x + \hat{j}y + \hat{k}z\right)$

in a frame that rotates at frequency $\vec{\Omega}$, we find

$\vec{v} = \vec{\Omega} \times \vec{r}$

$$\Rightarrow \boxed{\vec{K} = \sigma(\vec{\Omega} \times \vec{r}) = \sigma \Omega s \hat{\phi}}$$

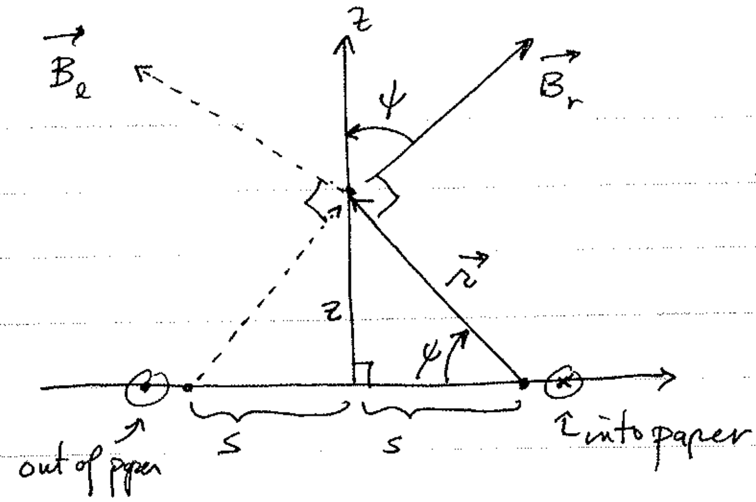
a) find $\vec{B}$



$\vec{K}\,dS = \vec{K}\,s\,d\phi\,ds$

and

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{K}\,dS \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$



$\Longrightarrow$ By symmetry, $\vec{B}_\perp = 0$, $\perp$ to z-axis

$\therefore$ Only need B_z

$$dB_z = \frac{\mu_0}{4\pi}\frac{|\vec{K}\,dS \times \hat{r}|}{r^2}\cos\psi \qquad \cos\psi = \frac{s}{r}$$

$$= \frac{\mu_0}{4\pi}\frac{\sigma\Omega s^2}{r^3}\underbrace{\sin\theta_{\phi r}}_{1 \text{ since } \vec{K}\perp\hat{r}}\, dS \qquad dS = s\,ds\,d\phi$$

$$B_z = \frac{\mu_0\sigma\Omega}{4\pi}\int\frac{s^2}{(s^2+z^2)^{3/2}}\,s\,ds\,d\phi$$

$$= \frac{\mu_0\sigma\Omega}{2}\int\frac{s^3\,ds}{(s^2+z^2)^{3/2}}$$

<u>let</u> $w = s^2+z^2 \Longrightarrow dw = 2s\,ds$ and $s^2 = (w - z^2)$

$$B_z = \frac{\mu_0\sigma\Omega}{2}\int\frac{1}{2}\frac{dw}{w^{3/2}}(w - z^2)$$

$$= \frac{\mu_0\sigma\Omega}{4}\int\frac{(w-z^2)\,dw}{w^{3/2}}$$

$$= \frac{\mu_0\sigma\Omega}{2}\left[\sqrt{a^2+z^2} - \sqrt{z^2} + \frac{z^2}{\sqrt{a^2+z^2}} - \frac{z^2}{\sqrt{z^2}}\right]$$

$\underline{z > 0}$

$$B_z = \frac{\mu_0 \sigma \Omega}{2}\left[\frac{a^2+2z^2}{\sqrt{a^2+z^2}} - 2z\right]$$

$\underline{z < 0}$

$$B_z = \frac{\mu_0 \sigma \Omega}{2}\left[\frac{a^2+2z^2}{\sqrt{a^2+z^2}} - 2|z|\right]$$

<u>Comment:</u>

$$\Delta B_z = \frac{\mu_0 \sigma \Omega}{2}\left[\frac{a^2+z^2 2}{\sqrt{a^2+z^2}} - 2z^+ - \frac{a^2+2z^2}{\sqrt{a^2+z^2}} + 2|z|\right]$$

$$= 0 \quad \text{at a current sheet}$$

So, (at least for this case), B_z is continuous at a charged sheet (current carrying sheet)

This is to be contrasted to

$$\Delta E_z = \sigma/\epsilon_0$$

at a charged sheet

A digression on units

In Electricity and Magnetism, we have:

$$\vec{F}_{es} = K_{es} \frac{Q_1 Q_2 \hat{r}}{r^2}$$

and

$$\vec{F}_{m} = K_{m} \frac{2 I_1 I_2}{D} \hat{D} \qquad \text{(for 2 wires)}$$

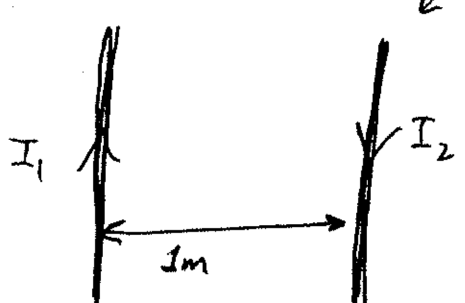
So, there are 2 "new" constants K_{es} and K_m; however, there really is only 1 "new" quantity, the charge.

$\Rightarrow$ can not define K_{es} and K_m independently!

The ampere is chosen to be the fourth unit (in addition to meters, kilograms, seconds)

and K_m is taken to be 10^{-7} using the ampere as a fundamental unit.

or, the ampere is defined as the current which produces a force per unit meter of 2×10^{-7} N/m for 2 infinite wires separated by 1m as



the ampere then defines Coulomb (charge) as

So, given the def^{n} of ampere, and

$$|\vec{F}_{es}| = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{R^2}$$

ϵ_0 must be adjusted to make the "force" agree w/expt.

In SI units, $\epsilon_0 = 8.854 \times 10^{-12}\, C^2/Nm^2$

(a) $$\frac{K_{es}}{K_m} = \frac{\frac{1}{4\pi\epsilon_0}}{\frac{\mu_0}{4\pi}} = \frac{1}{\epsilon_0\mu_0} = (\text{constant})^2 = c^2 \;!$$

speed of light (in vacuum)

(b) $\vec{F} = I d\vec{r} \times \vec{B}$ or $[N] = [A][m] \times [B]$

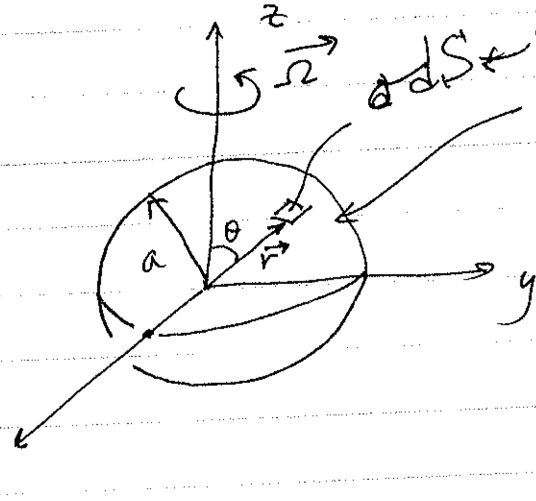
$$\Rightarrow [B] \equiv \frac{N}{A\text{-}m} \equiv \text{Tesla}$$

however, (for reasons which are obscure to me), but not really

the unit of magnetic field is given in c.g.s. units

or Gauss $= 10^{-4}$ Tesla

Consider a charged, rotating shell.



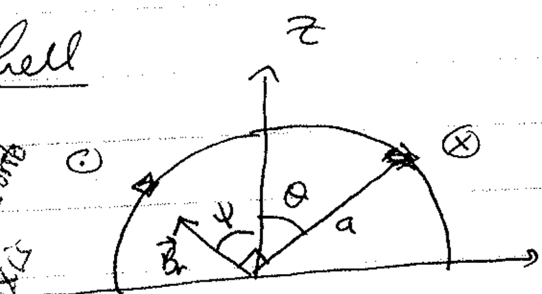
$dS \leftarrow \sigma a^2 \sin\theta\, d\phi\, d\theta$

$\sigma = \dfrac{Q}{4\pi a^2}$

$$\vec{K} \begin{cases} = \sigma \vec{v} \\ = \sigma(\vec{\Omega}\times\vec{r}) \\ = \sigma\Omega a \sin\theta\,\hat{\phi} \end{cases}$$

Find $\vec{B}$ at the center of the shell

By symmetry, we need only find B_z



$|\hat{K}\times\hat{r}| \Rightarrow \sin\frac{\pi}{2} = 1$

projection $d\vec{B}$ onto z axis

$$\Longrightarrow dB_z = \frac{\mu_0}{4\pi}\left|\frac{\vec{K}\,dS\times\hat{r}}{r^2}\right| \underbrace{\cos\left(\tfrac{\pi}{2}-\theta\right)}_{\cos\psi}$$

and

$$B_z = \frac{\mu_0}{4\pi}\sigma\Omega a\int \underbrace{\sin\theta}_{K}\ \underbrace{\frac{1}{a^2}}_{\frac{1}{r^2}}\ \underbrace{\sin\theta}_{\cos\frac{\pi}{2}-\theta}\ \underbrace{\left[\underbrace{\sin\theta\, d\theta\, d\phi\, a^2}_{\text{Area element}}\right]}_{dS}$$

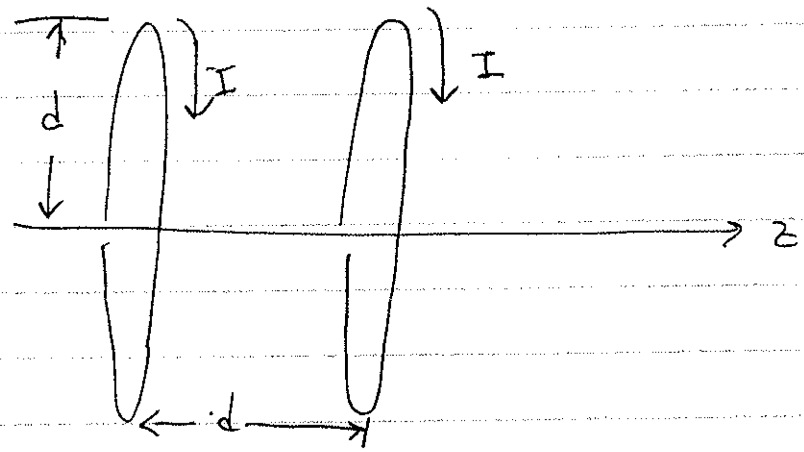
$$= \frac{\mu_0\sigma\Omega a}{4\pi}\int \sin^2\theta \sin\theta\, d\theta\, d\phi$$

$$= -\frac{\mu_0\sigma\Omega a}{4\pi}\int (1-\cos^2\theta)\, d(\cos\theta)\, d\phi$$

