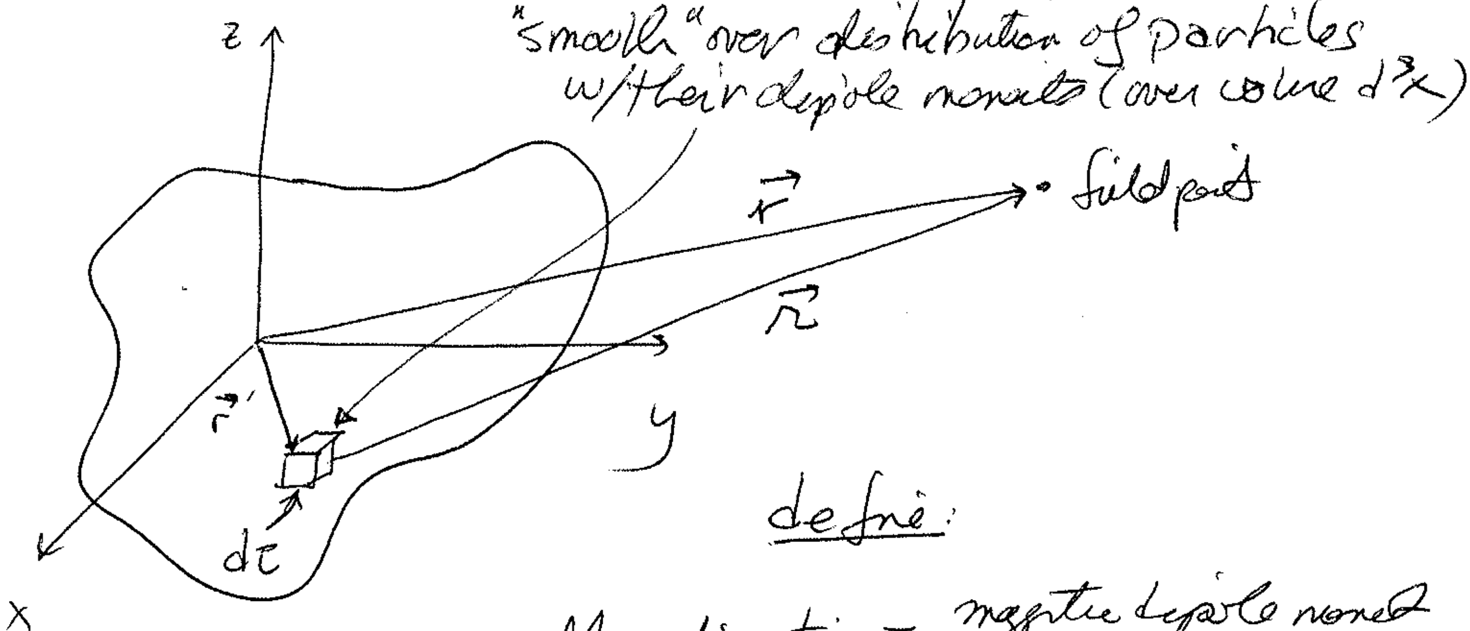
Field of a Magnetized Object



define:

Magnetization $\equiv$ magnetic dipole moment per unit volume

$$= \vec{M} \equiv \frac{\Sigma \vec{m}_i}{d\tau_i}$$

The contribution of volume $d\tau$ is

$$d\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{M}\, d\tau \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\rightarrow \boxed{\vec{A}_{dip} = \frac{\mu_0}{4\pi} \int \frac{\vec{M}\, d\tau \times \hat{\mathscr{r}}}{\mathscr{r}^2}}$$

let's fiddle w/this result

note: $\vec{\nabla}_{r'}\frac{1}{\mathscr{r}} = \vec{\nabla}_{r'}\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma'}}$ ← this looks ugly

$= \vec{\nabla}_{r'}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$

$= \frac{1}{\mathscr{r}^3}\left(\hat{x}'(x-x')+\hat{y}'(y-y')+\hat{z}'(z-z')\right)$

$= \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ $\qquad \vec{\nabla}_{r'}\times\left(\frac{\vec{M}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\vec{\nabla}_{r'}\times\vec{M} - \vec{M}\times\vec{\nabla}_{r'}\frac{1}{\mathscr{r}}$

$\rightarrow \vec{A}_{dip} = \frac{\mu_0}{4\pi}\int \vec{M}\,d\tau\times\vec{\nabla}_{r'}\left(\frac{1}{\mathscr{r}}\right)$

$\overset{\text{ID \#7}}{=} \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}\vec{\nabla}_{r'}\times\vec{M}\,d\tau - \int \vec{\nabla}_{r'}\times\left(\frac{\vec{M}}{\mathscr{r}}\right)d\tau\right\}$

(1,60b)

$= \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}(\vec{\nabla}_{r'}\times\vec{M})\,d\tau\right\} + \underbrace{\frac{\mu_0}{4\pi}\oint \frac{\vec{M}\times d\vec{S}}{\mathscr{r}}}_{\text{surface integral}}$

$\Rightarrow \boxed{\vec{J}_{bound} = (\vec{\nabla}_{r'}\times\vec{M}) \;\;\&\;\; \vec{K}_{bound} = \vec{M}\times\hat{n}}$

normal to the surface

$\boxed{1.60b}$

$$\boxed{\int (\vec{\nabla}\cdot\vec{C})\,d\tau = \oint \vec{C}\cdot d\vec{S}\text{, divergence thm.}}$$

<u>Let</u>: $\vec{C} = \vec{b}\times\vec{C}'$, $\vec{b}$ some constant vector

$\vec{C}'$ some arbitrary vector

$$\Rightarrow \int \left[\vec{\nabla}\cdot(\vec{b}\times\vec{C}')\right] d\tau = \oint (\vec{b}\times\vec{C}')\cdot d\vec{S}$$

$$\int \left[\vec{C}'\cdot \cancel{\vec{\nabla}\times\vec{b}}^{\,0} - \vec{b}\cdot\vec{\nabla}\times\vec{C}'\right] d\tau = \oint (\vec{b}\times\vec{C}')\cdot d\vec{S}$$

$\vec{b}$ is a constant vector

$$\longrightarrow \boxed{-\int \vec{b}\cdot(\vec{\nabla}\times\vec{C}')\,d\tau = \oint (\vec{b}\times\vec{C}')\cdot d\vec{S}}$$

<u>now</u>, $-\vec{b}\cdot\int(\vec{\nabla}\times\vec{C}')\,d\tau = \vec{b}\cdot\int \vec{C}'\times d\vec{S}$ (exchange $\cdot$ & $\times$)

$$\longrightarrow \boxed{\int \vec{\nabla}\times\vec{C}'\,d\tau = -\oint \vec{C}'\times d\vec{S}}$$

$\left(\frac{\vec{M}}{r}\right)$ $\left(\frac{\vec{M}}{r}\right)$

Again, as for polarization where we defined

$$\rho_b = -\vec{\nabla}\cdot\vec{P} \quad \& \quad \sigma_b = \vec{P}\cdot\hat{n}$$

we define

$$\vec{J}_b = \vec{\nabla}\times\vec{M} \quad \& \quad \vec{K}_b = \vec{M}\times\hat{n}$$

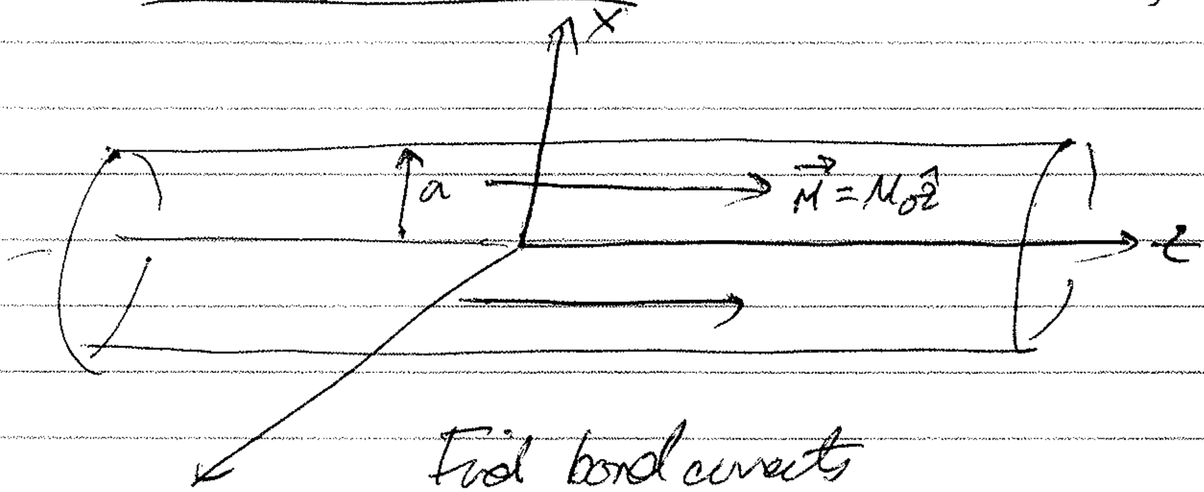
we can solve for the fields of a magnetized body in a straightforward manner.

For the following, assume that we have magnetized bodies; we do <u>not</u> worry about why they are magnetized.

Magnetised Objects

Prob6. 7

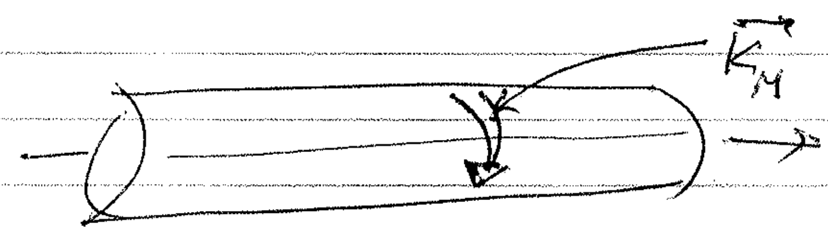
Infinite cylinder, radius a, with magnetization // to z-axis, $\vec{M} = M_0\hat{z}$. No free current, $\vec{J}_f = 0$



Find bound currents

$$\vec{J}_M = \vec{\nabla}\times\vec{M} = \underbrace{\frac{1}{r}\frac{\partial M_z}{\partial\phi}\hat{r} + \left(-\frac{\partial M_z}{\partial r}\right)\hat{\phi} + 0}_{=0}$$

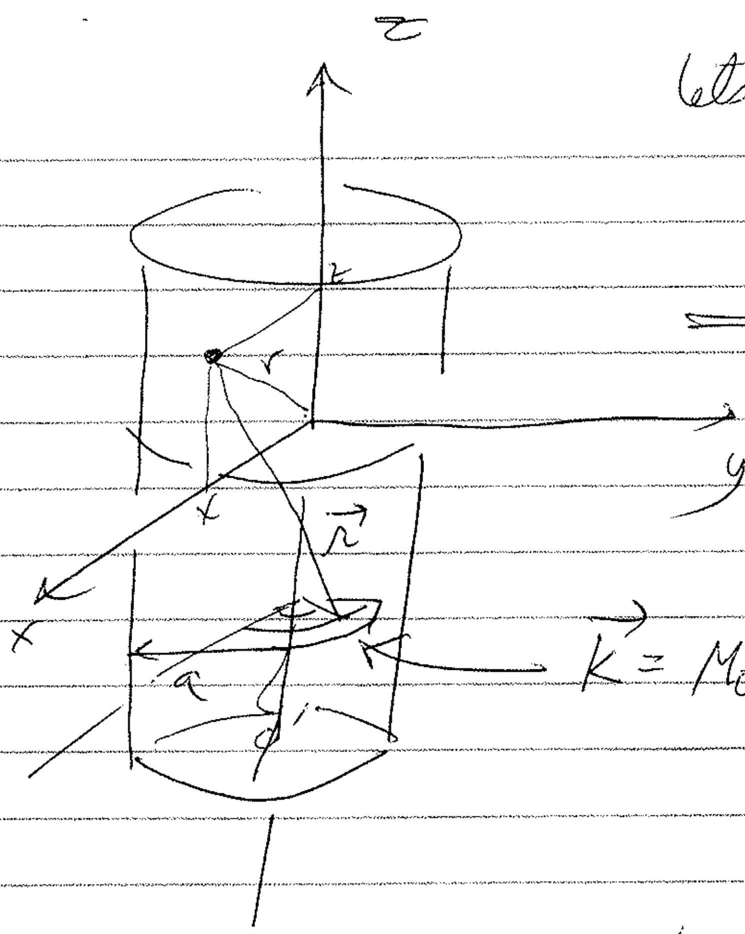
$$\vec{K}_M = \vec{M}\times\hat{n} = \vec{M}\times\hat{r} = M_0\hat{\phi}$$



$\Rightarrow$ Problem is an infinite solenoid w/

Surface current $= M_0\hat{\phi}$

and $\vec{B} = \begin{cases} 0, & r > a \\ \mu_0 M_0\hat{z}, & r < a \end{cases}$



Let's look at this in more detail

$$\Longrightarrow \vec{r} = (x - a\cos\phi', -a\sin\phi', z - z')$$

$$\vec{K} = M_0(-\sin\phi', \cos\phi', 0)$$

$$\Rightarrow d\vec{B} = \frac{\mu_0}{4\pi} M_0 \frac{(-\sin\phi', \cos\phi', 0) \times (x - a\cos\phi', -a\sin\phi', z - z')}{r^3}$$

$$= \frac{\mu_0}{4\pi} M_0 \frac{(\cos\phi'[\cancel{z} - z'], \sin\phi'[\cancel{z} - z'], a\sin^2\phi' + a\cos^2\phi' - x\cos\phi')}{r^3}$$

Because solenoid is ∞, let $z = 0$ (field point in xy plane)

$$d\vec{B} = \frac{\mu_0}{4\pi} M_0 \frac{(-z'\cos\phi', -z'\sin\phi', a - x\cos\phi')}{r^3}$$

$$r^3 \leftarrow \left[(x - a\cos\phi')^2 + a^2\sin^2\phi' + z'^2\right]^{3/2}$$

a) consider 2 loops, one at z' and one at $-z'$

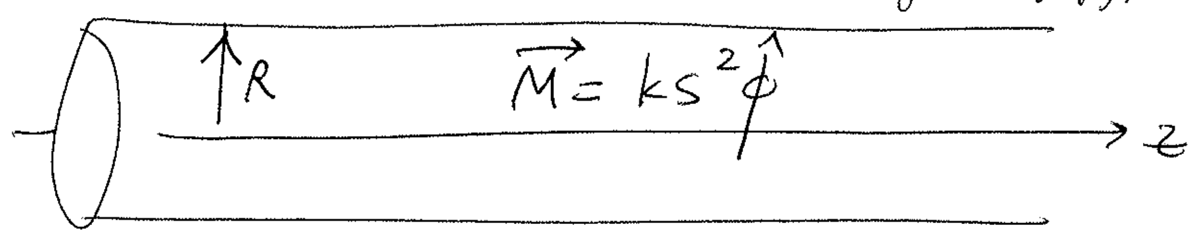
$\Rightarrow d\vec{B}_{x,y}$ cancel for these two loops ✓

and only field is in z-direction

$\Longrightarrow$ use Ampere's law

Prob 6.8

an infinite cylinder w/ radius R and an azimuthal magnetization, $\vec{M} = ks^2\hat{\phi}$



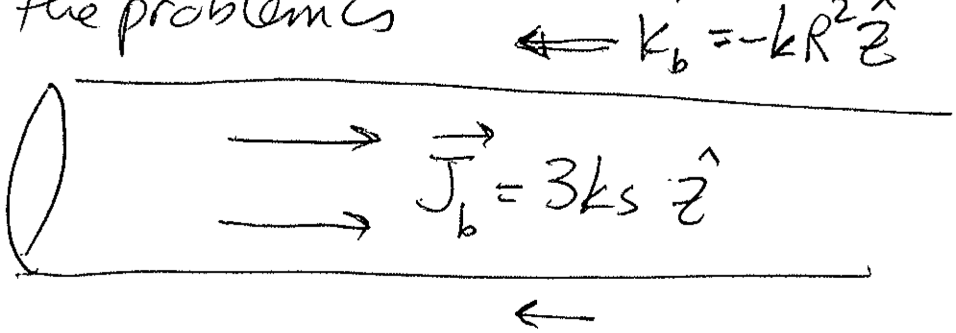
Find currents

$$\vec{J}_b = \vec{\nabla}\times\left(ks^2\hat{\phi}\right) = [0]\hat{s} + [0]\hat{\phi} + \left[\frac{1}{s}\frac{\partial}{\partial s}\left(ks^3\right)\right]\hat{z}$$

$$\boxed{\vec{J}_b = 3ks\,\hat{z}}$$

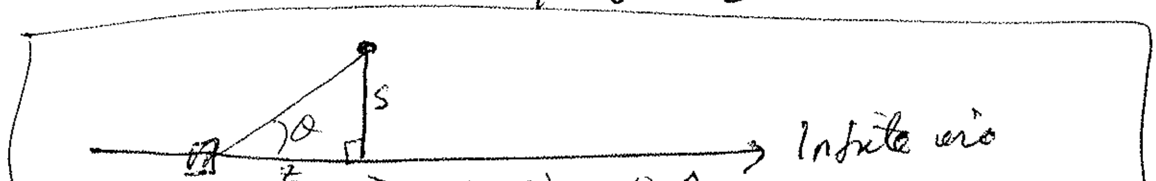
$$\boxed{\vec{K}_b = kR^2\hat{\phi}\times\hat{s} = -kR^2\hat{z}}$$

So, the problem is

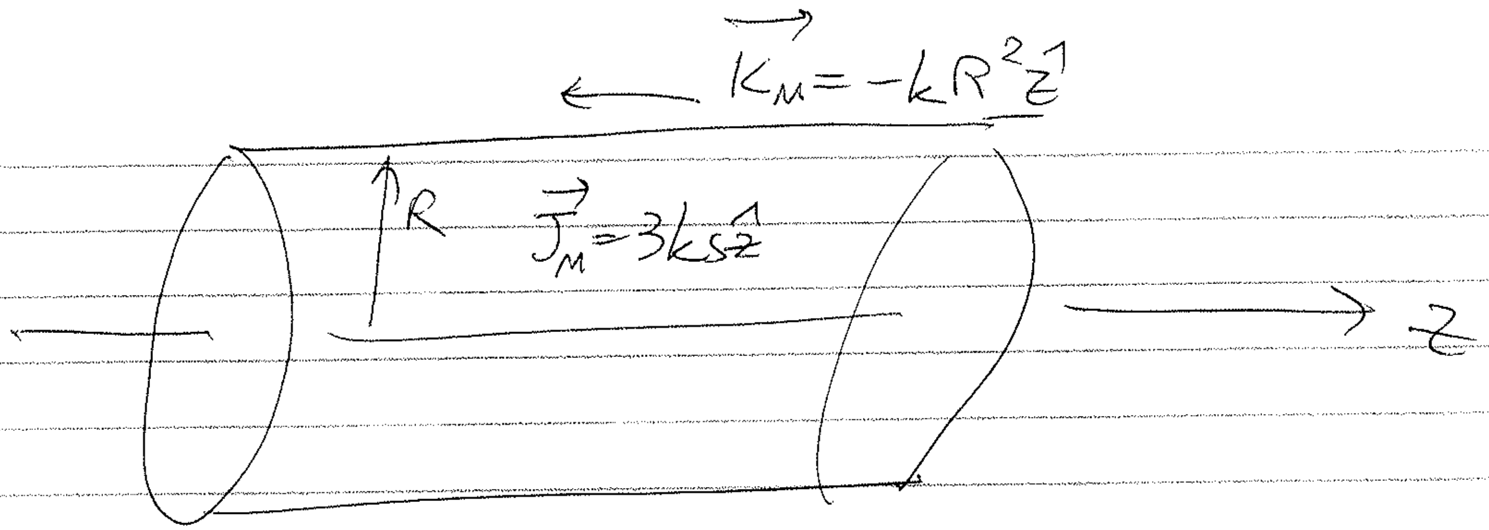


Because of symmetry, use Ampere's law,

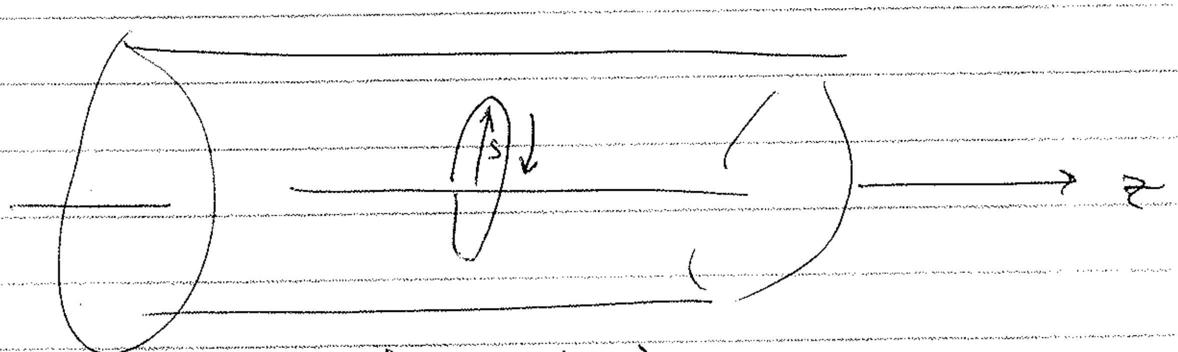
$$\oint \vec{B}\cdot d\vec{l} = \mu_0\int \vec{J}_b\cdot d\vec{S}$$



$$Idz \longrightarrow \vec{dB} = \frac{\mu_0}{4\pi}\frac{Idz\sin\theta}{(z^2+s^2)}\hat{\phi}$$



Use Ampere's law



a) $\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}_M \cdot d\vec{S} \Rightarrow B_\phi 2\pi s = \mu_0 3k \int s s\, d\phi\, ds$

$= \cancel{3}k\mu_0 2\pi \left(\frac{s^3}{\cancel{3}}\right)$

$$\Rightarrow \boxed{\vec{B}_\phi = \cancel{2\pi} k\mu_0 s^2 \hat{\phi}, \quad s<R}$$

b) $B_\phi 2\pi s = \overbrace{\mu_0 3k\, 2\pi \left(\frac{R^3}{3}\right)}^{\text{volume}}$, $s>R$

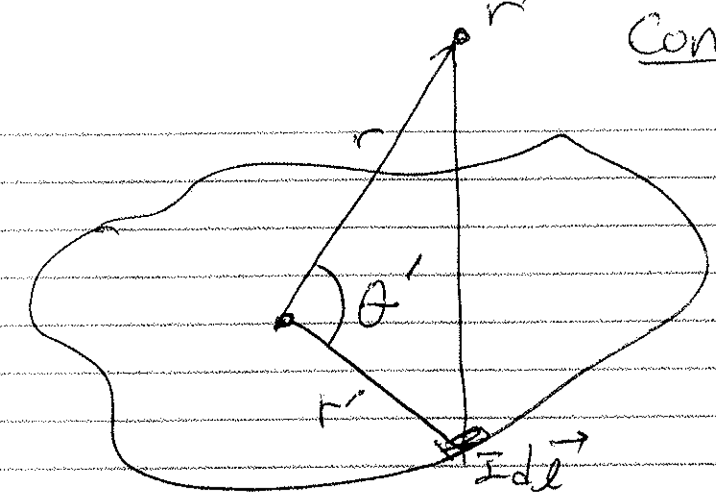
$+\mu_0 \int \vec{K}_m \cdot d\vec{S}$

$= \mu_0 2\pi k R^3 - \overbrace{\mu_0 k R^2 2\pi R}^{\text{surface integral}}$

$= 2\pi\mu_0 k R^3 - 2\pi \mu_0 k R^3 = 0$

$$\Rightarrow \boxed{B_\phi = 0, \quad s>R}$$

Comment:



$\boxed{\text{Fig. S.5}}$

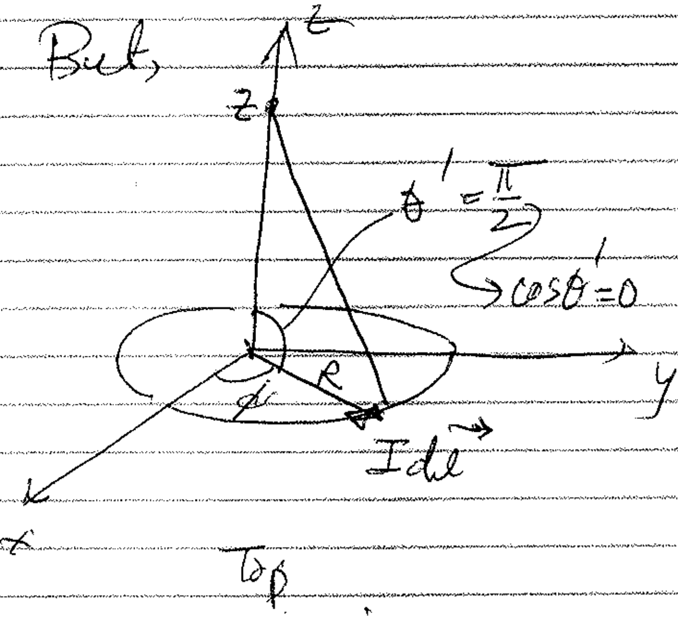
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}}{\mathcal{R}} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

if $r > r'$

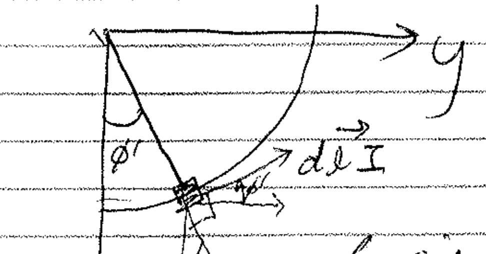
$$= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta')$$

$$= \frac{\mu_0 I}{4\pi r}\int \left(-\sin\phi', \cos\phi', 0\right) r'\,d\phi' \sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(0)$$

But,

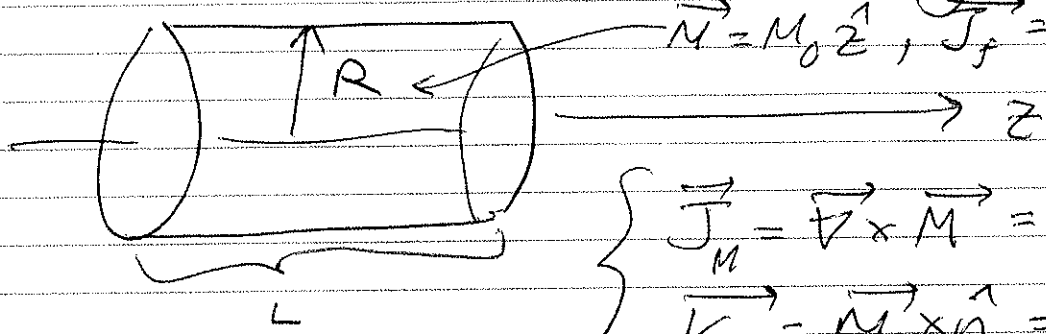


$\Rightarrow \begin{cases} \vec{A} = 0 \text{ on-axis of the} \\ \text{wire loop} \end{cases}$



$\rightarrow \left(I\sin\phi', I\cos\phi', 0\right) = I\,d\vec{l}$

Consider a finite, uniformly magnetized cylinder



$\vec{M} = M_0 \hat{z}$, $\vec{J}_f = 0$

$$\begin{cases} \vec{J}_M = \vec{\nabla} \times \vec{M} = 0 \\ \vec{K}_M = \vec{M} \times \hat{n} = M_0 \hat{\phi} \end{cases}$$

On-axis, the field is

$$\vec{B}_z = \frac{\mu_0 N I}{2} \hat{z} \frac{L}{\sqrt{R^2 + L^2/4}} \text{ at the midpoint}$$

away from this point, things get dicey. Are there things we can do to help us out here?