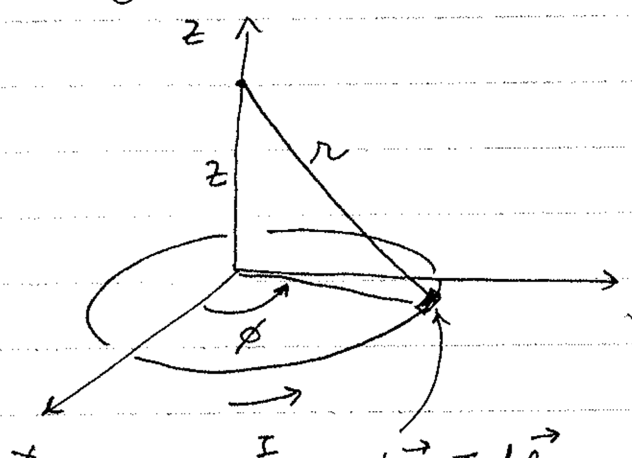
$$= -\frac{\mu_0\sigma\Omega a}{2}\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$\vec{B} = 2\mu_0\sigma\Omega a\,\hat{z} = \mu_0 Q\Omega\,\hat{z}$$

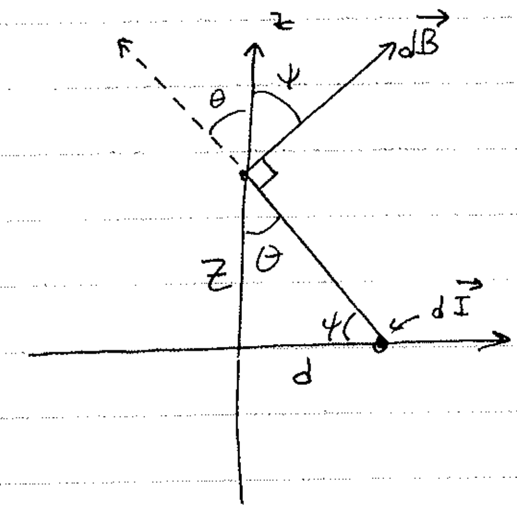
Helmholtz Coil



Find the $\vec{B}$-field on the z-axis at the midpoint of the above Helmholtz Coil.



$\Longrightarrow$



$$d\vec{I} = I_0 \, d\vec{l} = I_0 (d \, d\vec{\phi})$$

$$\rightarrow d\vec{B} = \frac{\mu_0}{4\pi} \frac{I_0 d (d\vec{\phi}) \times \hat{r}}{r^2}$$

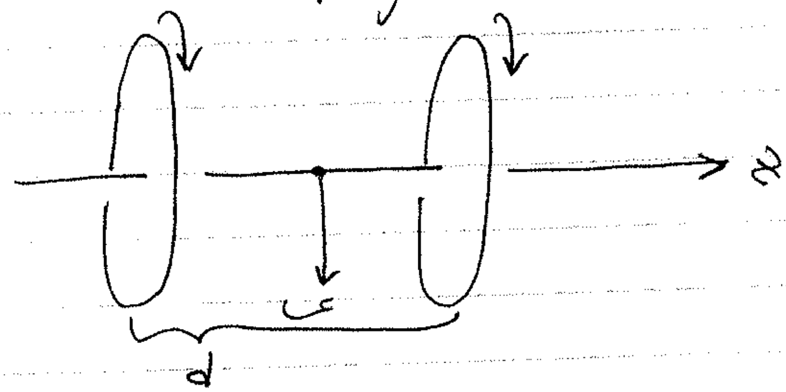
(a) We note that, by symmetry, only the z-component of the $d\vec{B}$ survives after integrating around the loop, and so,

$$dB_z = \frac{\mu_0 I_0 d}{4\pi} \frac{d\phi}{r^2} \cos\psi = \frac{\mu_0 I_0 d}{4\pi} \frac{d\phi}{r^2} \left(\frac{d}{r}\right)$$

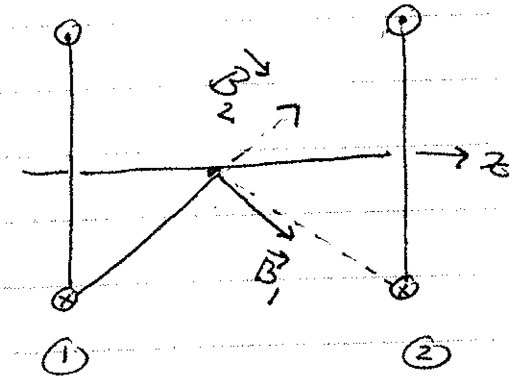
$$= \frac{\mu_0 I_0 d^2}{4\pi (d^2 + z^2)^{3/2}} d\phi$$

$$\longrightarrow B_z = \frac{\mu_0 I_0}{\ldots} \left[\frac{d^2}{(d^2 + z^2)^{3/2}} \right]$$

Return to 2 loop system



or



$$\vec{B}_{Tot} = \left[\frac{\mu_0 I}{2}\frac{d^2}{\left(d^2+\left\{z+\frac{d}{2}\right\}^2\right)^{3/2}} + \frac{\mu_0 I}{2}\frac{d^2}{\left(d^2+\left\{z-\frac{d}{2}\right\}^2\right)^{3/2}}\right]\hat{z}$$

expand around origin, $z=0$ or note that:

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots$$

at midpoint, $z=0$

$$\vec{B}_{Tot} = \frac{\mu_0 I}{d}\left[\left(\frac{4}{5}\right)^{3/2}\hat{z}\right]$$

More interestingly,

$$\frac{\partial B}{\partial z} = \frac{\mu_0 I}{2}\left[\frac{-3d^2\left(z+\frac{d}{2}\right)}{\left(d^2+\left[z+\frac{d}{2}\right]^2\right)^{5/2}} + \frac{-3d^2\left(z-\frac{d}{2}\right)}{\left(d^2+\left[z-\frac{d}{2}\right]^2\right)^{5/2}}\right]$$

at $z=0$

$$\frac{\partial B}{\partial z} = 0 \quad !$$

Further

$$\frac{\partial^2 B}{\partial z^2} = \frac{\mu_0 I}{2}\left(-3d^2\right)\left[\frac{1}{\left(d^2+\left[z+\frac{d}{2}\right]^2\right)^{5/2}} + \frac{1}{\left(d^2+\left[z-\frac{d}{2}\right]^2\right)^{5/2}}\right.$$

$$\left. - \frac{5\left(z+\frac{d}{2}\right)^2}{\left(d^2+\left[z+\frac{d}{2}\right]^2\right)^{7/2}} - \frac{5\left(z-\frac{d}{2}\right)^2}{\left(d^2+\left[z-\frac{d}{2}\right]^2\right)^{7/2}}\right]$$

at $z=0$

$$\frac{\partial^2 B}{\partial z^2} = -\frac{3}{2}\mu_0 I d^2\left[\frac{2}{\left(5d^2/4\right)^{5/2}} - \frac{(5/2)\,d^2}{\left(5d^2/4\right)^{7/2}}\right]$$

$$= -3\frac{\mu_0 I d^2}{\left(5d^2/4\right)^{5/2}}\left[1 - \frac{(5/4)\,d^2}{\left(5/4\, d^2\right)}\right]$$

$$= 0 \;!$$

So, the field in a Helmholtz coil at $z \approx 0$ is

$$B(z=0) = \frac{\mu_0 I}{d}\left(\frac{4}{5}\right)^{3/2}$$

and

$$B(z) = \frac{\mu_0 I}{d}\left(\frac{4}{5}\right)^{3/2} + \mathcal{O}(z^4)$$

in the neighborhood of $z=0$!

Vector Potential & Divergence of $\vec{B}$ & Curl of $\vec{B}$

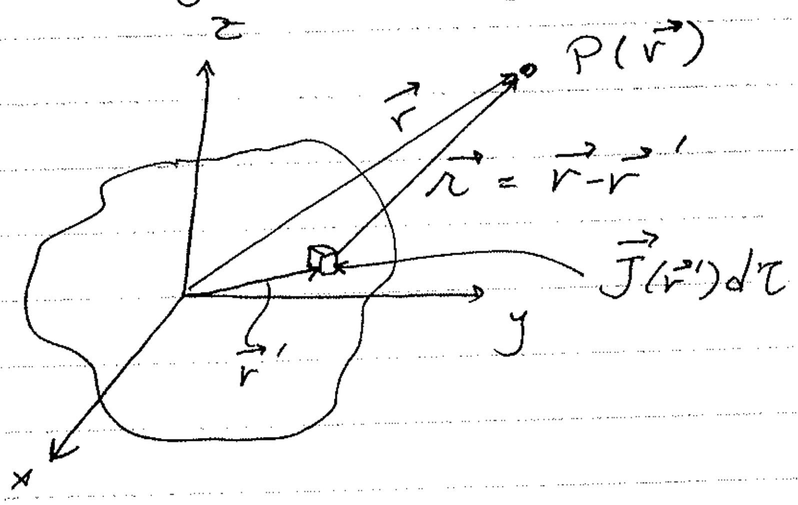
(i) In ES, $\vec{\nabla}\times\vec{E}=0$, $\vec{E}=-\vec{\nabla}V$, $\oint\vec{E}\cdot d\vec{l}=0$, $\vec{E}\cdot d\vec{r}=-dV$, $\int_A^B\vec{E}\cdot d\vec{r}=V(B)-V(A)$

(ii) **What about MS?**

Biot-Savart law

$$\vec{B}=\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau$$

where



Let's see what $\vec{\nabla}\cdot\vec{B}$ and $\vec{\nabla}\times\vec{B}$ are.

Divergence

$$\vec{\nabla}_r \cdot \vec{B} = \frac{\mu_0}{4\pi}\, \underbrace{\vec{\nabla}_r \cdot}_{} \int \vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau$$

derivative w/respect to field point $\vec{r}$

independent of $\vec{r}\,'$, the integration position vector

$$= \frac{\mu_0}{4\pi} \int \vec{\nabla}_r \cdot \left[\vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right] d\tau$$

ID#6

$$= \frac{\mu_0}{4\pi} \int \left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \cancelto{0}{\left(\vec{\nabla}_r \times \vec{J}(\vec{r}\,')\right)} - \vec{J}(\vec{r}\,') \cdot \left(\vec{\nabla}_r \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \right] d\tau$$

$\vec{J}$ depends only on $\vec{r}\,'$

Q: what is $\vec{\nabla}_r \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$?

$$\vec{\nabla}_r \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \left(\frac{x - x'}{\mathscr{r}^3}, \frac{y - y'}{\mathscr{r}^3}, \frac{z - z'}{\mathscr{r}^3} \right)$$

$$= \left[-\frac{(z - z')}{\mathscr{r}^5} \left(3[y - y'] \right) + \frac{(y - y')}{\mathscr{r}^5} 3(z - z'), \ \ldots, \ \ldots \right]$$

take limit as $x \to x'$, $y \to y'$, $z \to z'$

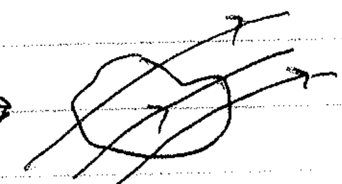
$$= (0, 0, 0)$$

$$\Longrightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0} \Longrightarrow$$ no sources or sinks of $\vec{B}$-field $\Rightarrow$ no magnetic charges, no monopoles