Define Auxiliary field $\vec{H}$. (as we did in electrostatics

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_f + \rho_b}{\epsilon_0} = \frac{\rho_f}{\epsilon_0} - \frac{\vec{\nabla} \cdot \vec{P}}{\epsilon_0}$$

$$\Longrightarrow \vec{\nabla} \cdot \left[\epsilon_0 \vec{E} + \vec{P}\right] = \rho_f$$

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f} \quad)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \mu_0 (\vec{J}_f + \vec{J}_M)$$

$$= \mu_0 (\vec{J}_f + \vec{\nabla} \times \vec{M})$$

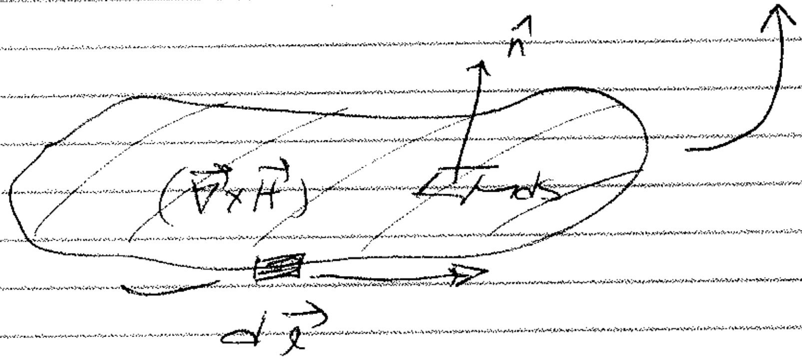
$$\Longrightarrow \vec{\nabla} \times \underbrace{\left[\frac{\vec{B}}{\mu_0} - \vec{M}\right]}_{\vec{H}} = \vec{J}_f$$

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f}$$

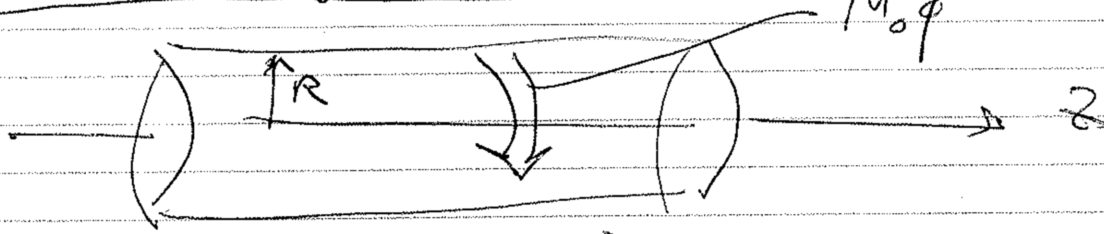
from $\vec{\nabla} \times \vec{H} = \vec{J}_f$, we form

$$\int (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \int \vec{J}_f \cdot d\vec{S}$$

$$\boxed{\oint \vec{H} \cdot d\vec{l} = \int \vec{J}_f \cdot d\vec{S}}$$

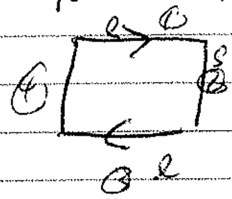


To get a feel for what we are doing, let $L \to \infty$ and deal w/ an infinite solenoid



(I) $\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}_M \cdot d\vec{S}$

outside solenoid



$$\Rightarrow \int_{0\,①}^{l} B_z^1\, dz + \int_{l\,③}^{0} B_z^2\, dz = 0$$

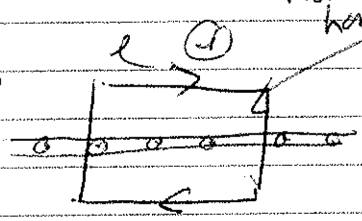
$$\Rightarrow B_z^1 = B_z^2$$

from right hand rule

if $B_z \to 0$ at ∞

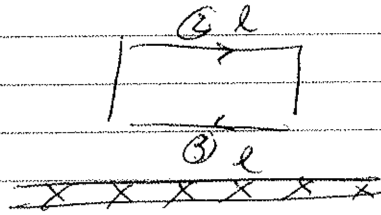
$$\Rightarrow B = 0 \text{ outside}$$

inside solenoid



$$\Rightarrow B_z^2(-l) = -\mu_0 M_0 l$$

$$B_z = \mu_0 M_0$$



$$\Rightarrow B_z^2 - B_z^1 = 0$$

$$\Rightarrow B_z^2 = B_z^1$$

$$\Rightarrow B_z = \mu_0 M_0 \text{ everywhere}$$

For infinite Solenoid

a) $\vec{B} = \mu_0 M_0 \hat{z}$, $\vec{M} = M_0 \hat{z}$

and $\vec{B} \parallel \vec{M}$ in solenoid

b) $\dfrac{\vec{B}}{\mu_0} - \vec{M} = \vec{H} \Rightarrow \dfrac{\mu_0 M_0 \hat{z}}{\mu_0} - M_0\hat{z} = 0 = \vec{H}$

$\vec{B}$ is not parallel to $\vec{H}$

II) Could we have found $\vec{B}$ more easily?

$$\oint \vec{H}\cdot d\vec{l} = \int \vec{J}_f \cdot d\vec{S}$$

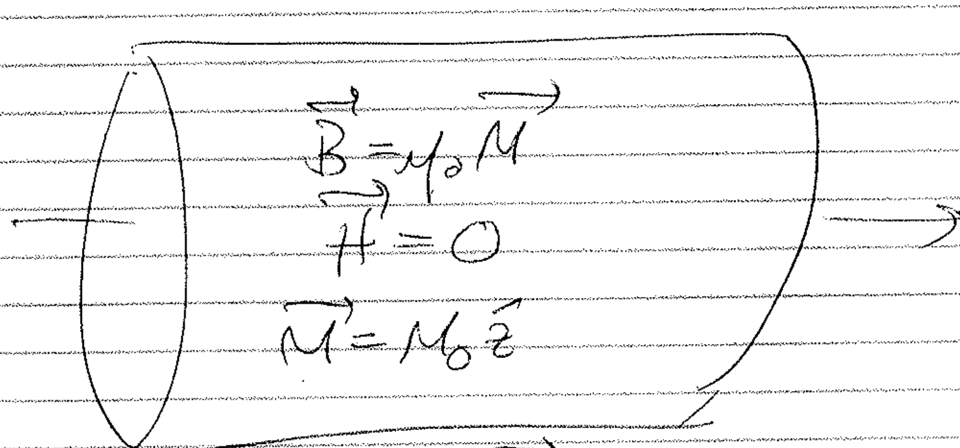
a) $\vec{J}_f = 0 \Rightarrow \int \vec{H}\cdot d\vec{l} = 0 \Rightarrow \vec{H}$ is uniform everywhere.

If $\vec{B} \to 0$ at $\infty \Rightarrow \vec{H} = 0$

since $\vec{H} = \dfrac{\vec{B}}{\mu_0}$ outside the soloniod

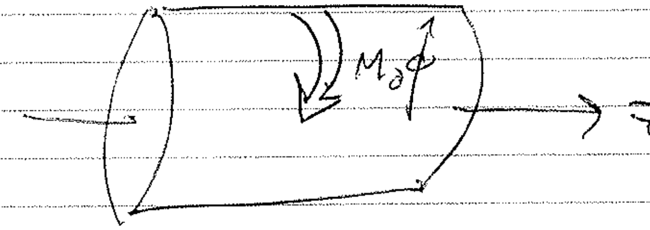
b) In Soloniod,

$$\vec{H} = 0 = \frac{\vec{B}}{\mu_0} - \vec{M} \Longrightarrow \boxed{\vec{B} = \mu_0 \vec{M}}$$



$\vec{B} = 0,\ \vec{H} = 0,\ \vec{M} = 0$

Return to finite cylinder



$\Rightarrow B_z^L(z=0, s=0) = \dfrac{\mu_0 M_0 L}{2\sqrt{R^2+L^2/4}}$

(I) $$\frac{B_z^L(z=0, s=0)}{B_z^\infty} = \frac{\dfrac{\mu_0 M_0 L}{2\sqrt{R^2+L^2/4}}}{\mu_0 M_0} = \frac{L}{2\sqrt{R^2+L^2/4}} < 1$$

$\Rightarrow$ B_z^L weaker than for ∞ solenoid

Hmm, means that

a) $H_z^L = \dfrac{B_z^L}{\mu_0} - M = \dfrac{B_z^L}{\mu_0} - M_0 < H_z^\infty (=0)$

$\Rightarrow H_z^L < 0 \Rightarrow H_z^L$ is in opposite direction to B_z^L.

b) Outside of cylinder $\vec{M} = 0$

$\Rightarrow \vec{H} = \dfrac{1}{\mu_0}\vec{B} \Rightarrow \vec{B}$ and $\vec{H}$ are parallel

(II) Can we get a more quantitative handle on this answer?

yes, $\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J}_f + \vec{J}_M)$

$$\longrightarrow \underline{\vec{\nabla} \times \vec{H} = \vec{J}_f}$$

now, $\vec{J}_f = 0 \Longrightarrow \vec{\nabla} \times \vec{H} = 0 \Longrightarrow \boxed{\vec{H} = -\vec{\nabla} V_M}$

<u>oh good, is this useful?</u> Consider

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \Big[\underbrace{\mu_0 (\vec{H} + \vec{M})}_{\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}} \Big] \Rightarrow \boxed{\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}}$$

Define $\rho_M = -\vec{\nabla} \cdot \vec{M} \Longrightarrow \vec{\nabla} \cdot \vec{H} = \rho_M$

and

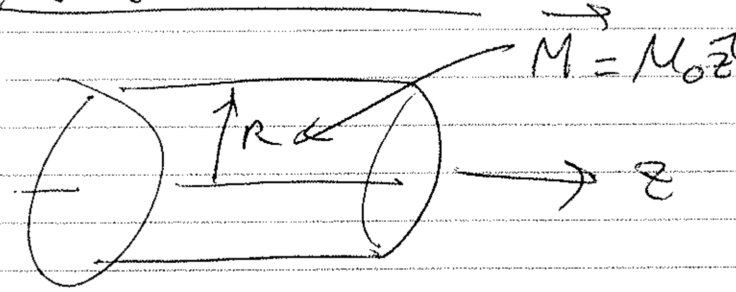
$$\rightarrow \vec{\nabla} \cdot \left[-\vec{\nabla} V_M \right] = \rho_M \Longrightarrow \boxed{\nabla^2 V_M = -\rho_M}$$

and

$$V_M = \frac{1}{4\pi} \int \frac{\rho_M \, d^3x}{r} + \frac{1}{4\pi} \int \frac{\sigma_M \, dS}{r} \qquad \sigma_M = \vec{M} \cdot \hat{n}$$

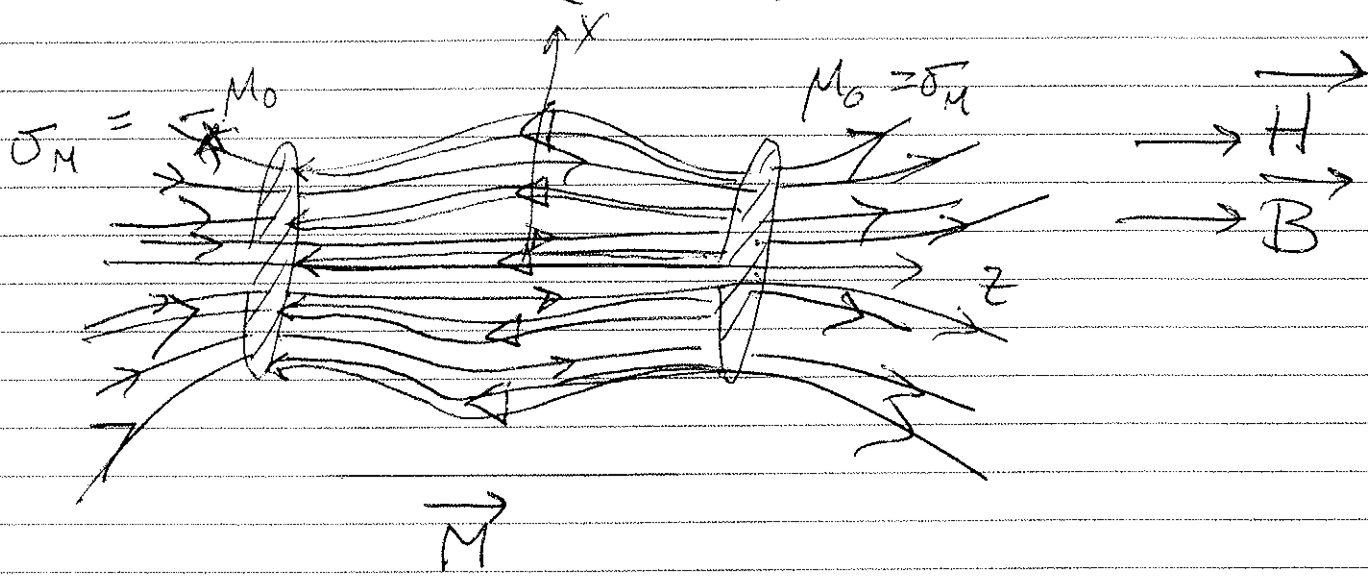
Now, the machinery developed for V in electrostatics can be brought to bear

So, for finite solenoid



$\rho_M = -\vec{\nabla}\cdot\vec{M} = 0$, no "magnetic charges"

$$\sigma_M = \vec{M}\cdot\hat{n} = \begin{cases} M_0 & , z = L/2 \\ -M_0 & , z = -L/2 \end{cases}$$



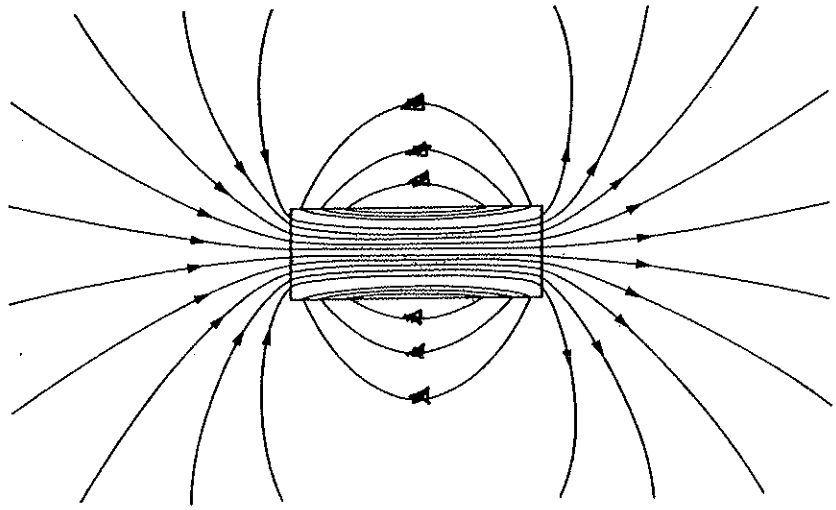
for $\vec{J}_f = 0 \Rightarrow$ a) $\vec{\nabla}\times\vec{H} = 0 \Rightarrow \boxed{-\vec{\nabla}V_M = \vec{H}\ !}$

b) $\vec{\nabla}\cdot\vec{B} = 0 = \vec{\nabla}\cdot\left[\mu_0(\vec{H}+\vec{M})\right]$

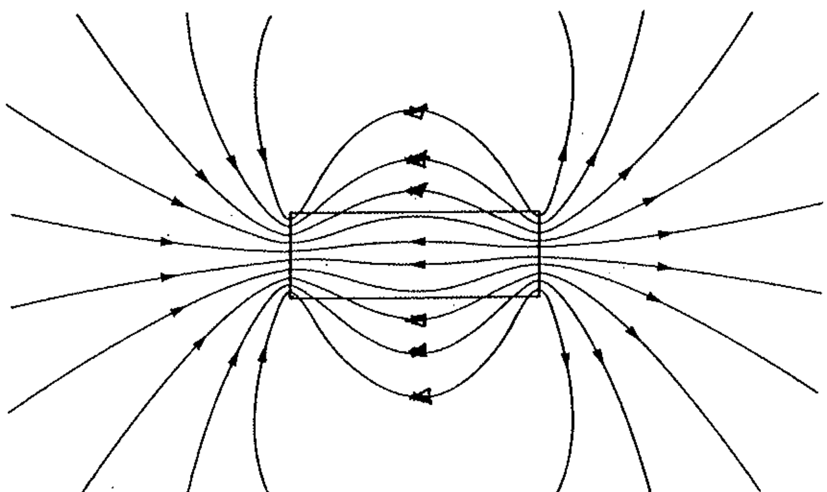
$\Rightarrow \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} = \rho_M$

and so,

$$\boxed{\nabla^2 V_M = -\rho_M}$$



B field



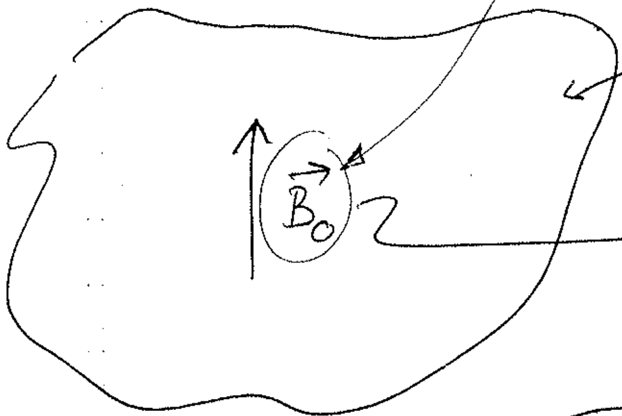
H field

FIGURE 7-13 Fields **B** and **H** for a rod with constant magnetization along its axis

Prob 6.13

A large piece of magnetic material has field $\vec{B}_0$ in its interior

Magnetic material

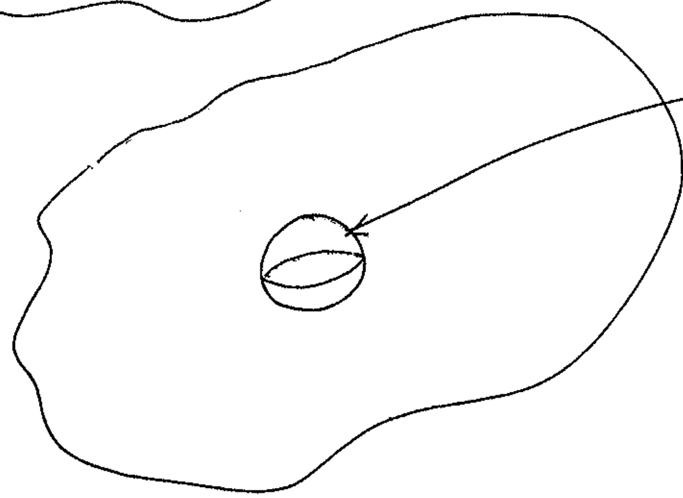


$$\vec{H}_0 = \frac{\vec{B}_0}{\mu_0} - \vec{M}$$

from $\vec{B}$ we know that we have

Hollowed a spherical cavity

a)



Q: what is the field at the center of the cavity in terms of $\vec{B}_0$ & $\vec{M}$?

Q: Find $\vec{H}$ at the center of the cavity in terms of $\vec{H}_0$ and $\vec{M}$.

Sol^n

(a) Suppose $\vec{J}_f = 0 \rightarrow \vec{\nabla} \times \vec{H} = 0$

Consequently,

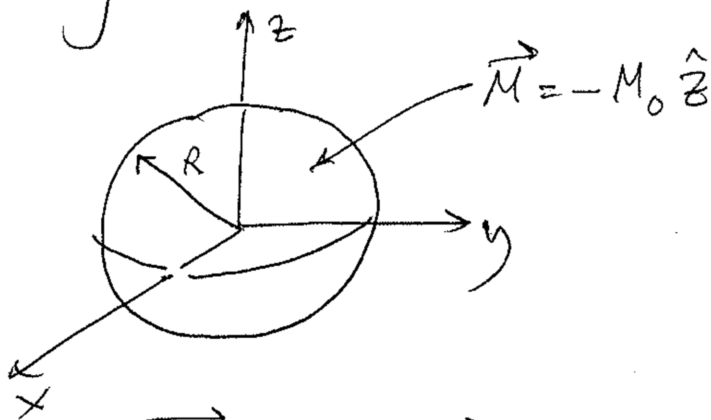
$$\vec{H} = -\vec{\nabla} V_M \quad \& \quad \rho_M = -\vec{\nabla}\cdot\vec{M}, \ \sigma_M = \vec{M}\cdot\hat{n}$$

Can go this route, but let's try another tactic.

Solve by Superposition

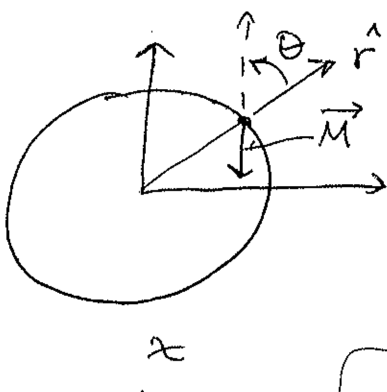
(iii) Let the cavity have $-\vec{M}_0$ ($\Longleftrightarrow \vec{M} = 0$ in the cavity as desired)

<u>Q</u>: what is field of a spherical magnetized object?



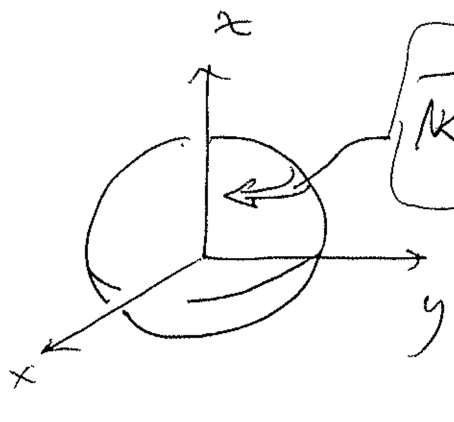
$$\longrightarrow \vec{J}_M = \vec{\nabla} \times \vec{M} = 0$$

$$\vec{K}_M = \ ? \ = \vec{M} \times \hat{r}$$



$$\Longrightarrow \vec{K}_M = -M_0 \sin(\pi - \theta)\,\hat{\phi}$$
$$= -M_0 \sin\theta\,\hat{\phi}$$

and



$$\boxed{\vec{K}_M = -M_0 \sin\theta\,\hat{\phi}}$$

The problem reduces to finding the field of a spherical shell of charge which rotates about the z-axis (Example 5.11).

If we let $\vec{K} = \sigma\omega R \sin\theta\,\hat{\phi} \longrightarrow \vec{K}_M = -M_0 \sin\theta\,\hat{\phi}$
then accord'g to (5.68)

$$\vec{B} = \frac{2}{3}\mu_0 \sigma R\omega\,\hat{z} \quad \text{in the cavity}$$

$$\longrightarrow \vec{B}_M = -\frac{2}{3}\mu_0 M_0\,\hat{z}$$

(1) So, the total field in the cavity is

$$\vec{B}_{Total} = \mu_0\left[\vec{H}_0 + \vec{M}\right] + \left[-\frac{2}{3}\mu_0 \underbrace{M_0\,\hat{z}}_{\vec{M}}\right]$$

$$\boxed{\vec{B}_{Total} = \mu_0\vec{H}_0 + \frac{1}{3}\mu_0\vec{M}}$$

oh, Problem asks for $\vec{B}_{Total}$ in terms of $\vec{B}_0$ & $\vec{M}$

$$\rightarrow \vec{B}_{Total} = \mu_0\left[\frac{\vec{B}_0}{\mu_0} - \vec{M}\right] + \frac{1}{3}\mu_0\vec{M}$$

$$= \vec{B}_0 - \frac{2}{3}\mu_0\vec{M}$$

(2) So, the $\vec{H}$ in the cavity is

$$\vec{H} = \left(\frac{\vec{B}_0}{\mu_0} - \vec{M}\right) + \left(\underbrace{\frac{-\frac{2}{3}\mu_0\vec{M}}{\mu_0}}_{\vec{B}/\mu_0} - \underbrace{(\vec{M})}_{\vec{M}_{cavity}}\right)$$

Magnetic Susceptibility χ_m

Linear Materials and Nonlinear Materials

We write, $\vec{M}$ $\left(\equiv \frac{\text{dipole moment}}{\text{volume}}\right)$, as

$$\vec{M} = \chi_m \vec{H}$$

magnetic susceptibility (scalar)
(see Table 6.1)

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.6×10^{-4}	Oxygen	1.9×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.1×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.8×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.8×10^{-4}
Carbon Dioxide	-1.2×10^{-8}	Liquid Oxygen (−200° C)	3.9×10^{-3}
Hydrogen	-2.2×10^{-9}	Gadolinium	4.8×10^{-1}

Table 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, 20° C). *Source: Handbook of Chemistry and Physics,* 67th ed. (Boca Raton: CRC Press, Inc., 1986).

for linear materials (recall: $\vec{P} = \epsilon_0 \chi_e \vec{E}$).

For linear materials,

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu_0} - \chi_m \vec{H}$$

and

$$\vec{H} = \frac{1}{\mu_0}\vec{B}\Big/(1+\chi_m)$$

$$\Rightarrow \vec{B} = \underbrace{\mu_0(1+\chi_m)}\vec{H}$$

$$= \mu \vec{H}$$

permeability

note: $\mu_0 \approx \mu$

for Linear Materials

Comment:

Recall: ① $\rho_p = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\epsilon_0\chi_e\vec{E} = -\epsilon_0\chi_e\,\vec{\nabla}\cdot\vec{D}/\epsilon$

$= -\dfrac{\epsilon_0\chi_e}{\epsilon}\rho_f$

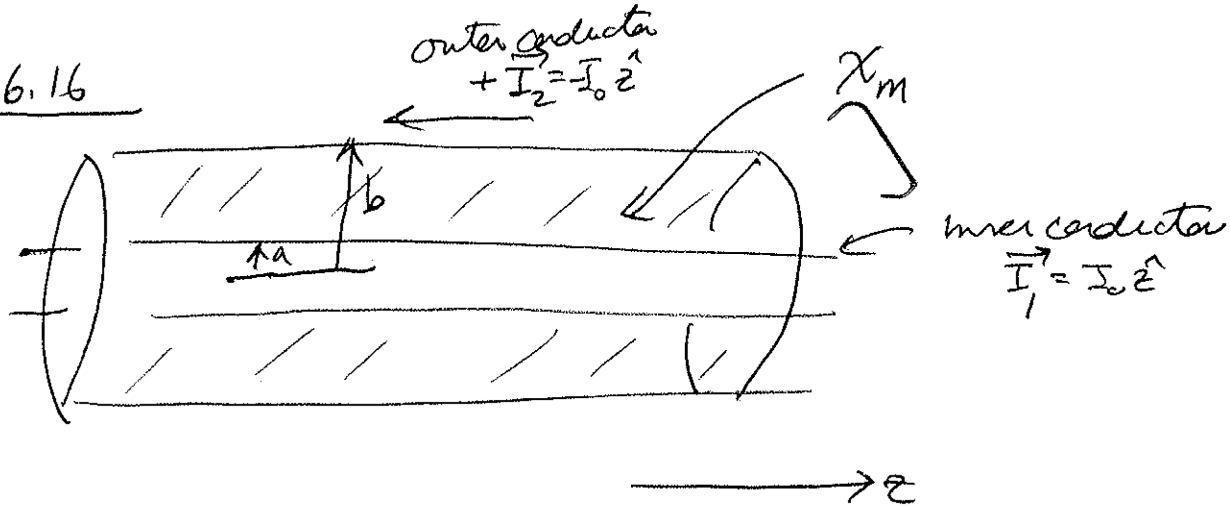
for a LIH dielectric

② $\vec{J}_M = \vec{\nabla}\times\vec{M} = \vec{\nabla}\times\chi_m\vec{H} = \chi_m(\vec{\nabla}\times\vec{H})$

$= \chi_m\vec{J}_f$

$\Rightarrow$ if $\vec{J}_f = 0 \Rightarrow \vec{J}_M = 0$ for an LIH magnetic material.

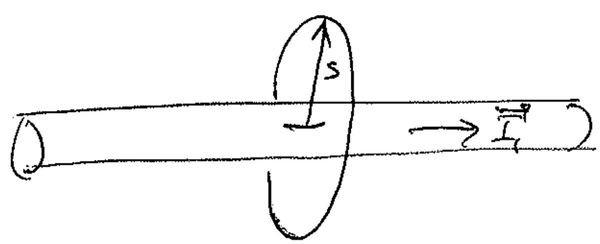
Prob 6.16



a) Determine the magnetic field in the region between the conductors. We have free $I_0\hat z$ on the inner cylinder; $-I_0\hat z$ on outer cylinder

Because of symmetry, we expect that we can use

$$\oint \vec H \cdot d\vec l = \int \vec J_f \cdot d\vec S$$



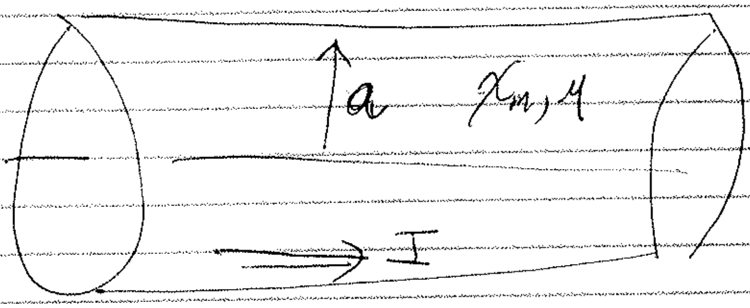
a) $$\oint \vec H \cdot d\vec l = H_\phi 2\pi s = I_0 \Rightarrow H_\phi = \frac{I_0}{2\pi s}, \quad \begin{cases} s < b \\ s > a \end{cases}$$

and

$$H_\phi 2\pi s = 0 \Rightarrow H_\phi = 0, \quad s > b$$

$$\vec H = \frac{I_0}{2\pi} \begin{cases} 0 & s < a \\ \hat\phi / s & a < s < b \\ 0 & b < s \end{cases}$$

Prob 6.17



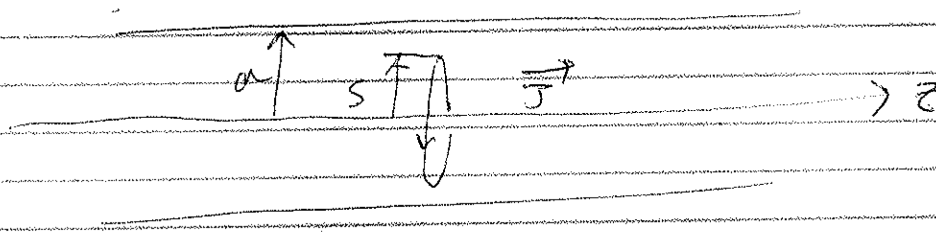
a current I flows down a long straight wire of radius a.

a) The material has susceptibility χ_m (μ). What is $\vec{B}$ at distance s from axis?

b) Find $\vec{J}_M$.