"Gauss's Theorem"

$$\int (\vec{\nabla} \cdot \vec{B})\, d\tau = 0$$

$$\boxed{\oint \vec{B} \cdot d\vec{S} = 0}$$; Closed surface $\Rightarrow$



Curl

$$\vec{\nabla}_{\vec{r}} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla}_{\vec{r}} \times \int \left[\vec{J}(\vec{r}\,') \times \frac{\hat{\imath}}{\imath^2} d\tau \right]$$

ID #8

$$= \frac{\mu_0}{4\pi} \int \Bigg[\cancelto{0}{\left(\frac{\hat{\imath}}{\imath^2} \cdot \vec{\nabla}_r\right) \vec{J}(\vec{r}\,')} - \overbrace{\left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla}_r\right) \frac{\hat{\imath}}{\imath^2}}^{?} + \vec{J}(\vec{r}\,') \underbrace{\left(\vec{\nabla}_r \cdot \frac{\hat{\imath}}{\imath^2}\right)}_{4\pi \delta^3(\vec{\imath})} - \frac{\hat{\imath}}{\imath^2} \underbrace{\cancelto{0}{\left(\vec{\nabla}_r \cdot \vec{J}(\vec{r}\,')\right)}}_{\text{if steady current}} \Bigg] d\tau$$

depends only on $\vec{r}\,'$

if steady current $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

Q: what is

$$-\frac{\mu_0}{4\pi} \int \left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla}_r\right) \frac{\hat{\imath}}{\imath^2} d\tau \quad ?$$

note: $\left[\vec{J}(\vec{r}\,') \cdot \vec{\nabla}_r\right] \frac{\hat{\imath}}{\imath^2} = -\left[\vec{J}(\vec{r}\,') \cdot \vec{\nabla}_{r'}\right] \frac{\hat{\imath}}{\imath^2}$, because $\vec{\imath} = \vec{r} - \vec{r}\,'$

and so, we have

$$\frac{\mu_0}{4\pi} \int \left[\vec{J}(\vec{r}\,') \cdot \vec{\nabla}_{r'}\right] \frac{\hat{\imath}}{\imath^2} d\tau = \; ? \quad \longleftarrow \text{look at } \left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla}_{r'}\right) \left(\frac{x - x'}{\imath^3}\right)$$

x-comp: Use vector ID, $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + (\vec{A} \cdot \vec{\nabla}) f$

$$\left(\vec{J} \cdot \vec{\nabla}_{r'}\right)\left(\frac{x - x'}{\imath^3}\right) = \vec{\nabla}_{r'} \cdot \left[\frac{x - x'}{\imath^3} \vec{J}\right] - \left(\frac{x - x'}{\imath^3}\right) \underbrace{\vec{\nabla}_{r'} \cdot \vec{J}(\vec{r}\,')}$$

recall:

Continuity Equation $\quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

$\rightarrow \vec{\nabla} \cdot \vec{J} = 0$ in steady state

$$(\vec{J}\cdot\vec{\nabla}_{r'})\left(\frac{x-x'}{\imath^3}\right) = \vec{\nabla}_{r'}\cdot\left[\frac{(x-x')}{\imath^3}\vec{J}\right]$$

and the x-comp integral is

$$-\frac{\mu_0}{4\pi}\int \vec{\nabla}_{r'}\cdot\left[\frac{x-x'}{\imath^3}\vec{J}\right]d\tau$$

$$= -\frac{\mu_0}{4\pi}\oint\left(\frac{x-x'}{\imath^3}\right)\vec{J}\cdot d\vec{S}$$

if Volume encloses all $\vec{J} \Rightarrow \vec{J}\cdot d\vec{S} = 0$ and

$\Rightarrow \oint = 0$

$$\Rightarrow -\frac{\mu_0}{4\pi}\int\left[\vec{J}(\vec{r}\,')\cdot\vec{\nabla}_r\right]\frac{\hat{\imath}}{\imath^2}d\tau = 0$$

and

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\,4\pi\,\delta^3(\vec{\imath})\,d\tau$$

note: $\vec{\imath} = 0 \Longrightarrow \vec{J}$ is evaluated at $\vec{r}\,' = \vec{r}$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})}$$, $\vec{J}(\vec{r})$ doesn't extend to ∞ and $\frac{\partial}{\partial\tau} = 0$

Ampere's Law

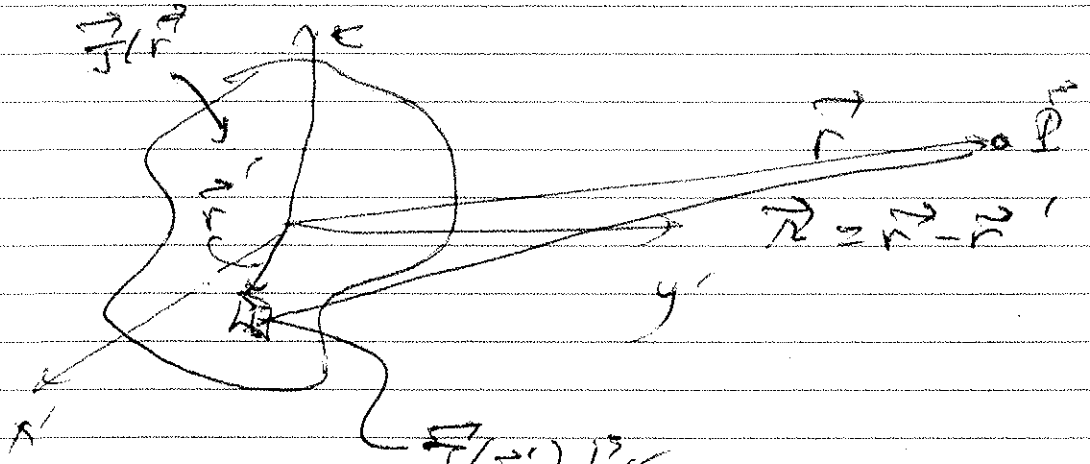
Stokes's thm $\Longrightarrow \oint(\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \mu_0\int\vec{J}(\vec{r})\cdot d\vec{S}$

Stokes' theorem $\Rightarrow$ $\boxed{\oint\vec{B}\cdot d\vec{l} = \mu_0\int\vec{J}(\vec{r})\cdot d\vec{S}}$

Biot-Savart law

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,d^3x' \times \hat{\mathcal{R}}}{\mathcal{R}^2}$$

is useful in that it gives the field from an arbitrary current distribution. However, as in electrostatics it is useful to have differential forms to get $\vec{B}$.



Recall: $\boxed{\dfrac{\hat{\mathcal{R}}}{\mathcal{R}^2} = -\vec{\nabla}_r\left(\dfrac{1}{\mathcal{R}}\right)}$

reverse order and pull out $\vec{\nabla}_r \times$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\int \left(\vec{J}(\vec{r}')\,d^3x'\right) \times \left(-\vec{\nabla}_r \frac{1}{\mathcal{R}}\right)$$

independent of $\vec{r}'$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}_r \times \int \frac{\vec{J}(\vec{r}')\,d^3x'}{\mathcal{R}}}$$

$\mathcal{R}$ ← must stay inside $\int$ sign

① $\underline{\vec{\nabla}\cdot\vec{B}}$

$$\vec{\nabla}_r\cdot\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}_r\cdot\left[\vec{\nabla}_r\times\int\frac{\vec{J}(\vec{r}')\,d^3x'}{\mathcal{R}}\right]$$

$\longrightarrow 0$

and $\boxed{\vec{\nabla}_r\cdot\vec{B} = 0}$

② $\underline{\vec{\nabla}\times\vec{B}}$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}_r\times\left[\vec{\nabla}_r\times\int\frac{\vec{J}(\vec{r}')\,d^3x'}{\mathcal{R}}\right]$$

$$\boxed{\text{ID}\#11\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{A}}$$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left\{\underbrace{\vec{\nabla}_r\left(\vec{\nabla}_r\cdot\int\frac{\vec{J}(\vec{r}')\,d^3x'}{\mathcal{R}}\right)}_{a} - \underbrace{(\vec{\nabla}\cdot\vec{\nabla})\int\frac{\vec{J}(\vec{r}')\,d^3x'}{\mathcal{R}}}_{b}\right\}$$

ⓐ $\vec{\nabla}_r\cdot\int\frac{\vec{J}(\vec{r}')\,d^3x'}{\mathcal{R}} = \int\left(\vec{J}(\vec{r}')\,d^3x'\cdot\underbrace{\vec{\nabla}_r\right)\frac{1}{\mathcal{R}}}$

$$\vec{\nabla}_r\frac{1}{\mathcal{R}} = -\vec{\nabla}_{r'}\frac{1}{\mathcal{R}}$$

$$\Rightarrow -\int \vec{J}(\vec{r}')\, d^3x' \cdot \vec{\nabla}_{r'}\left(\frac{1}{r}\right)$$

(b) $\vec{\nabla}_r \cdot \vec{\nabla}_r \int \frac{\vec{J}(\vec{r}')\, d^3x'}{r}$

$$= \int \vec{J}(\vec{r}')\, d^3x'\, \vec{\nabla}_r \cdot \vec{\nabla}_r\left(\frac{1}{r}\right)$$

$$= \int \vec{J}(\vec{r}')\, d^3x'\, \underbrace{\vec{\nabla}_r \cdot \left(\frac{\hat{r}}{r^2}\right)}_{4\pi\delta^3(\vec{r})}$$

and we have

$$\vec{\nabla}\times\vec{B} = -\frac{\mu_0}{4\pi}\vec{\nabla}_r \int \underbrace{\vec{J}(\vec{r}')\, d^3x' \cdot \vec{\nabla}_{r'}\left(\frac{1}{r}\right)} + \mu_0 \vec{J}(\vec{r})$$

note: $\vec{\nabla}_{r'} \cdot \left[\vec{J}(\vec{r}')\frac{1}{r}\right] = \frac{1}{r}\vec{\nabla}_{r'} \cdot \vec{J}(\vec{r}') + \vec{J} \cdot \vec{\nabla}_{r'}\frac{1}{r}$

$$\rightarrow \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}_r \int \left\{ \underbrace{\frac{\vec{\nabla}_{r'} \cdot \vec{J}(\vec{r}')\, d^3x'}{r}}_{\rightarrow 0,\ \frac{\partial\rho}{\partial t} = 0} - \underbrace{\vec{\nabla}_{r'} \cdot \left(\frac{\vec{J}(\vec{r}')}{r}\right) d^3x'}_{\rightarrow \oint \frac{\vec{J}(\vec{r}')}{r} \cdot d\vec{S}' \rightarrow 0,\ \because \text{ no enclose } \vec{J}} \right\} + \mu_0 \vec{J}$$