Soln

(a) Infinite wire w/ $\vec{J} = \frac{I\hat{z}}{\pi a^2}$ = uniform. Use symmetry arguments to argue that $\vec{B}$ is azimuthal (ϕ) and apply Ampere's law for $\vec{H}$ ($\nabla \times \vec{H} = \vec{J}_f \rightarrow \int \vec{H}\cdot d\vec{l} = \int \vec{J}_f \cdot d\vec{S}$)



(i) $\oint \vec{H}\cdot d\vec{l} = \int \vec{J}_f \cdot d\vec{S} \Rightarrow H_\phi 2\pi s = J_f \pi s^2$ $\quad s<a$

$$\boxed{\vec{H}_\phi = \frac{s}{2} J_f \hat{\phi}}$$

(ii) $\Rightarrow H_\phi 2\pi s = J_f \pi a^2$ $\quad s>a$

$$\boxed{\vec{H}_\phi = \frac{1}{2}\frac{a^2}{s} J_f \hat{\phi}}$$

(b) $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu_0} - \chi_m \vec{H} \Rightarrow \boxed{\vec{B} = \underbrace{\mu_0 (1+\chi_m)\vec{H}}_{\mu\vec{H}}}$

$$\vec{B}_\phi = \mu_0 \begin{cases} (1+\chi_m)\frac{s}{2} J_f \hat{\phi} & s<a \\ \frac{a^2}{2s} J_f \hat{\phi} & s>a \end{cases}$$

(c) $\vec{M} = \chi_M \vec{H}$

$$= \begin{cases} \frac{\chi_m J_f}{2} s \hat{\phi} & s < a \\ 0 & s > a \end{cases}$$

(d) $\vec{J}_M = \vec{\nabla} \times \vec{M} = \left(\frac{1}{s} \frac{\partial M_z}{\partial \phi} - \frac{\partial M_\phi}{\partial z} \right) \hat{s}$

$$+ \left(\frac{\partial M_s}{\partial z} - \frac{\partial M_z}{\partial s} \right) \hat{\phi} + \frac{1}{s} \left(\frac{\partial}{\partial s} \left[s M_\phi \right] - \frac{\partial M_s}{\partial \phi} \right) \hat{z}$$

$$= 0 \hat{s} + 0 \hat{\phi} + \chi_m J_f \hat{z}$$

(e) $\vec{K}_M = \vec{M} \times \hat{s} = M_a \hat{\phi} \quad w/ \ s = a$

$$= \chi_m \frac{a}{2} J_f (-\hat{z})$$

(f) Total bound current:

(i) $|\vec{J}_f| = \frac{I}{\pi a^2} \Rightarrow |\vec{J}_M| = \underbrace{\chi_m \frac{I}{\pi a^2}}_{\text{in } +z \text{ direction}} \Rightarrow I_M = \chi_m I$

(ii) $\Rightarrow |\vec{K}_M| = \underbrace{\chi_m \frac{a}{2} \frac{I}{\pi a^2}}_{\text{in } -z \text{ direction}} \Rightarrow I_M = \chi_m I$

$\Rightarrow$ Total Bound Current is 0

Magnetic Scalar Potential

a) $\vec{\nabla} \times \vec{H} = \vec{J}_f \Rightarrow \vec{H} = -\vec{\nabla} V_m$ if $\vec{J}_f = 0$

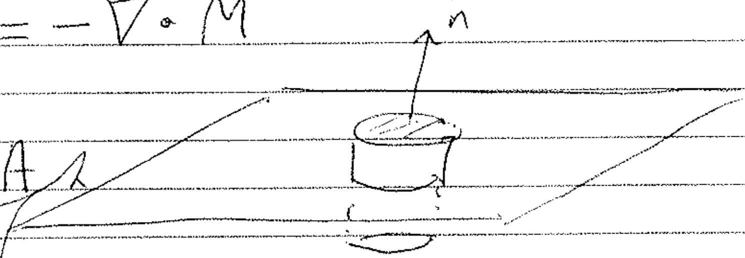
b) $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left[\mu_0 \vec{H} + \mu_0 \vec{M} \right] = 0 \Rightarrow \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} = \rho_M$

a) & b) $\Rightarrow \vec{\nabla} \cdot \vec{H} = \vec{\nabla} \cdot \left(-\vec{\nabla} V_M \right) = \rho_M$

$$\Rightarrow \boxed{\nabla^2 V_M = -\rho_M}$$

Boundary Conditions

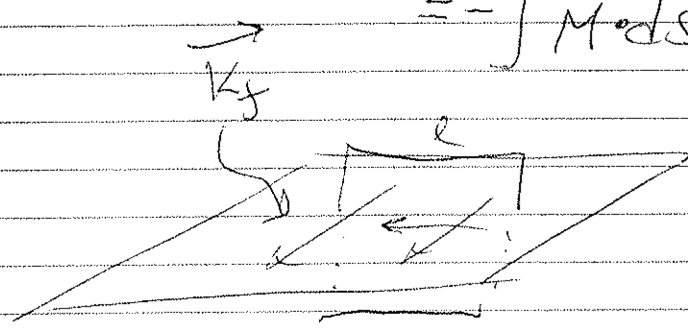
a) $\vec{\nabla} \cdot \vec{H} = \rho_M = -\vec{\nabla} \cdot \vec{M}$



$H_\perp^+ A - H_\perp^- A = -M_\perp^+ A + M_\perp^- A$

$$\int (\vec{\nabla} \cdot \vec{H})\, d^3x = \int \vec{H} \cdot d\vec{S} = \int \rho_M\, d^3x = -\int \vec{M} \cdot d\vec{S}$$

$$\Rightarrow \boxed{\Delta H_\perp = -\Delta M_\perp}$$



b) $\vec{\nabla} \times \vec{H} = 0 = \vec{J}_f$

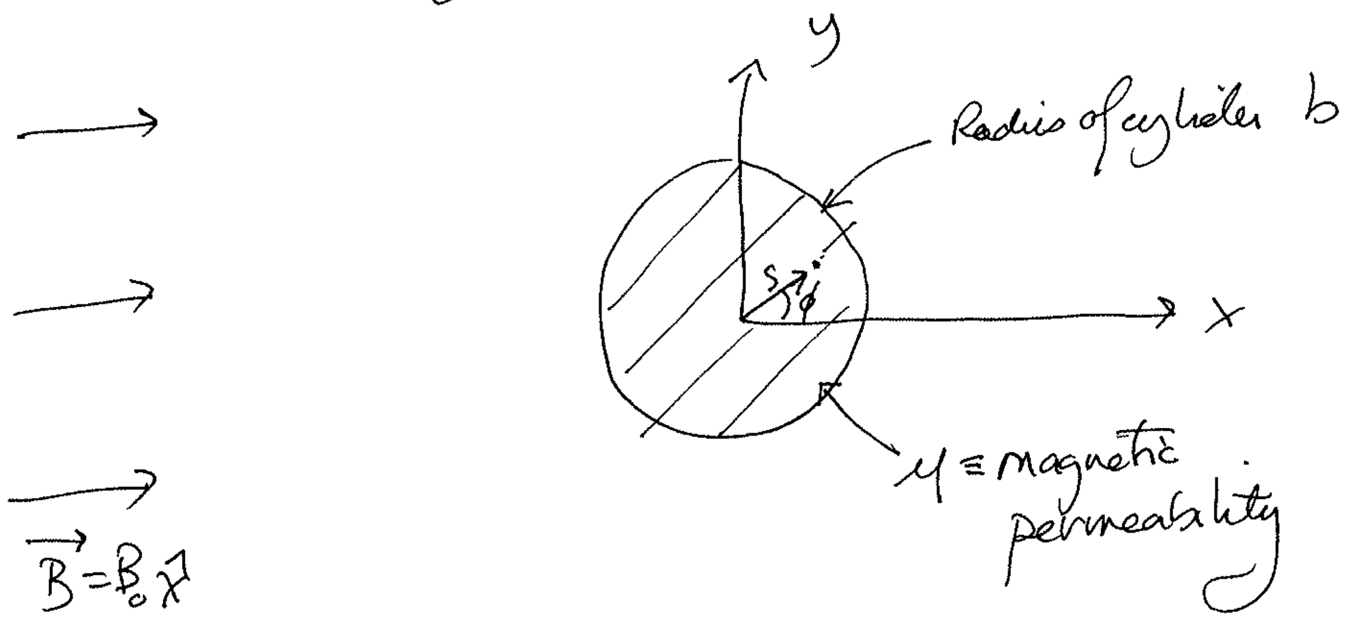
$$\int (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \int \vec{J}_f \cdot d\vec{S}$$

$$\int \vec{H} \cdot d\vec{l} = \int \vec{K}_f \cdot d\vec{l} = H_\parallel^+ (-l) + H_\parallel^- l = K_f (-l$$

$$\boxed{\begin{array}{l} \Delta H_\parallel = K_f \\ \Delta \vec{H}_\parallel = \vec{K}_f \times \hat{n} \end{array}}$$

Consider an infinite magnetizable cylinder placed into an otherwise uniform $\vec{B}$-field.

Find $\vec{B}$ everywhere



No free $\vec{J} \Rightarrow \vec{J}_M = 0$ and only bound current will be found at surface of the cylinder.

Solve this problem as follows:

Magnetic Scalar Potential

a) $\vec{\nabla} \times \vec{H} = \vec{J}_f$; $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left[\mu_0 \vec{H} + \mu_0 \vec{M} \right] = 0$

$$\Longrightarrow \vec{\nabla} \cdot \vec{H} = -\underbrace{\vec{\nabla} \cdot \vec{M}}_{\text{"magnetic" charge, } \rho_M}$$

If $\vec{J}_f = 0$,

$\vec{\nabla} \times \vec{H} = 0 \Longrightarrow \vec{H}$ can be expressed as the $-\vec{\nabla} V_M \left(= -\vec{\nabla} W\right.$, Prob 6.15$\left.\right)$

So,

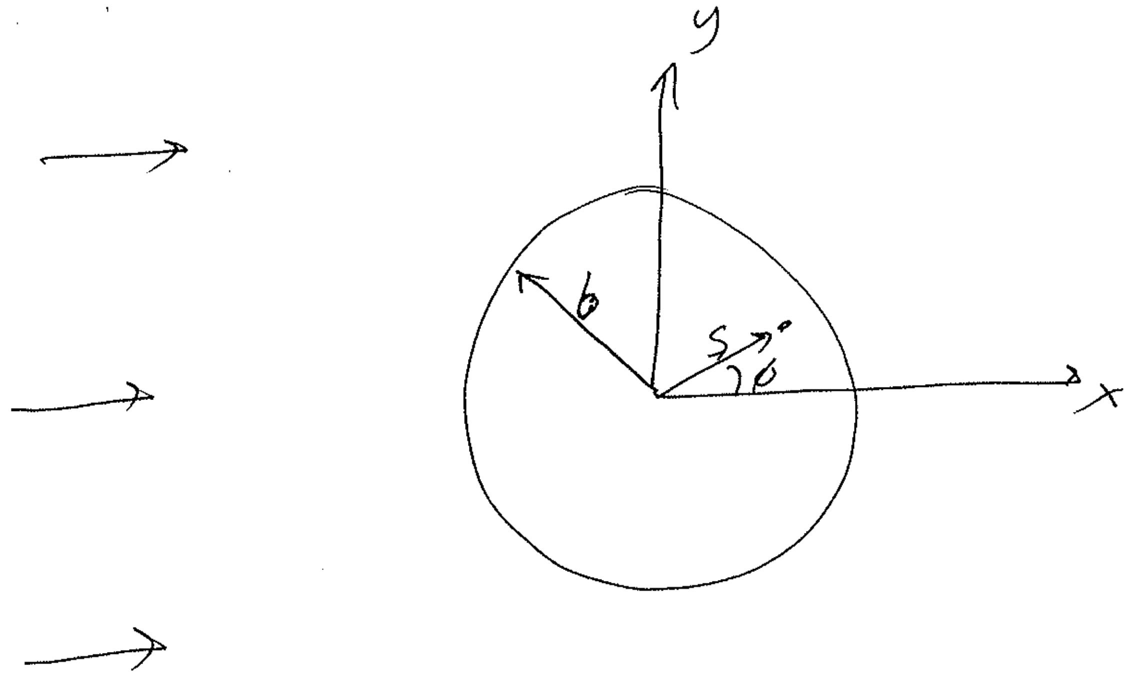
$$\vec{\nabla} \cdot \vec{H} = \vec{\nabla} \cdot \left(-\vec{\nabla} V_M\right) = -\vec{\nabla} \cdot \vec{M} = +\rho_M$$

$$\boxed{\nabla^2 V_M = -\rho_M}$$

Oh, we now see that we use the machinery developed for the scalar potential, V! If $\rho_M = 0$

$$\Longrightarrow \nabla^2 V_M = 0$$

and we can use our machinery for Laplace's Eqn in regions where $\rho_M = 0$.



$\vec{B} = B_0 \hat{x}$

Boundary Conditions

a) at ∞, $\vec{B}$ is uniform,

$\vec{B} = B_0 \hat{x}$

in vacuum

Because $\vec{B} = \mu \vec{H} = \mu_0 \vec{H}$ ad $\vec{H} = -\vec{\nabla} V_M$

$$\Longrightarrow V_M = -\frac{B_0}{\mu_0} x = -\frac{B_0}{\mu_0}(s\cos\phi)$$

at infinity

b) Now, consider conditions on the surface of the cylinder.