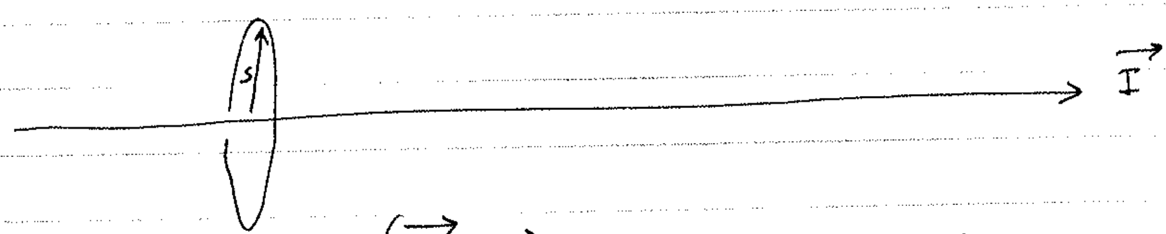
$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}} \Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Ampere's law

Examples

Ampere's law (plays role in Gauss's law in magnetostatics)

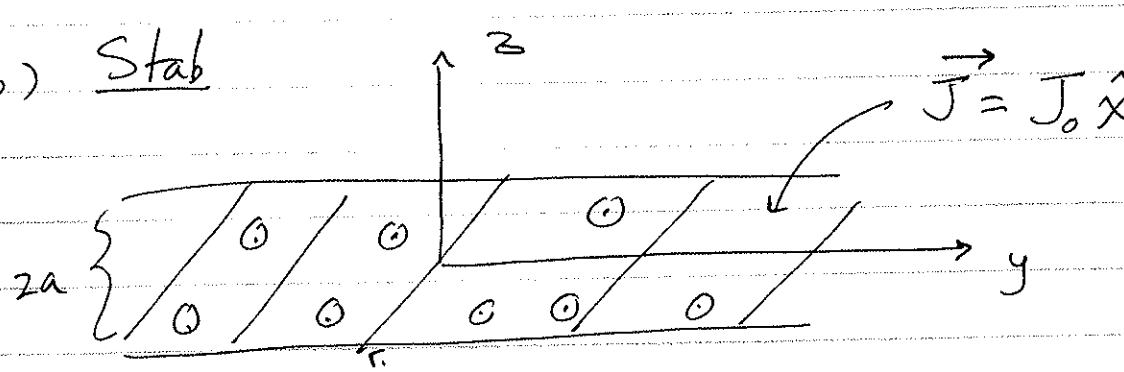
a) line current (Wire) $\Longrightarrow B_\phi$!



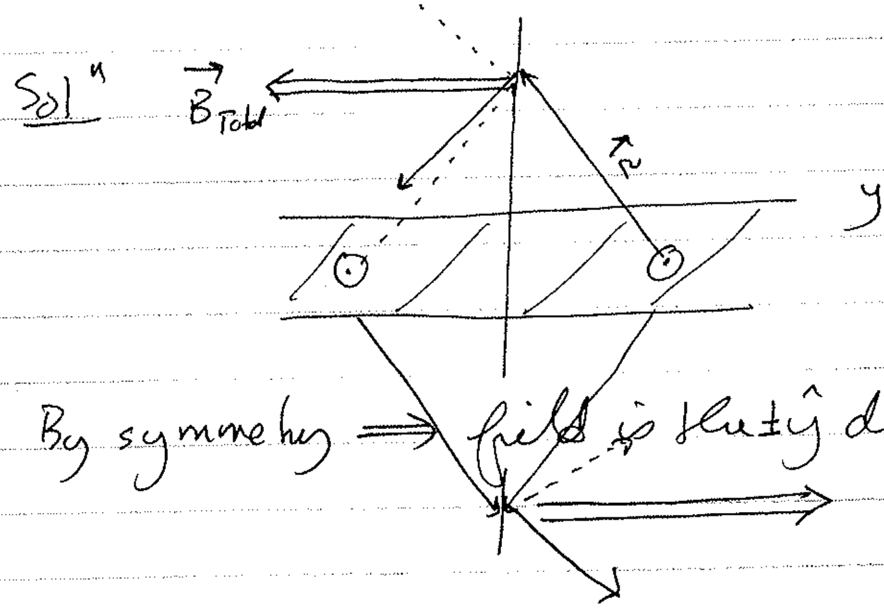
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{s} \Rightarrow B_\phi\, 2\pi s = \mu_0 I$$

$$\boxed{\vec{B}_\phi = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi}}$$

b) Slab



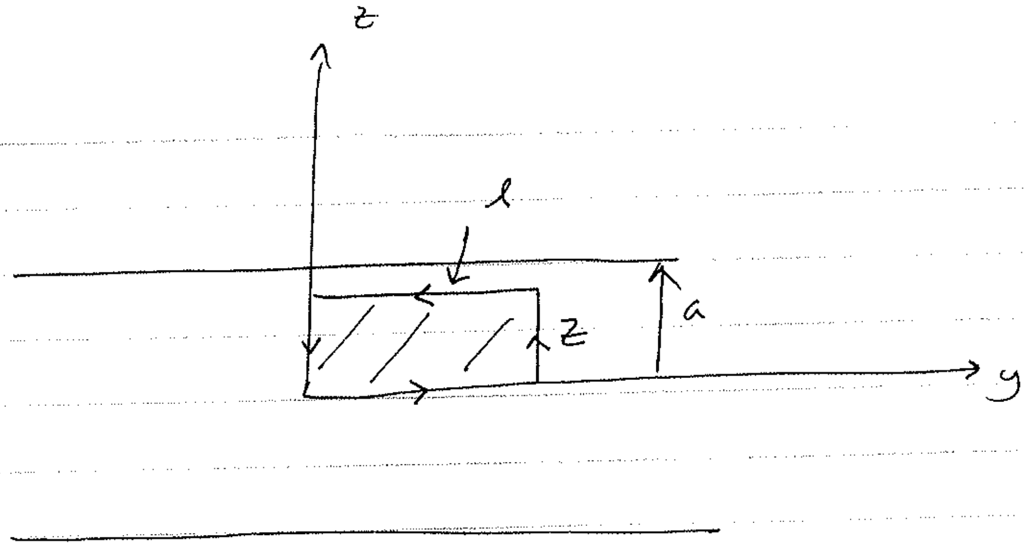
Sol^n



By symmetry $\Longrightarrow$ field is the $\pm\hat{y}$ direction

$\Longrightarrow B_y(z=0) = 0$ by symmetry

So,



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S} = \mu_0 J_0 l \begin{cases} z, & 0<z<a \\ a, & a<z \end{cases}$$

$$-B_y(z)\, l - \cancelto{0}{B_y(z=0)}\, l = \mu_0 J_0 l \begin{cases} z & 0<z<a \\ a & a<z \end{cases}$$

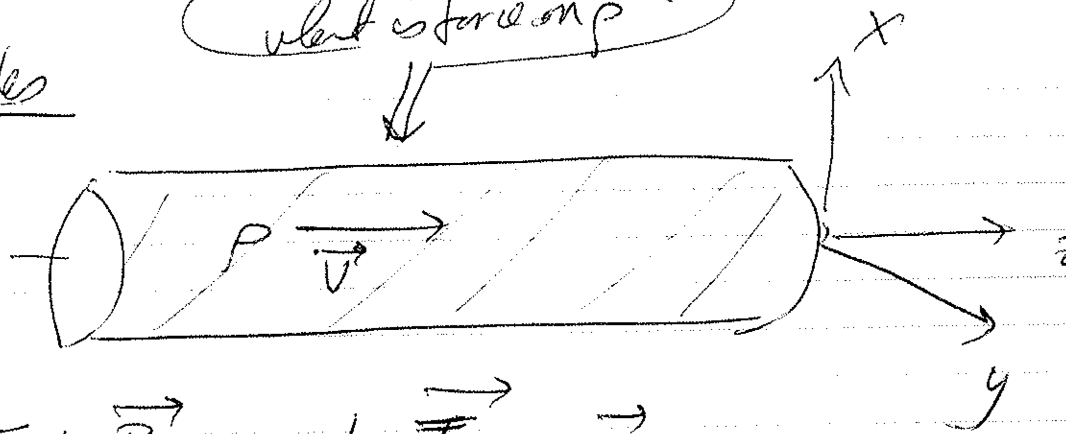
$$\rightarrow -B_y(z) = \begin{cases} \mu_0 J_0 z & 0<z<a \\ \mu_0 J_0 a & a<z \end{cases}$$

<u>by symmetry</u>

$$+B_y(z) = \begin{cases} -\mu_0 J_0 z & 0>z>-a \\ \mu_0 J_0 a & -a>z \end{cases}$$

what is force on ρ?

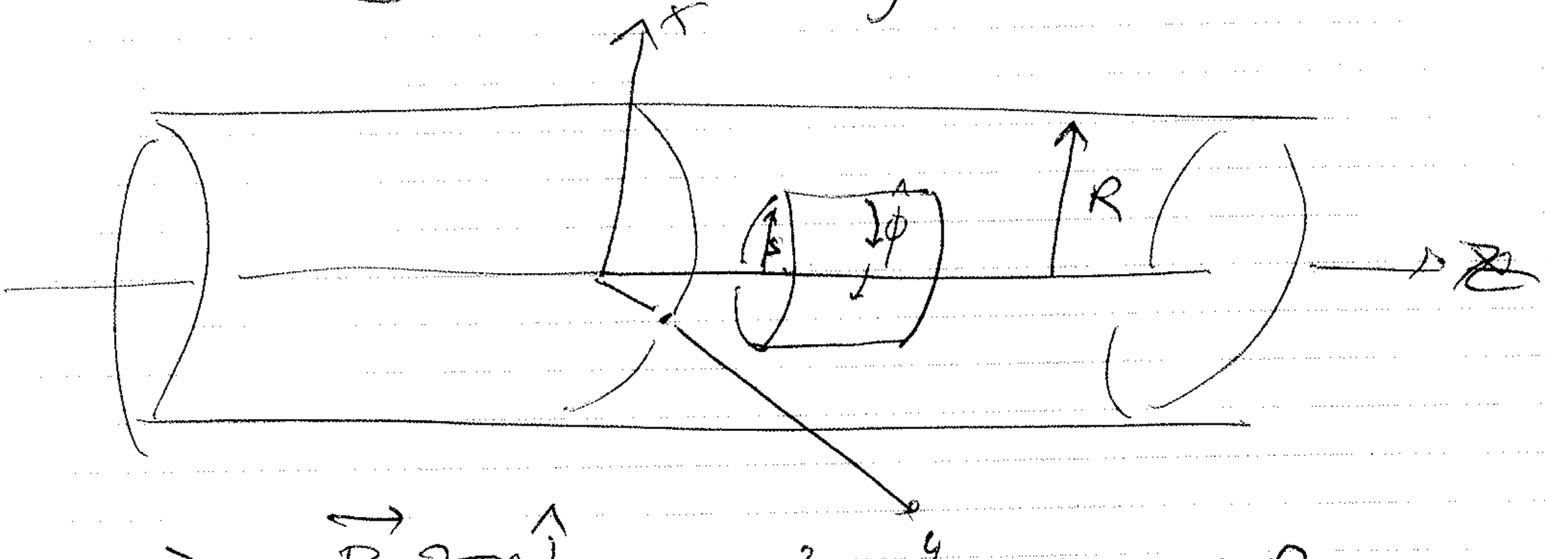
Examples



ⓐ Find $\vec{B}$ due to $\vec{J} = \rho\vec{v}$

(i) Use Ampere's law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S}$$



$\Rightarrow$ a) $\vec{B}_\phi \cdot 2\pi s\hat{\phi} = \rho\pi s^2 v\mu_0$; $s < R$

and $B_\phi = \frac{\rho v\pi s^2}{2\pi s}\mu_0 \Rightarrow \vec{B}_\phi = \mu_0\frac{\rho v s}{2}\hat{\phi}$

b) $s > R \Rightarrow B_\phi = \frac{\rho v \pi R^2}{2\pi s}\mu_0 \Rightarrow \vec{B}_\phi = \mu_0\frac{\rho v R^2}{2s}\hat{\phi}$

Force :

(a) $d\vec{F} = \vec{J}\, d^3x \times \vec{B}$

$$\Rightarrow d\vec{F} = -\rho v\, d^3x \left(\mu_0 \frac{\rho v}{2} s\right) \hat{s}$$

force inward (radially directed)

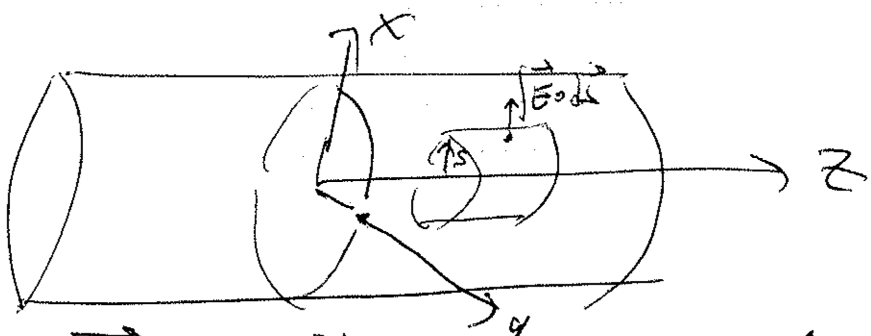
Lorentz force

$$\vec{F}_s = -\mu_0 \frac{\rho^2 v^2}{2} \int \underbrace{(s\, d\phi\, ds\, dz)}_{d^3x} s\, \hat{s}$$

$$= -\mu_0 \frac{1}{2} \rho^2 v^2 \left[2\pi L \frac{s^3}{3} \right] \hat{s}$$

$$\boxed{\vec{F}_{s}/L = -\mu_0 \frac{\rho^2 v^2 \pi s^3}{3} \hat{s}}$$ ← force per unit length

b)



$$\vec{E}_s = \frac{\rho \pi s^2 L}{\epsilon_0 \underbrace{2\pi s L}_{\text{area}}} \hat{s} = \frac{\rho s}{2\epsilon_0} \hat{s}$$ Gauss's law

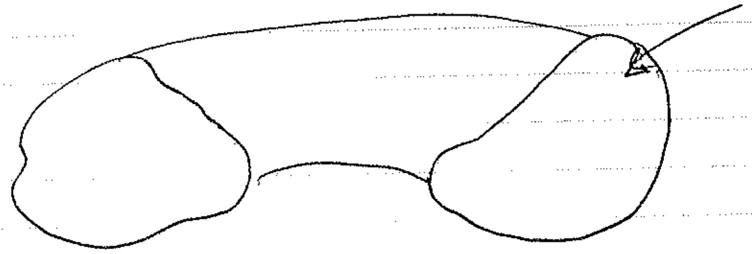
coulomb

$$\vec{F} = \int \frac{1}{2\epsilon_0} \rho s \rho (s\, d\phi\, ds\, dz)$$

$$= \pi \rho^2 \left[\frac{s^3}{3} L \right] \hat{s} \implies \boxed{\frac{\vec{F}_s}{L} = \frac{\pi \rho^2}{3\epsilon_0} s^3 \hat{s}}$$

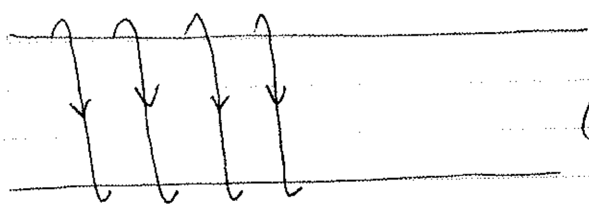
"Toroidal (Donut)" loop

(a)



(b) wrap coils of wire tightly around perimeter (about the meridional slice) so that there is no current in ϕ (around the loop). Q. How do we accomplish this?

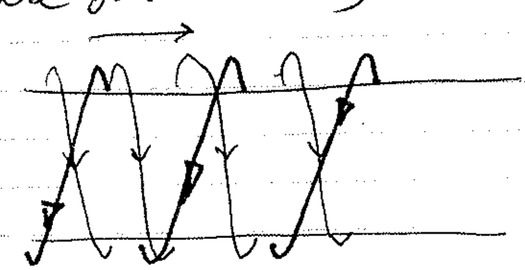
(i)



(i) However, I has ↻ + $\longrightarrow$

that is, 2 components

(ii) we "fix" this by

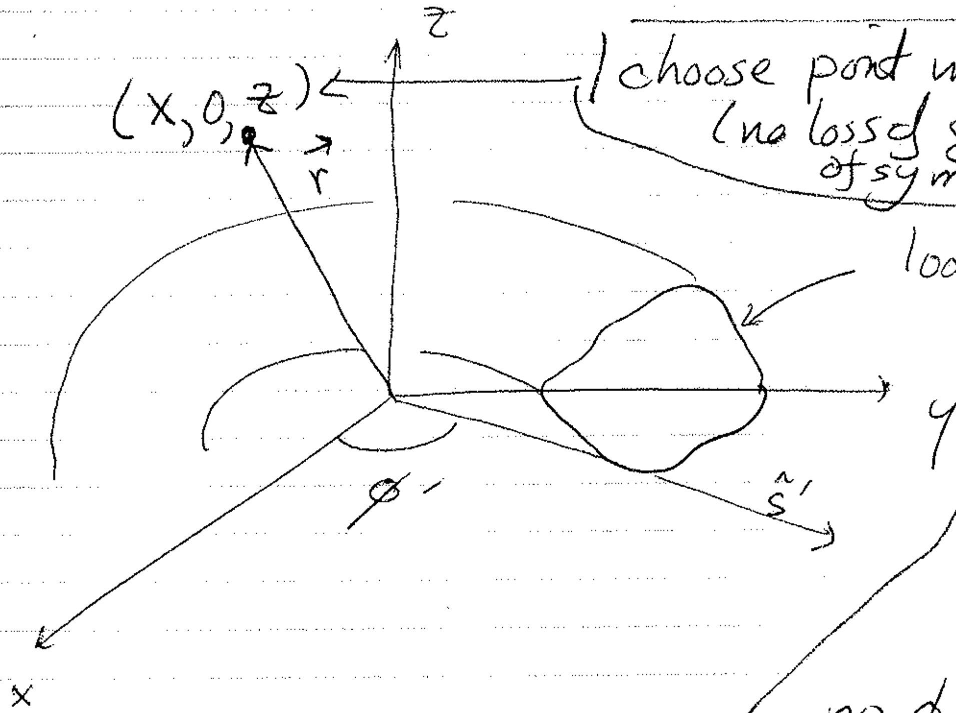


ii) I has ↻ + $\longleftarrow$

again, 2 components

(i) + (ii) $\Rightarrow$ $\leftrightarrows$ cancel, but ↻ add

Okay let's find $\vec{B}$ due to the donut



choose point in x-z plane (no loss of generality because of symmetry)

loop at ϕ' has current

$\vec{I} = (I_s \cos\phi', I_s \sin\phi', I_z)$

no ϕ'-component

Biot-Savart law

(a) $$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} dl$$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ $\quad y = 0$

$= (x - x', y - y', z - z')$

(b) write down $\vec{I} \times \vec{\mathscr{r}}$

$$\boxed{\begin{aligned} \vec{I} \times \vec{\mathscr{r}} &= (I_s \cos\phi', I_s \sin\phi', I_z) \times (x - x', -y', z - z') \\ &= \big(I_s \sin\phi' [z - z'] + I_z y', \; I_z [x - x'] - I_s \cos\phi' [z - z'], \\ &\qquad I_s \cos\phi' [-y'] - I_s \sin\phi' [x - x'] \big) \end{aligned}}$$

(c) Integrate $\vec{I} \times \vec{\mathscr{r}}$ over loops

(d) first note that

$$(x', y', z') = (s'\cos\phi', s'\sin\phi', z')$$

$$\Rightarrow \vec{I}\times\vec{r} = \Big(\sin\phi'\left[I_s(z-z') + I_z s'\right],$$

$$\cos\phi'\left[-s'I_z - I_s(z-z')\right] + xI_z,$$

$$-\sin\phi'\left[I_s x\right] + \cos\phi'\sin\phi'\underbrace{\left[-s'I_s + I_s s'\right]}_{\to 0}\Big)$$

(e) over a meridional slice, ϕ' is fixed. So, when we do the integrals, s always runs over the same interval as does z' and don't change integral

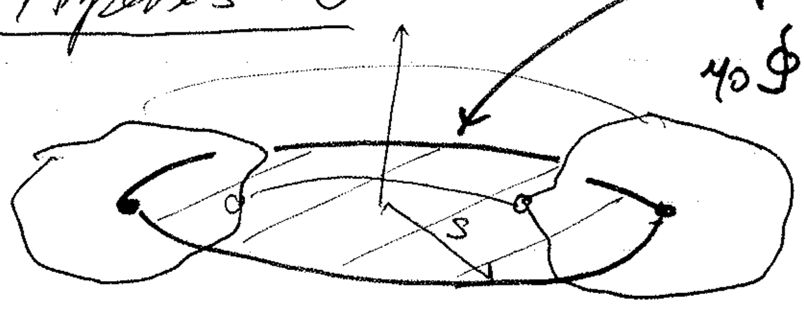
~~ϕ' runs from $0 \to 2\pi \Rightarrow \sin\phi', \cos\phi'$ alike ... $\to$ cancel over the integration~~

parts that survive integration

(f) ~~$\Rightarrow \vec{I}\times\vec{r} = (0, xI_z, 0)$~~

$\longrightarrow$ field points in ϕ direction

(g) Use Ampere's law

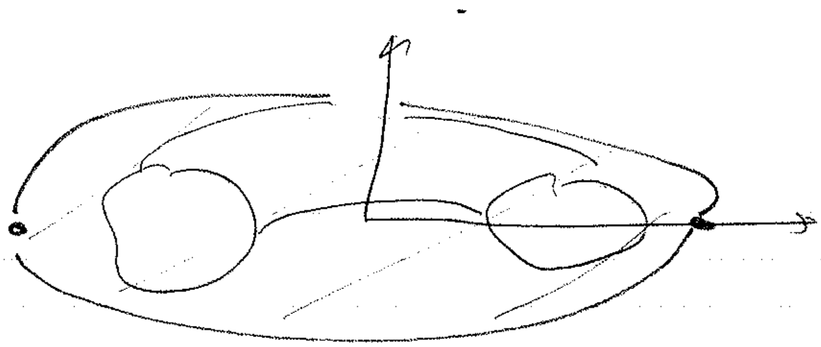


Amperian loop

$$\mu_0 \oint \vec{J}\cdot d\vec{S} = \oint \vec{B}\cdot d\vec{l}$$

$$\mu_0 N I = B_s 2\pi s$$

$$B_s = \mu_0 \frac{IN}{2\pi s}$$



$$\mu_0 \oint \vec{J}\cdot d\vec{S} = 0$$
$$= B_s\, 2\pi s$$
$$\Longrightarrow B_s = 0$$

and the field is

$$\vec{B} = \hat{s}\begin{cases} \dfrac{\mu_0 N I}{2\pi s} & , s \text{ inside} \\ 0 & , s \text{ outside} \end{cases}$$