(ii) We already know that

$\vec{B} = \mu \vec{H}$

$$\Delta B_n = 0 \text{ ; no magnetic monopoles.}$$

$$\Rightarrow -\mu_0 \frac{\partial V_M^>}{\partial s} + \mu \frac{\partial V_M^<}{\partial s} = 0 \Rightarrow \boxed{\frac{\partial V_M^>}{\partial s} = \frac{\mu}{\mu_0} \frac{\partial V_M^<}{\partial s}}$$

(iii) Now, because $\vec{\nabla} \times \vec{H} = \vec{J}_f = 0$

$$\Rightarrow \Delta H_T = 0 \text{ , no free currents.}$$

$$-\frac{1}{s}\frac{\partial}{\partial \phi} V_M^> + \frac{1}{s}\frac{\partial}{\partial \phi} V_M^< = 0$$

$$\boxed{\frac{\partial V_M^>}{\partial \phi} = \frac{\partial V_M^<}{\partial \phi}} \Rightarrow V_M^> = V_M^< + \text{consta}$$

(iv) Further, V_M must be continuous at the boundaries.

$$\boxed{V_M^> = V_M^< \text{ at boundaries}}$$

because of (xii) (set constant to 0).

By topology, let's use (cylindrical coord sol$^{\underline{n}}$ to Laplace Eq$^{\underline{n}}$),

$$V_M = A_0 + A_1 \ln s + \sum_{m=1}^{\infty}\left(A_m s^m + \frac{B_m}{s^m}\right)(C_m \cos m\phi + D_m \sin m\phi)$$

first, because

$$V_M(s,\phi) \equiv V_M(s,-\phi)$$

by symmetry, $\Rightarrow D_m = 0$ (sin mφ terms go away)

$$\Rightarrow V_M = A_0 + A_1 \ln s + \sum_{m=1}^{\infty}\left(A_m s^m + \frac{B_m}{s^m}\right)\cos m\phi$$

BC's

(i) at $x \to \pm\infty$, $V_M \to -\frac{B_0}{\mu_0}(s\cos\phi)$

$$\Rightarrow V_M(s>b) = A_1^{>} s\cos\phi + \frac{B_1^{>}}{s}\cos\phi$$

can explicitly show B_m's, A_m's → 0 for $m>1$

$$\boxed{V_M(s>b) = -\frac{B_0}{\mu_0} s\cos\phi + \frac{B_1^{>}}{s}\cos\phi}$$

(ii) at $s=0$, V_M must be finite

$$\Rightarrow \boxed{V_M(s<b) = A_0^{<} + A_1^{<} s\cos\phi}$$ ditto

Nonlinear Material

Ferromagnetism

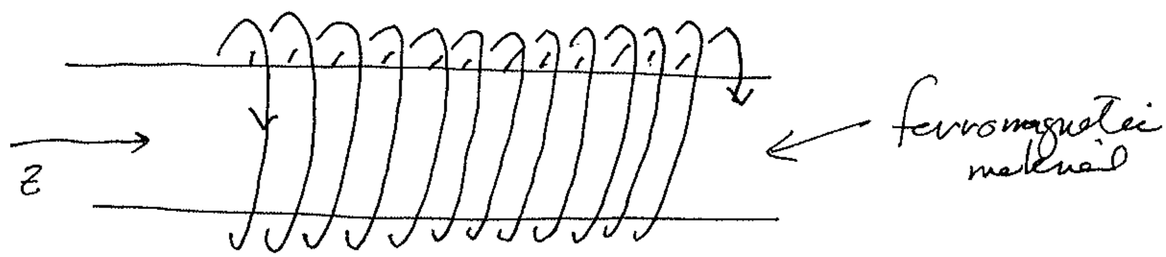
Certain materials (iron, cobalt, nickel) and some rare earths and compounds (CrO_2) exhibit

"Ferromagnetism"

In Ferromagnets,

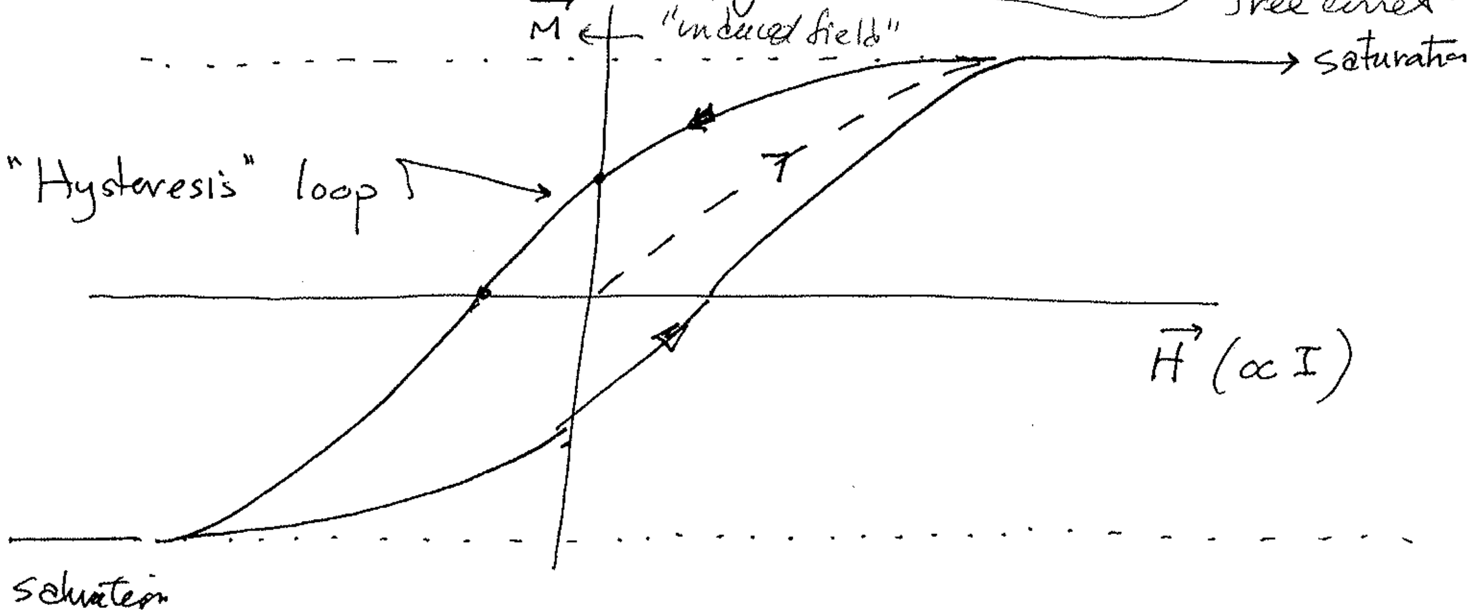
$$\vec{M} = \chi_m \vec{H} \quad (A) \longrightarrow \vec{B} = \mu_0 (1 + \chi_m) \vec{H}$$

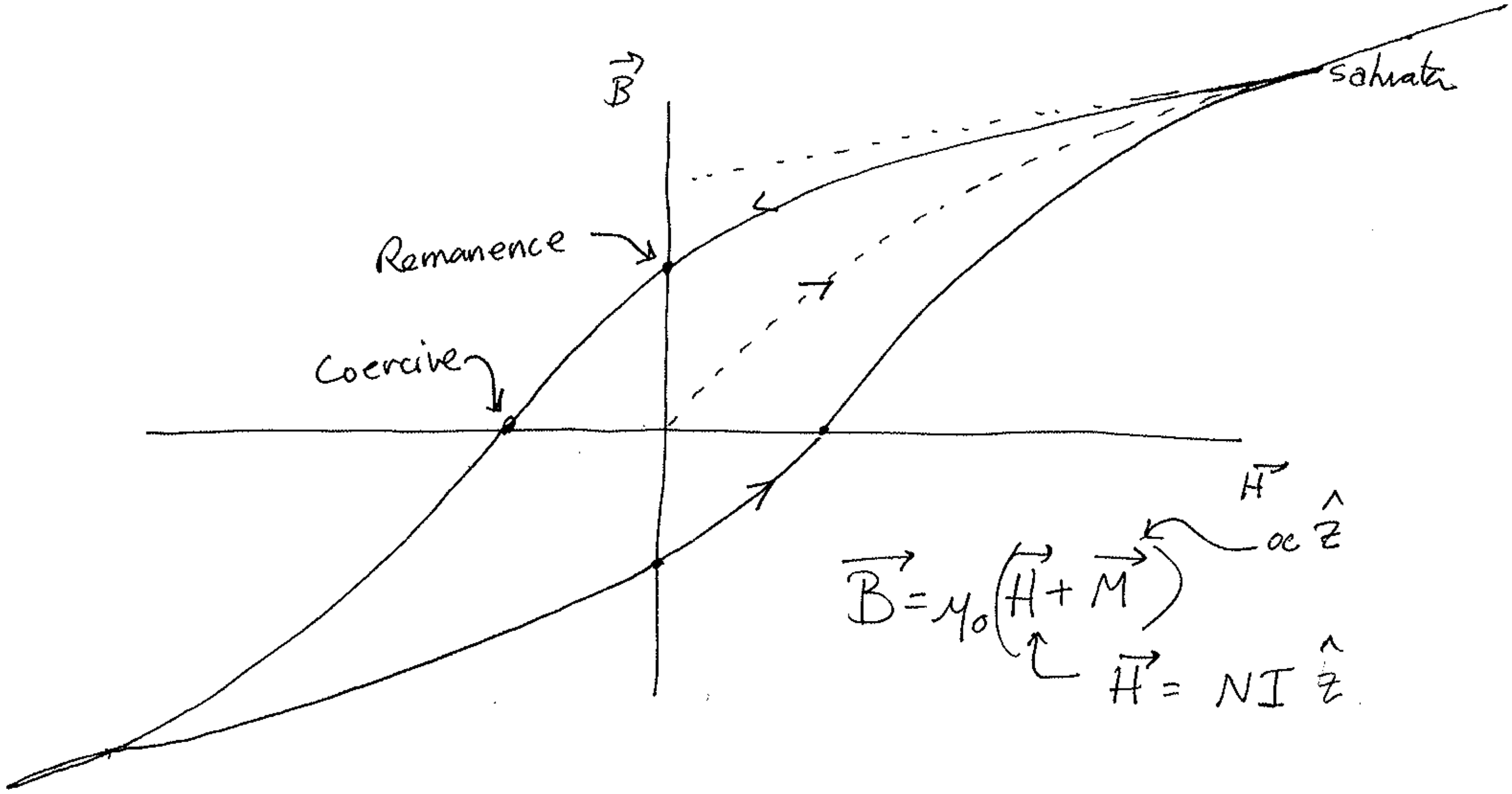
Have $\chi_m(I, \vec{H}, \text{dir}^n, \text{history})$. Can define χ_m as change in $\vec{M}$ for a given change in $\vec{H}$, not as given in (A)!



Wrap a coil around the bar with $K = NI$. From

$\oint \vec{H} \cdot d\vec{l} = \int \vec{J}_f \cdot d\vec{S}$, we get $\boxed{\vec{H}_{coil} = \hat{z} N I}$ field due only to free current





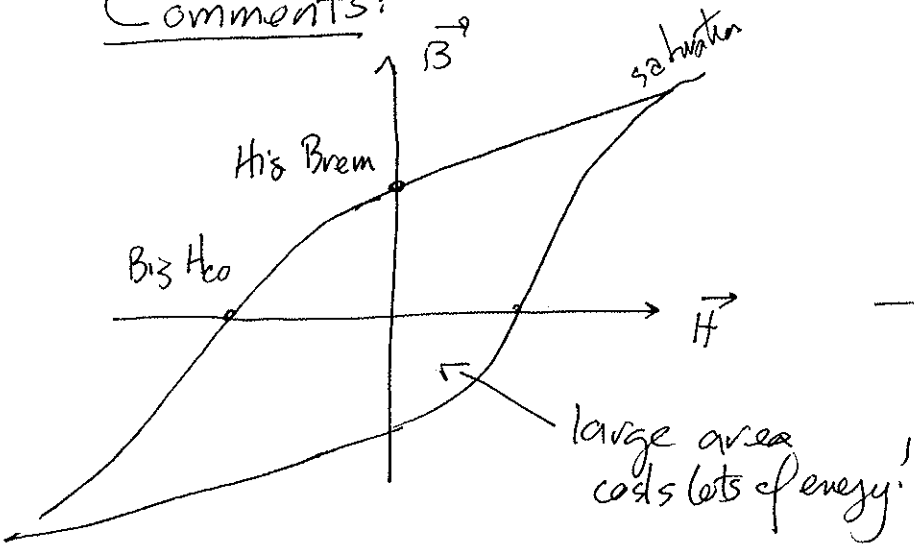
$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

$\vec{M} \propto \hat{z}$

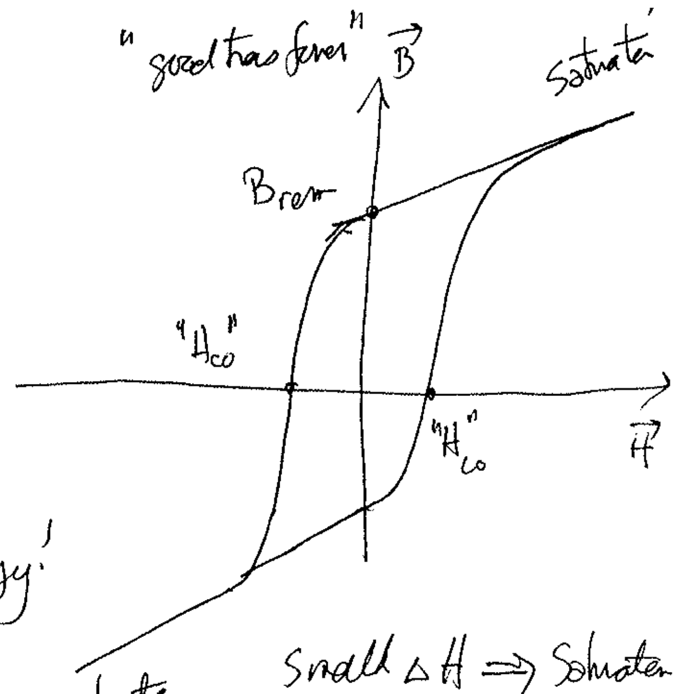
$\vec{H} = NI\,\hat{z}$

Magnetic Materials	Remanence, B(T)	Coercive H($\frac{A}{m}$)
Alnico	1.25	44,000
Silicon Steel	1.20	4
Iron	0.4	50
Supermalloy	0.5	0.32

Comments:



"good permanent magnet"