(1) Tokomaks work on this idea

$$\left.\begin{array}{l} \text{a) instabilities} \\ \text{b) curvature drift} \\ \text{c) gradient drift} \\ \text{d)} \quad \vdots \end{array}\right\} \text{bad}$$

bad $\Longrightarrow$ Tokomaks use helical fields:

(i) Current flow ρv_ϕ $\Longrightarrow$ poloidal field

(ii) Solenoidal field $\Longrightarrow$ toroidal field

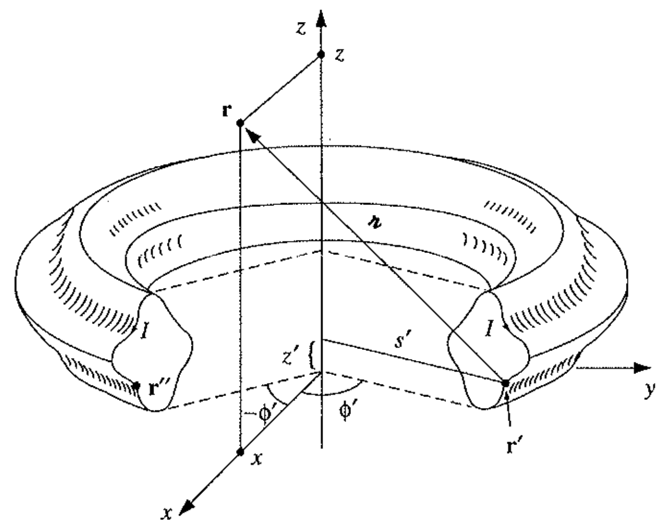


Figure 5.39

Accordingly,

$$\mathbf{I} \times \boldsymbol{\mathscr{r}} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ I_s \cos\phi' & I_s \sin\phi' & I_z \\ (x - s'\cos\phi') & (-s'\sin\phi') & (z - z') \end{bmatrix}$$

$$= [\sin\phi'(I_s(z - z') + s'I_z)]\,\hat{\mathbf{x}}$$

$$+ [I_z(x - s'\cos\phi') - I_s\cos\phi'(z - z')]\,\hat{\mathbf{y}} + [-I_s x \sin\phi']\,\hat{\mathbf{z}}.$$

But there is a symmetrically situated current element at $\mathbf{r}''$, with the same s', the same $\mathscr{r}$, the same dl', the same I_s, and the same I_z, *but negative* ϕ' (Fig. 5.39). Because $\sin\phi'$ changes sign, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ contributions from $\mathbf{r}'$ and $\mathbf{r}''$ cancel, leaving only a $\hat{\mathbf{y}}$ term. Thus the field at $\mathbf{r}$ is in the $\hat{\mathbf{y}}$ direction, and in general the field points in the $\hat{\boldsymbol{\phi}}$ direction. qed

Now that we know the field is circumferential, determining its magnitude is ridiculously easy. Just apply Ampère's law to a circle of radius s about the axis of the toroid:

$$B 2\pi s = \mu_0 I_{\text{enc}},$$

and hence

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil,} \\ 0, & \text{for points outside the coil,} \end{cases} \tag{5.58}$$

where N is the total number of turns.

Prob 5.41 $\Longrightarrow$ problem $\Longrightarrow$ $\vec{B}(s) = B_z(s)\,\hat{z}$

q, m, v_0

if m escapes radially, show that

$$\oint \vec{B}(s)\cdot d\vec{A} = 0$$

a)
$$\begin{cases} \ddot{x} = \dot{y}\left(\frac{qB(s)}{m}\right) \\ \ddot{y} = -\dot{x}\left(\frac{qB(s)}{m}\right) \\ \ddot{z} = 0 \end{cases}$$

from earlier lecture

b) Torque, $\vec{N} = \vec{r} \times \vec{F}$

$$= \left(0, 0, -x\dot{x}\left[\frac{qB(s)}{m}\right] - y\dot{y}\left[\frac{qB(s)}{m}\right]\right)$$

$\vec{L} = \vec{r} \times \vec{p}$

$$\frac{d\vec{L}}{dt} = \left(0, 0, -\left[\frac{qB(s)}{m}\right]\frac{1}{2}\frac{d}{dt}\left(x^2 + y^2\right)\right)$$

$$\int \frac{dL_z}{dt}\,dt = -\frac{q}{2m}\int B(s)\,\frac{d}{dt}\left(x^2+y^2\right)dt$$

$$L_z\Big|_{t=0}^{t_{esc}} = -\frac{q}{2m}\int_{s=0}^{R} B(s)\, s\, ds$$

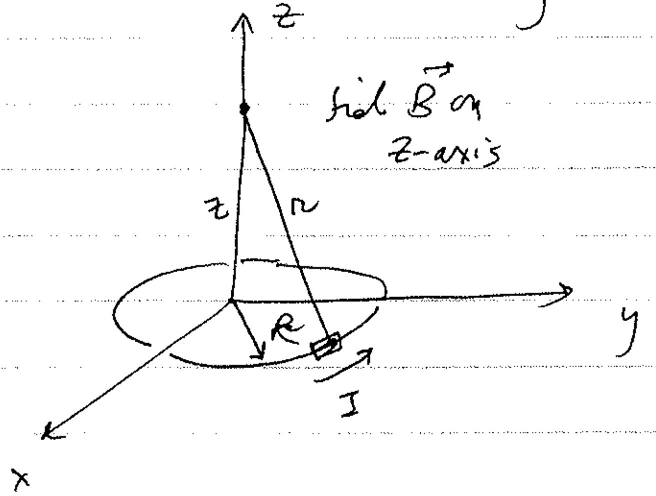
$$\underbrace{L_z(R) - L_z(0)}_{0} = -\frac{q}{4\pi m}\int \vec{B}(s)\cdot d\vec{A}$$

$$\Longrightarrow \int \vec{B}(s)\cdot d\vec{A} = 0$$

m moves radially
$\Rightarrow \vec{L}$ about origin is 0

Approximation Scheme

a) look at the field of a wire loop



$$dB_z = \frac{\mu_0}{4\pi}\,\frac{I R\,d\phi\, R}{r^3}$$

$$\rightarrow \boxed{\vec{B_z} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\,\hat{z}}$$

b) Can we infer something about the field near the z-axis?

Yes, let's use $\vec{\nabla}\cdot\vec{B}$ & $\vec{\nabla}\times\vec{B}$

(i) $\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{s}\frac{\partial}{\partial s}(sB_s) + \underbrace{\cancel{\frac{1}{s}\frac{\partial}{\partial \phi}(B_\phi)}}_{0\text{, by symmetry}} + \frac{\partial}{\partial z}B_z = 0$$

we know $\vec{B}$ on the z-axis ($s=0$). Use this to find B_s & B_z off-axis

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{s}\frac{\partial}{\partial s}(sB_s) + \frac{\partial}{\partial z}(B_z) = 0$$

$$\rightarrow \frac{1}{s}\frac{\partial}{\partial s}(sB_s) = -\frac{\partial B_z}{\partial z} = -\frac{\mu_0 I R^2}{2}\left(-\frac{\frac{3}{2}\,2z}{(R^2+z^2)^{5/2}}\right)$$

$$= -\frac{3\mu_0 I R^2}{2}\left(\frac{-z}{(R^2+z^2)^{5/2}}\right)$$

$$\rightarrow s B_s \Big|_0^s \approx \frac{3\mu_0 I R^2}{2}\left[\frac{z}{(R^2+z^2)^{5/2}}\right]\frac{s^2}{2}\Big|_0^s$$

$$\rightarrow \boxed{\vec{B}_s(s,z) \simeq \frac{3\mu_0 I R^2}{4}\left[\frac{sz}{(R^2+z^2)^{5/2}}\right]\hat{s}}$$

(ii) Use $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} = 0$ to find $\vec{B}_z(s,z)$

$$\vec{\nabla}\times\vec{B} = 0 + \hat{\phi}\left[\frac{\partial}{\partial z}B_s - \frac{\partial}{\partial s}B_z\right] + 0$$

$$\rightarrow \frac{\partial B_s}{\partial z} = \frac{\partial B_z}{\partial s}$$

from above

$$= \frac{3\mu_0 I R^2}{4}\left[s\left(\frac{R^2-4z^2}{(R^2+z^2)^{7/2}}\right)\right]$$

integrate

$$B_z\Big|_0^s = \frac{3\mu_0 I R^2}{8}s^2\left[\frac{R^2-4z^2}{(R^2+z^2)^{7/2}}\right]\Big|_0^s$$

$$B_z(s,z) = B_z(0,z) + \frac{3\mu_0 I R^2}{8}\left(\frac{R^2-4z^2}{[R^2+z^2]^{7/2}}\right)s^2$$

plug in for on-axis field

$$\boxed{\vec{B}_z = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\left[1+\frac{3}{4}\frac{(R^2-4z^2)}{(R^2+z^2)^2}s^2\right]\hat{z}}$$

Steady State Maxwell Equations

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\oint\vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int\rho\, d\tau \qquad \oint\vec{B}\cdot d\vec{S} = 0$$

$$\oint\vec{E}\cdot d\vec{l} = 0 \qquad \oint\vec{B}\cdot d\vec{l} = \mu_0\int\vec{J}\cdot d\vec{S}$$

Comment, $\vec{\nabla}\cdot$, $\vec{\nabla}\times$ are sufficient to describe an arbitrary vector field (Appendix B: Helmholtz Theorem) for given BCs.