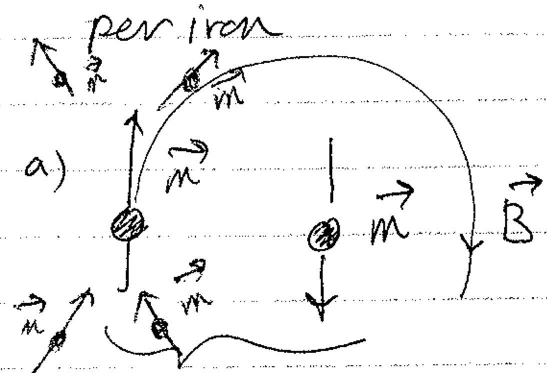
Small $\Delta H \Rightarrow$ Saturation

Ferromagnetism, "Theory"

Consider iron. In Iron, there are 2 unpaired outer e^-'s $\longrightarrow$

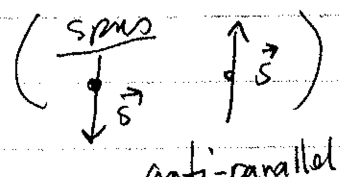
$$\vec{m} = 2\mu_{Bohr}$$

per iron

a) 

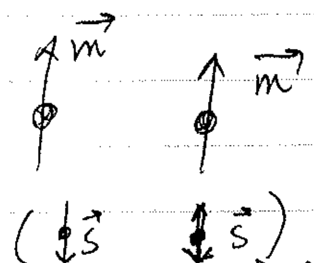
lowest Eng state according to

$$U = -\vec{m}\cdot\vec{B}$$



in "isolated" vapor

b) there is an QM effect which overwhelms this. There is a spin alignment torque which forces



{and all atoms want to line-up (moments)
want to line-up

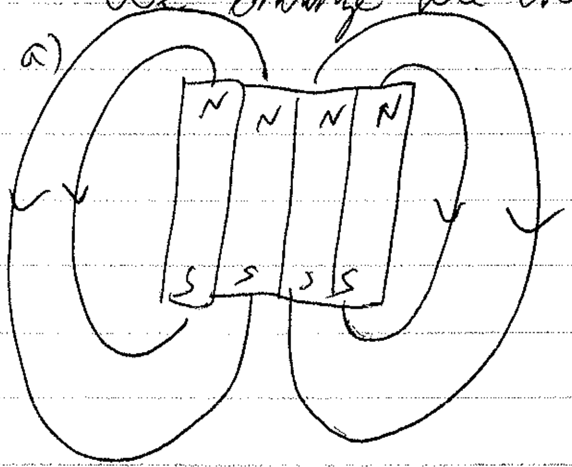
$\Longrightarrow$ why aren't all ferromagnets ~~saturated~~ so that

$$\vec{M} \approx N_{Fe}\,(2\mu_{Bohr})\ ?$$

number density of Iron atoms

We minimize the internal energy but

a)

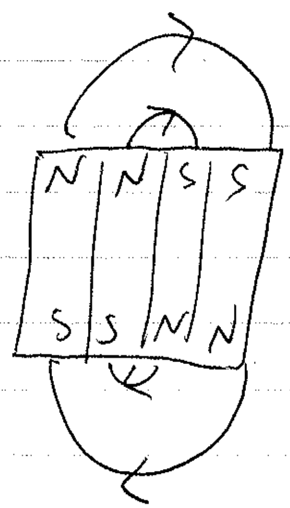


$\Rightarrow$ large external $\vec{B}$

(and since $W_B \approx \frac{B^2}{2}$)

$\rightarrow$ large energy (external)

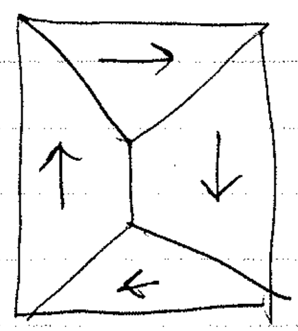
b)



$\Rightarrow$ weaker $\vec{B}$

$\Rightarrow$ less energy (external)

c)



$\Rightarrow$ No external field

$\rightarrow$ low energy

$\Rightarrow$ domains are formed in which $\vec{m}$'s align

$\Rightarrow$ U_B local is minimized

domains arrange themselves to make U_B external minimum

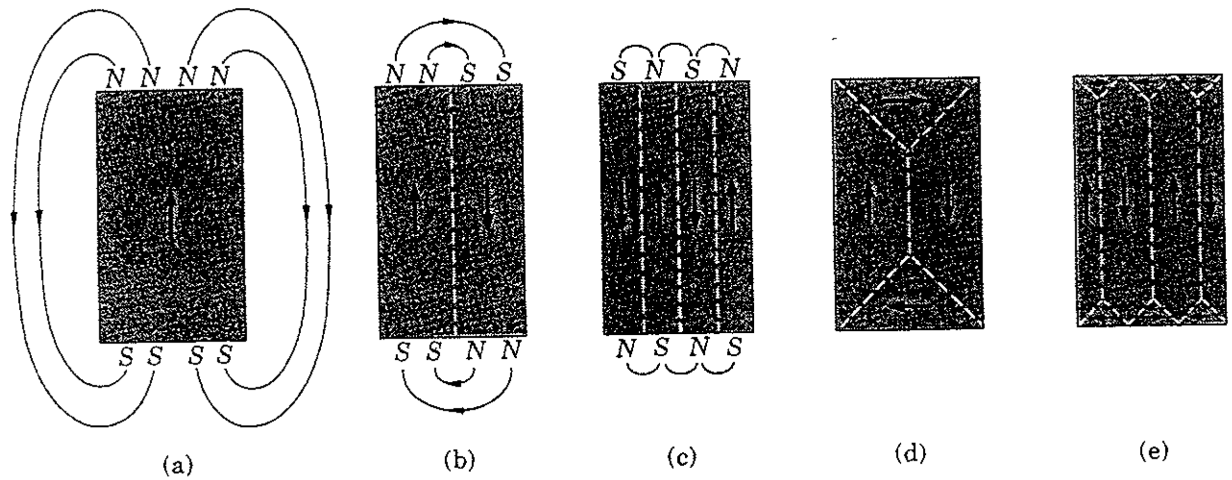


Figure 37 The origin of domains.

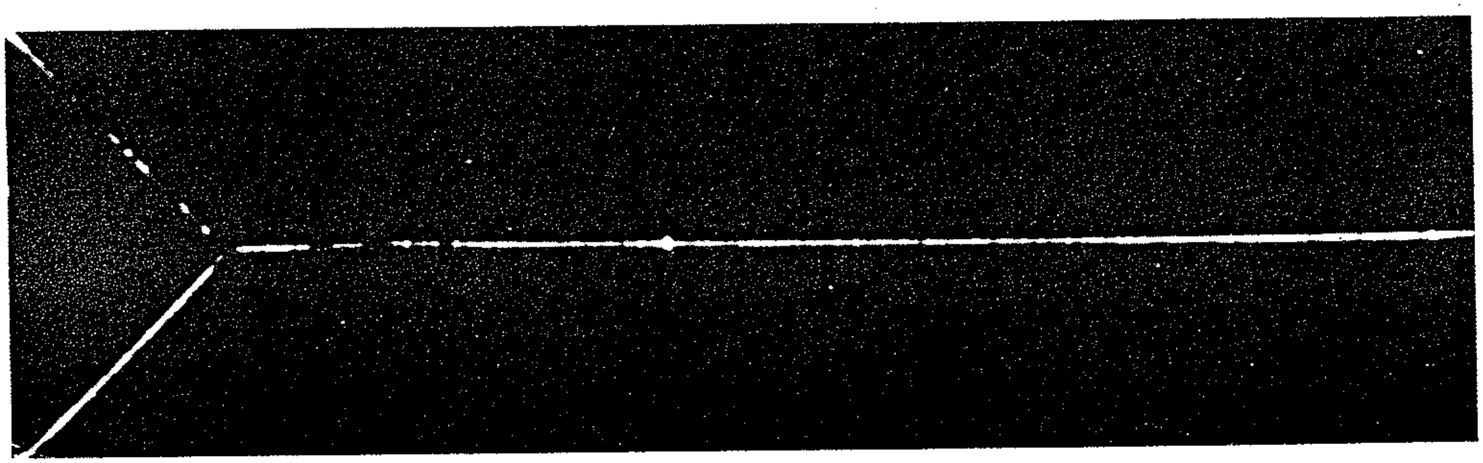
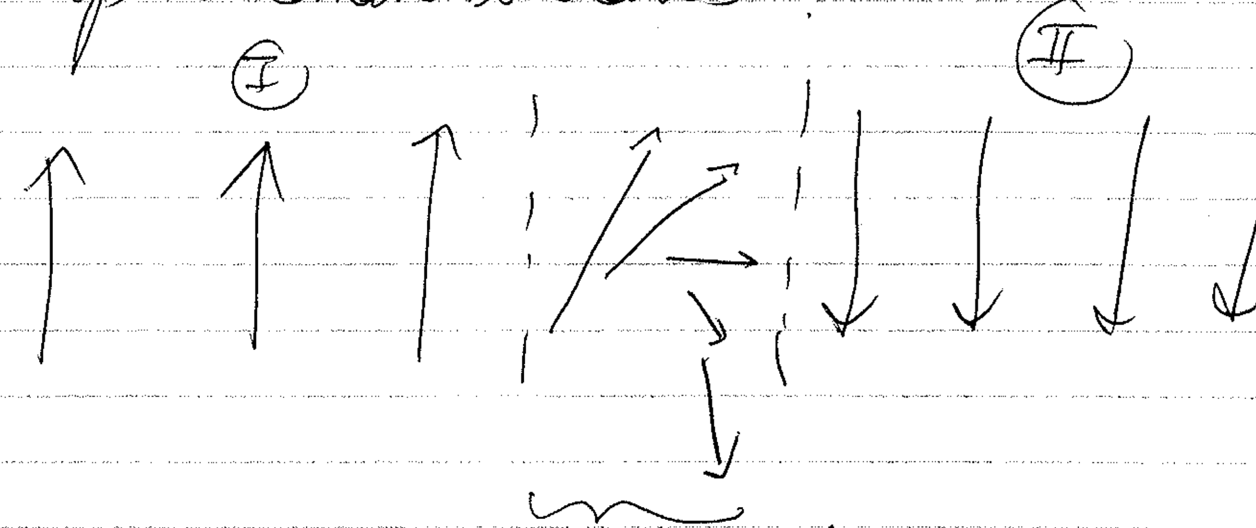


Figure 38 Domain of closure at the end of a single crystal iron whisker. The face is a (100) plane; the whisker axis is [001]. (Courtesy of R. V. Coleman, C. G. Scott, and A. Isin.)

We cannot have an infinite # of domains, however, because there is a "surface tension" that gets set up at domain boundaries



$\vec{m}$'s are rotated

Typical Domain Size:

a) 10^{17}-10^{21} atoms

b) 10^{-12}-10^{-8} m^3

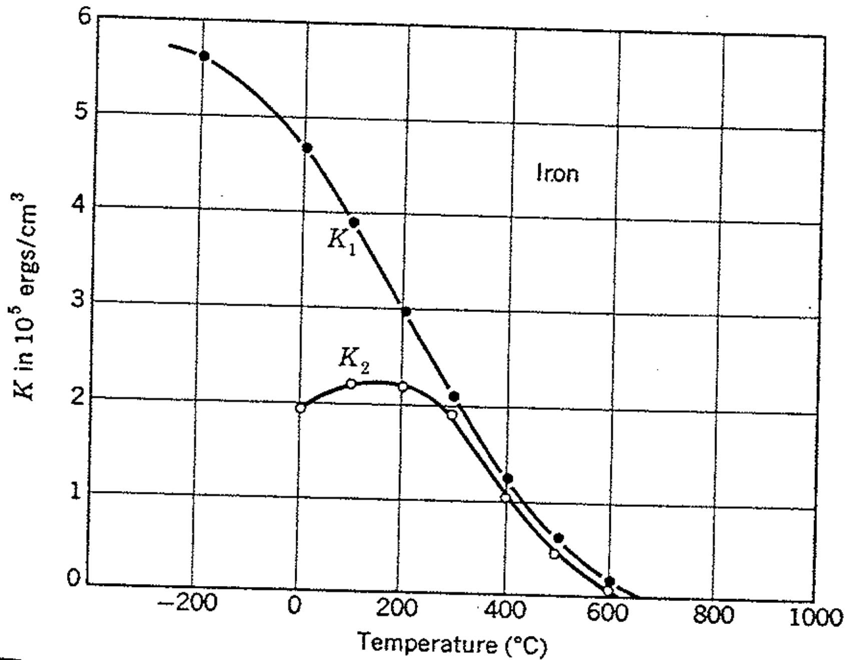
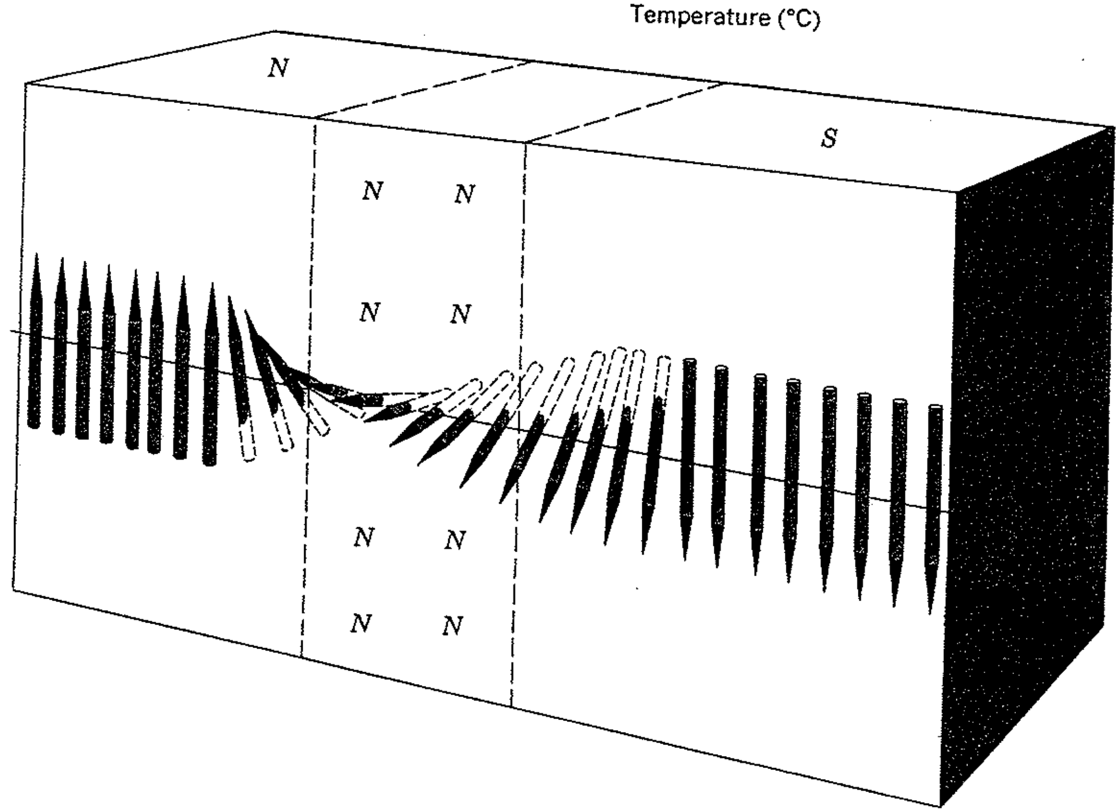


Figure 35 Temperature dependence of anisotropy constants of iron.