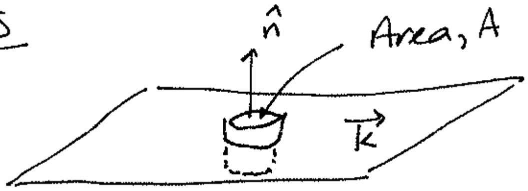
Boundary Conditions

ES

(i) $\Delta E_n = \sigma/\epsilon_0$, $\Delta E_T = 0$

MS

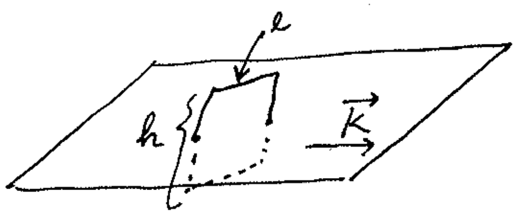
(i)



$$\vec{\nabla}\cdot\vec{B} = 0$$

$$B_n^+ A - B_n^- A = \int \vec{K}\cdot d\vec{S} = 0 \Rightarrow \Delta B_n = 0$$

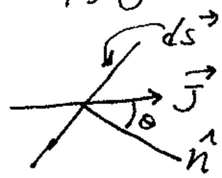
(ii)



$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S} \leftarrow$$ the normal to dS is θ

Top $\Longrightarrow$



$\Rightarrow \vec{J}\cdot d\vec{S} = J\,dS\cos\theta$
$= K l \cos\theta$

↖ In the plane of the sheet, $\measuredangle(\vec{J}, d\vec{S}) = \theta$

$$+B_T^+ l - B_T^- l = \mu_0 K l \cos\theta$$

$$\Delta B_T = \begin{cases} 0 & \vec{K} \parallel \vec{B} \\ \mu_0 K & \vec{K} \perp \vec{B} \end{cases}$$

Vector Potential

Exploit $\vec{\nabla} \cdot \vec{B} = 0$

Define a fcn. which is $\perp$ to $\vec{B}$,

$$\vec{B} = \vec{\nabla} \times \vec{A} \leftarrow \vec{A} \text{ is the vector potential}$$

$$\rightarrow \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \overset{\text{by def}^{n}}{=} 0$$

a) Is this it? No, similar to the scalar potential V, where

$$\vec{E} = -\vec{\nabla} V$$

we could add any $\overset{\text{constant}}{\text{scalar}}$ to V and leave $\vec{E}$ untouched,

$$\vec{E} = -\vec{\nabla}(V + f_0) = -\vec{\nabla} V - \cancelto{0}{\vec{\nabla} f_0}, \quad f_0 = \text{const}$$

b) Add $\vec{\nabla} f$ to $\vec{A}$. Note that

$$\vec{\nabla} \times (\vec{\nabla} f) = 0 \rightarrow \vec{B} \text{ is unchanged}$$

"Gauge freedom"

Q: How do we set the gauge?

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$$

Ib #11

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \underbrace{(\vec{\nabla}\cdot\vec{\nabla})}\vec{A} = \mu_0\vec{J}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

*

define $\vec{A} \ni \vec{\nabla}\cdot\vec{A} = 0$. This can be accomplished because of $\vec{\nabla}f \leftarrow$ gauge freedom.

okay $\vec{\nabla}\cdot\vec{A} = 0$ is the "Coulomb gauge"

and we have

$$\Longrightarrow \vec{\nabla}\cdot\vec{\nabla}\vec{A} = \nabla^2\vec{A} = -\mu_0\vec{J},$$ "Poisson's Eqn" for $\vec{A}$

$$\Longrightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\,d\tau}{r}}$$

as long as $\vec{J}$ doesn't extend to infinity.

Connect

$\int(\vec{\nabla}\cdot\vec{A})\,d\tau = 0$, from gauge condition

$\longrightarrow \oint\vec{A}\cdot d\vec{S} = 0$

* Addition to explain $\boxed{}$ comment on previous page

To see this,

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\Big[\overbrace{\vec{A}_0 + \vec{\nabla} f}^{\vec{A}}\Big]$$

f ← some arbitrary function

$$\Rightarrow \vec{B} = \left\{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}_0) - \nabla^2\vec{A}_0\right\} + \left\{\vec{\nabla}(\vec{\nabla}\cdot\vec{\nabla} f) - \nabla^2(\vec{\nabla} f)\right\}$$

$$= \left\{\vec{\nabla}\Big[\vec{\nabla}\cdot\big(\overbrace{\vec{A}_0 + \vec{\nabla} f}^{\vec{A}}\big)\Big] - \nabla^2\Big[\overbrace{A_0 + \vec{\nabla} f}^{\vec{A}}\Big]\right\}$$

now, $\vec{B} = \vec{\nabla}\times\vec{A}$ contains the $\nabla\times$ of our def^{n} of the vector field $\vec{A}$, however, we still need to specify the $\vec{\nabla}\cdot$ behavior of $\vec{A}$ to contain $\vec{A}$.

we $\boxed{\text{choose}}$ to set $\boxed{\vec{\nabla}\cdot\vec{A} = 0}$. This can be accomplished because of $\vec{\nabla} f$

$$\Rightarrow \vec{\nabla}\cdot\left[\vec{A}_0 + \vec{\nabla} f\right] = 0 \Longrightarrow \nabla^2 f = -\underbrace{\vec{\nabla}\cdot\vec{A}_0}_{\rho}$$

and
$$\boxed{f = -\frac{1}{4\pi}\int\frac{(\vec{\nabla}\cdot\vec{A}_0)}{r}\,d^3x'}$$

$\Rightarrow$ if $(\vec{\nabla}\cdot A_0) \Longrightarrow 0$ at ∞, we are okay

Coulomb Gauge: Comments

Let's digress for a minute

a) $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ Gauss's law

b) $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ Faraday's law (complete Maxwell Equation)

$$= -\frac{\partial}{\partial t}\left[\vec{\nabla} \times \vec{A}\right]$$

$$\Rightarrow \vec{\nabla} \times \left[\underbrace{\vec{E} + \frac{\partial}{\partial t}\vec{A}}_{= -\vec{\nabla} V}\right] = 0$$

Now, plug in for $\vec{E}$ in Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \left[-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right] = \rho/\epsilon_0$$

$$-\nabla^2 V - \frac{\partial}{\partial t}\left[\vec{\nabla} \cdot \vec{A}\right] = \rho/\epsilon_0$$

$$\Longrightarrow \nabla^2 V = -\rho/\epsilon_0$$

in Coulomb gauge $(\vec{\nabla} \cdot \vec{A} = 0)$

Unfortunately, $\vec{E} \neq -\vec{\nabla} V(r,t)$,

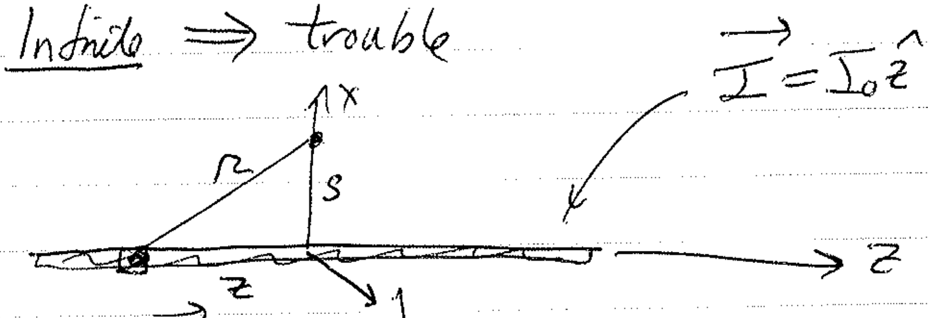
$$\vec{E} = -\vec{\nabla} V(r,t) - \frac{\partial \vec{A}}{\partial t}$$

and it turns out $\frac{\partial \vec{A}}{\partial t}$ is hard to find in general! In the time-dependent case, we use a more convenient gauge,

$$\left(\vec{\nabla} \cdot \vec{A}\right) = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \quad \text{"Lorentz Gauge"}$$

Find $\vec{A}$ for an infinite line current

Infinite $\Rightarrow$ trouble



(1) $\vec{A}$ will be independent of z, by symmetry and so let's evaluate $\vec{A}$ at $z=0$.

(2) To get around ∞, consider a wire of length $2L$ and then let $L \to \infty$

$$A_z = \frac{\mu_0}{4\pi}\int_{-L}^{L} \frac{I\,dz}{(s^2+z^2)^{1/2}}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left[z + \sqrt{z^2+s^2}\right]_{-L}^{L}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left\{\frac{L + \sqrt{L^2+s^2}}{-L+\sqrt{s^2+L^2}}\right\}$$

let $L \to \infty$ ($\Rightarrow L \gg s$)

$$\simeq \frac{\mu_0 I}{4\pi} \ln\left\{\frac{L + L\left(1+\frac{1}{2}\frac{s^2}{L^2}\right)}{-L + L\left(1+\frac{1}{2}\frac{s^2}{L^2}\right)}\right\}$$

$$A_z \simeq \frac{\mu_0 I}{4\pi}\left[\ln\left(2+\frac{1}{2}\frac{s^2}{L^2}\right)-\ln\left(\frac{1}{2}\frac{s^2}{L^2}\right)\right]$$

$$=\frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{1}{2L^2}\left[4L^2+s^2\right]\right)-\ln\left(\frac{1}{2L^2}\left[s^2\right]\right)\right]$$

$$=\frac{\mu_0 I}{4\pi}\left[\ln\left(4L^2+s^2\right)-\ln s^2\right]$$

$$=\frac{\mu_0 I}{4\pi}\left[\ln 4L^2+\underbrace{\ln\left(1+\frac{s^2}{4L^2}\right)}_{\to 0}-\ln s^2\right]$$

$$A_z \simeq \frac{\mu_0 I}{4\pi}\left[\underbrace{\ln 4L^2}_{\to\infty}-\ln s^2\right]$$

as $L \to \infty$!

However, it is a constant $\Longrightarrow$ $\vec{\nabla}\times\vec{A}_z$ is not affected by it and so can be dropped

$$\Longrightarrow \boxed{A_z \simeq -\frac{\mu_0 I}{2\pi}\ln s}$$

as $L \to \infty$

(a) $\vec{B} = \vec{\nabla} \times \vec{A}$

$$= \hat{s}\left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)$$

$$+ \hat{z}\,\frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial}{\partial \phi}A_s\right)$$

$$= \hat{s}\left[0 - 0\right] + \hat{\phi}\left[0 + \frac{\mu_0 I}{2\pi s}\right] + \hat{z}\left[0 - 0\right]$$

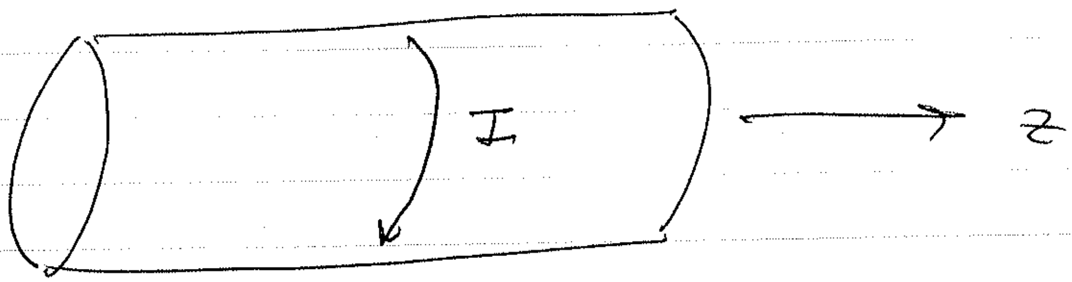
$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$$

(b) $\vec{\nabla} \cdot \vec{A} = \frac{1}{s}\frac{\partial}{\partial s} s A_s + \frac{1}{s}\frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z$

$$= 0 + 0 + 0$$

$$\boxed{\vec{\nabla} \cdot \vec{A} = 0}$$

Infinite Solenoid: find $\vec{A}$



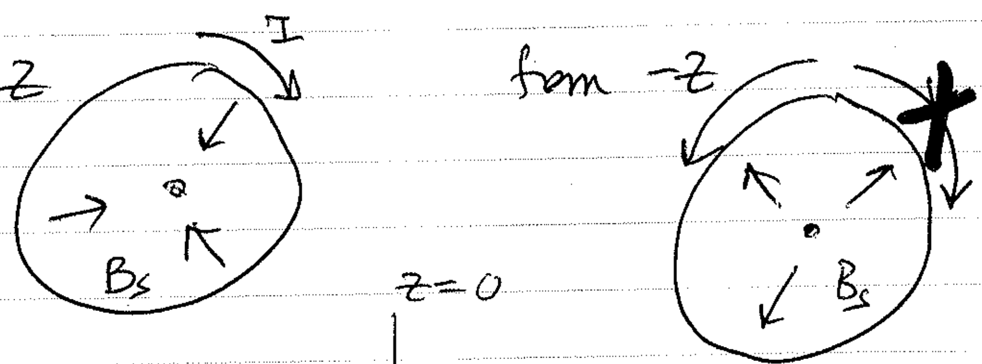
a) N = # of turns per unit length

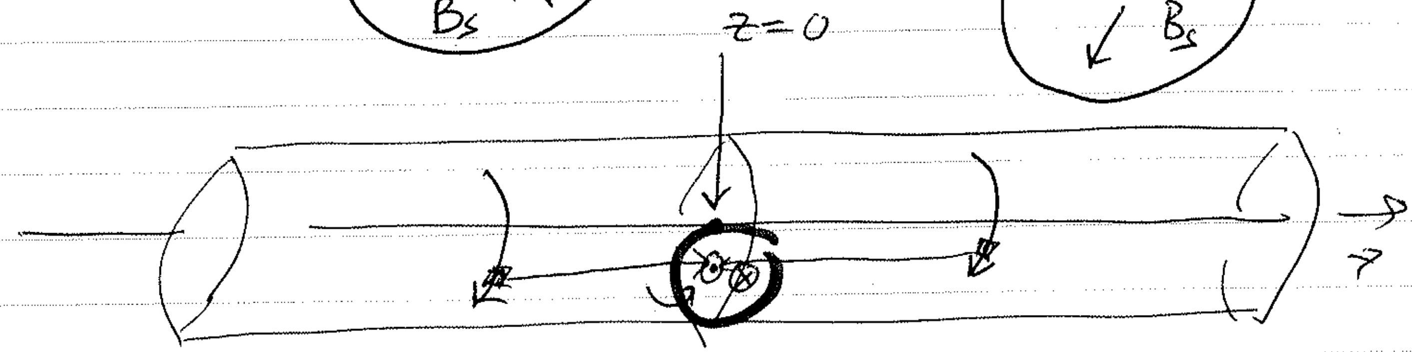
b) 2 layers of helical winds w/ opposite pitch
$\Rightarrow$ only circular I survives

c) Use Ampere's law to find $\vec{B}$ everywhere and then find $\vec{A}$.

d) ~~Symmetry~~ Biot-Savart law $\Longrightarrow$ $B_\phi = 0$ because I is azimuthal.

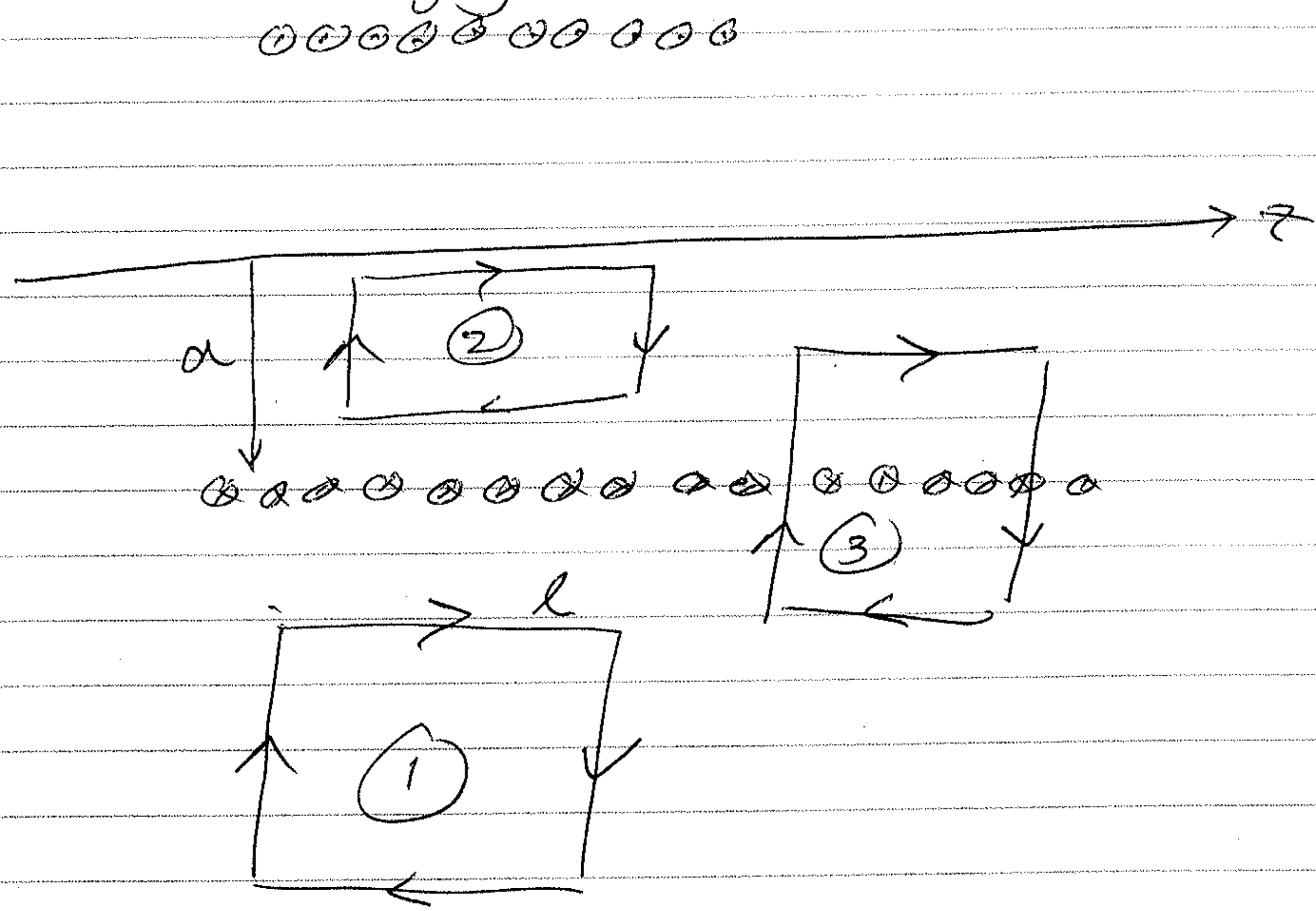
e) from $+z$ from $-z$





only way for this to be consistent is if
$B_s = 0$ (b/c $z=0$ is arbitrary)

$\Rightarrow$ B is going to be in z-direction



① $B^1_{z,out}\, l + 0 - B^2_{z,out}\, l + 0 = \mu_0 \int \vec{J}\cdot d\vec{S} = 0$

0 at $\infty \Rightarrow B_{z,out} = 0$

$\Rightarrow B^1_{z,out} = B^2_{z,out} \Rightarrow \cancel{B_{z,out}}$ is constant for $\rho > a$

② $B^1_{z,in}\, l + 0 - B^2_{z,in}\, l + 0 = \mu_0 \int \vec{J}\cdot d\vec{S} = 0$

$\Rightarrow B^1_{z,in} = B^2_{z,in} \Rightarrow B_{z,in}$ is constant

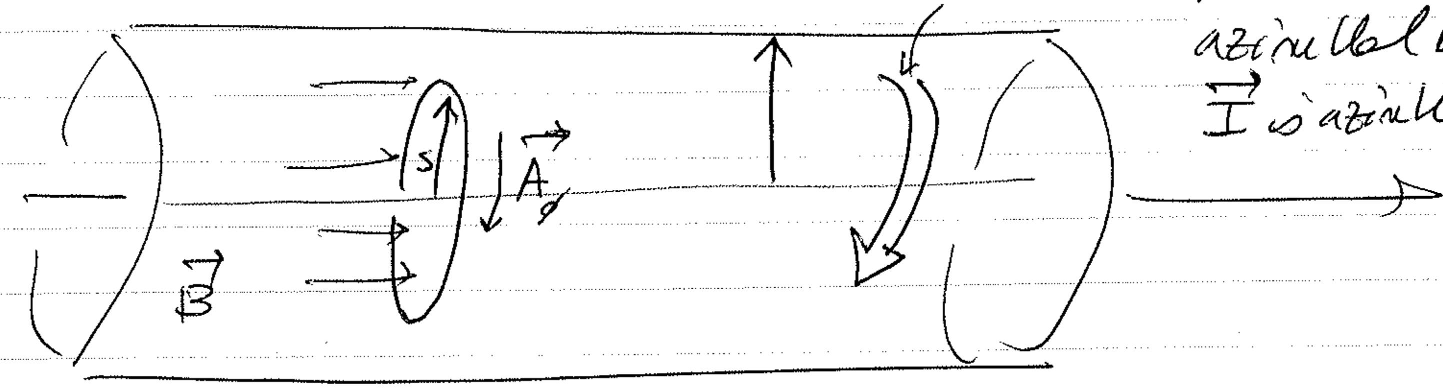
③ $B_{z,in}\, l + 0 - \overset{0}{\cancel{B_{z,out}}}\, l = \mu_0 I N l \Rightarrow \boxed{B_{z,in} = \mu_0 N I}$

ⓐ find $\vec{A}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\int \vec{B} \cdot d\vec{S} = \int \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l}$$

$\vec{A}$ will be azimuthal b/c $\vec{I}$ is azimuthal



a) $\int \vec{B} \cdot d\vec{S} = (\mu_0 N I) \pi s^2 = A_\phi 2\pi s, \quad s < a$

$$\Rightarrow \boxed{A_\phi^< = \frac{\mu_0 N I}{2} s \quad . \quad s < a}$$

b) $\int_0^a \vec{B} \cdot d\vec{S} = (\mu_0 N I) \pi a^2 = A_\phi 2\pi s, \quad s > a$

$$\Rightarrow \boxed{A_\phi^> = \frac{\mu_0 N I a^2}{2s}, \quad s > a}$$

<u>note:</u>

$$\left.\begin{array}{l} \vec{B} = 0 \text{ for } s > a \\ \vec{A} \neq 0 \text{ for } s > a \end{array}\right\} \text{ which is "real" field}$$

6 $(\vec{\nabla}\cdot\vec{A}) = \frac{1}{s}\frac{\partial}{\partial s} s A_s + \frac{1}{s}\frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z$

$= 0 + 0 + 0$

$$\boxed{\vec{\nabla}\cdot\vec{A} = 0}$$