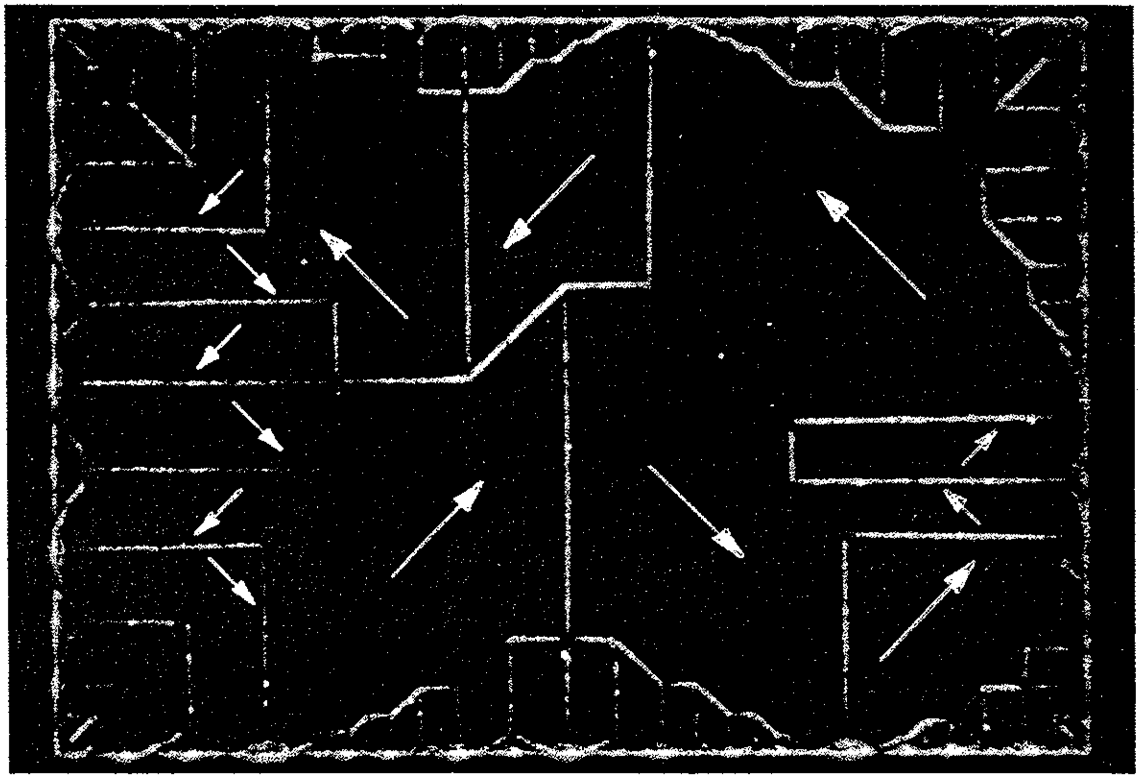


Figure 30 Ferromagnetic domain pattern on the surface of a single crystal platelet of nickel. The domain boundaries are made visible by the Bitter technique. The direction of magnetization within a domain is determined by observing growth or contraction of the domain in an applied magnetic field, as in Fig. 31a. (Courtesy of R. W. De Blois.)

Transitions along hysteresis loop are actually changes in discrete steps (due to Quantum nature of problem),

"Barkhausen Effect"

<u>Comments:</u> "Saturation" field

In iron, the two outer e's give $m_{iron} = 2\mu_{Bohr}$

<u>Saturation</u>: $\vec{M}_{sat} = N_V (2\mu_{Bohr})$

Number of irons per volume

$$= \frac{N_{avogadro}\,\rho_{Fe}}{A}(2\mu_{Bohr})$$

mass density of iron

atomic weight of the most abundant form of iron (≈ 56

$$\approx \frac{6.02\times10^{26}\times 7.9\times10^{3}}{55.6}\left(2\times 9.3\times10^{-24}\right)$$

$$= 1.59\times10^{6}\,\frac{A}{m}$$

Because, $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$, $\vec{M}$ is uniform in a "saturated chunk" of iron (all domains align)

$\vec{J}_f = 0$

$$\Rightarrow \vec{B}_{sat} = \mu_0\vec{M} + \mu_0\vec{H} \quad (\text{and } \vec{M} = \chi_m \vec{H})$$

χ_m large $\Rightarrow \vec{M} \gg \vec{H}$

$= 2.0\ T$

$\mu_0 \vec{H}$ can be ignored

Measured saturation field is

$\vec{B}_{sat,\,expt} \approx 2.15\ T$!

<u>Comments:</u> <u>Curie Temperature</u>, for $T > 770\ ^\circ C$ and iron, $\vec{B}$ goes <u>away</u>. Transition is abrupt;

"Phase Transition"

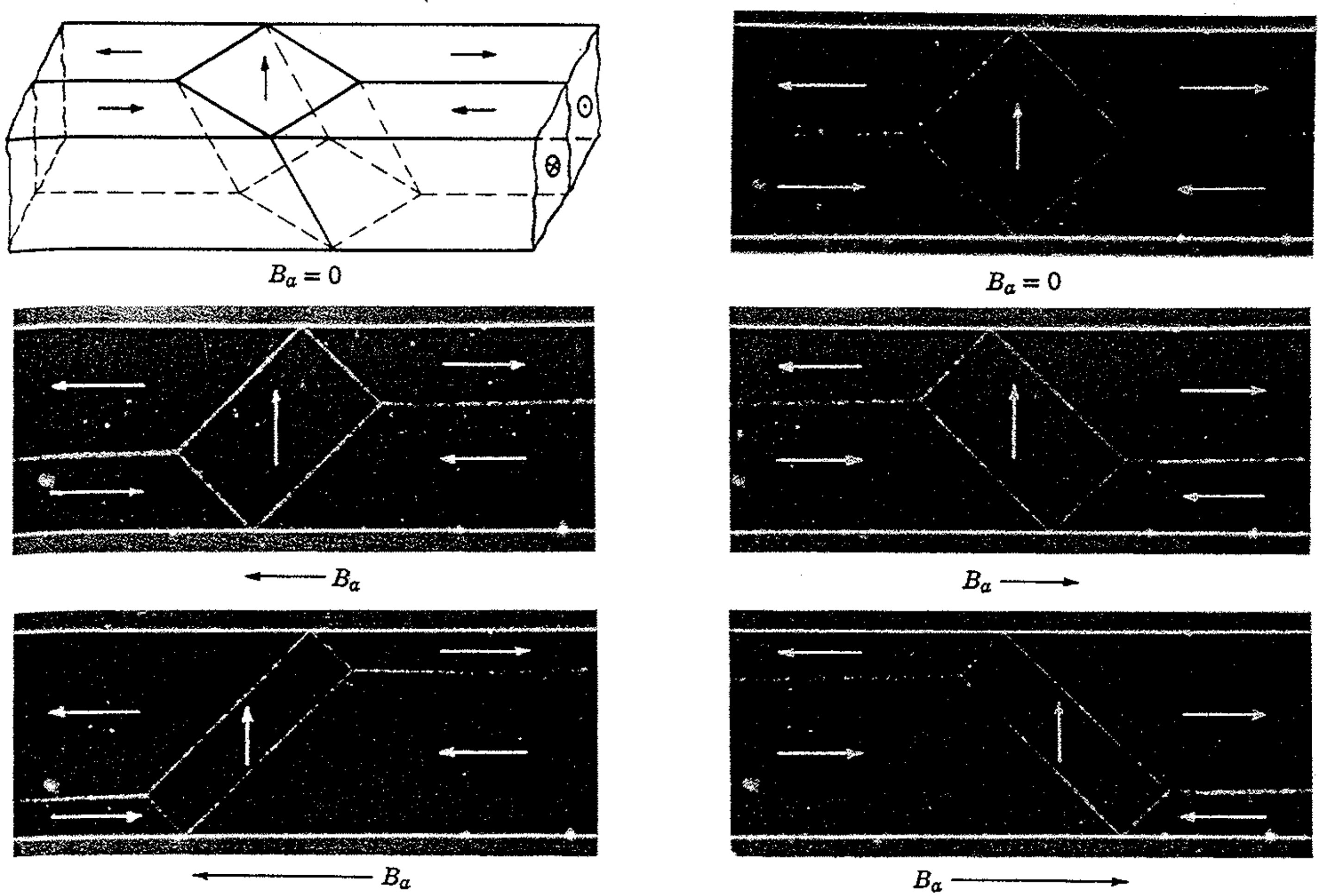
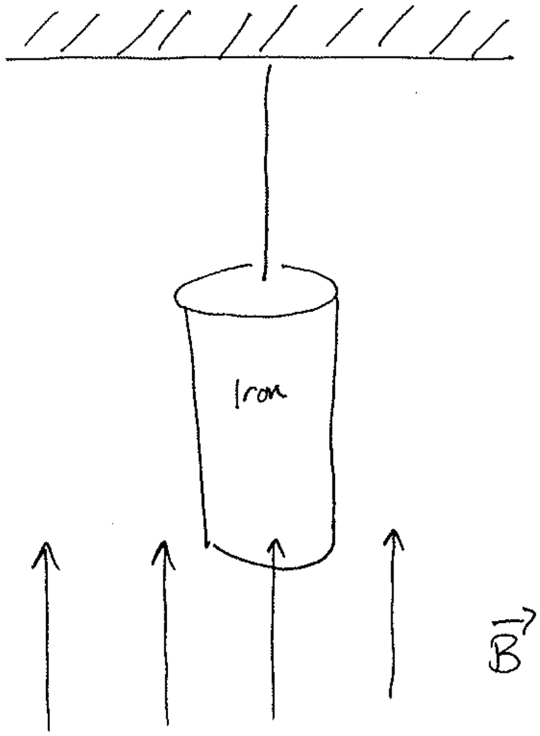


Figure 31a Smooth reversible domain wall motion in an iron crystal. The domains oriented in the direction of the applied field grow at the expense of the other domains. The field is in a [001] direction; the surface is a (100) plane; the maximum applied field is about 10 G. The crystal is a whisker (Chapter 20) about 10^{-2} cm on a side. (Courtesy of R. W. De Blois and C. D. Graham.)

Q: Is ferromagnetism due to spin?



attach a cylindrical block of iron to the ceiling using an inextensible, plastic string.

Turn on a "solenoidal" field,

$$\vec{B} = \mu_0 N I.$$

$\vec{B}$ ($\vec{H}$) then induces $\vec{M}$. Reverse the field so that $\vec{M}$ reverses.

If $\vec{B}$($\vec{H}$) was increased so that $\vec{M}$ saturated, we should see the iron rod start to spin.

Idea (Einstein - de Haas Effect)

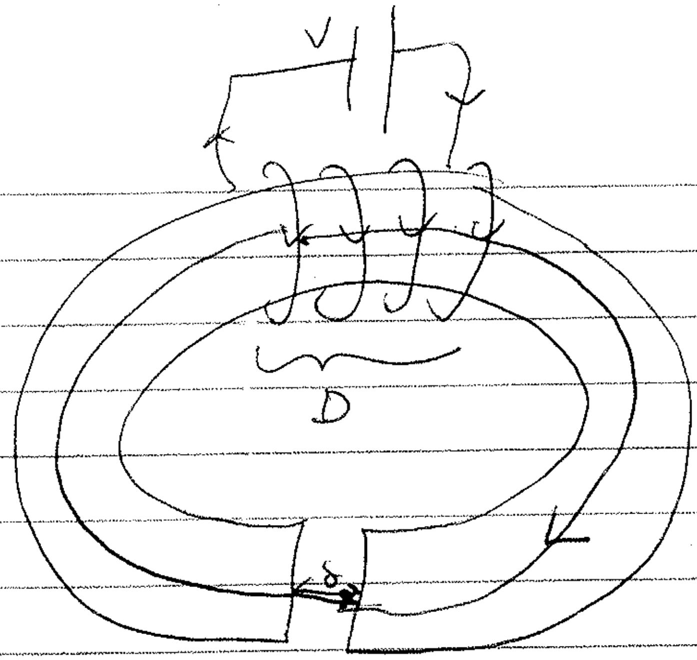
$$\vec{M} = N_i \left(2\mu_{Bohr}\right) \Rightarrow \vec{L} = \left(2\frac{\hbar}{2}\right) \leftarrow e^- \text{ spin}$$

$$\rightarrow \frac{\Delta m}{\Delta L} = \frac{(2\mu_{Bohr})}{(\hbar)} = 2g\frac{\frac{\hbar e}{2m_e}}{\hbar} = g\frac{e}{m_e}$$

and so,

$$g = \underbrace{\left(\frac{m_e}{e}\right)\left(\frac{\Delta m}{\Delta L}\right)}$$

Expt. finds — is ~ 2 (as predicted if ferromagnetism is due to e^- spin alignment)



circumference = $l + \delta$

N loops/length cd coil width D

$$\oint \vec{H} \cdot d\vec{l} = \oint \vec{J}_f \cdot d\vec{S}$$

$$\boxed{\underbrace{H_{\phi,\mu}\, l}_{\text{"magnet"}} + \underbrace{H_{\phi,\mu_0}\, \delta}_{\text{gap}} = \underbrace{NID}_{\text{enclosed current}}}$$

(B) We are interested the $\vec{B}$ in the gap, B_{ϕ,μ_0}

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \Delta B_n = 0 \rightarrow$ (in our case)

B_ϕ is continuous across boundary

$$\boxed{B_{\phi,\mu} = B_{\phi,\mu_0}} \quad (a)$$

note: $\vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M}_0 \Rightarrow \mu_0(\vec{H} + \vec{M}_0) = \mu_0(1 + \chi_m)\vec{H}$

$= \mu\vec{H}$

or $\vec{H} = \dfrac{\vec{B}}{\mu}$

recall that we found that

$$H_{\phi,\mu}\, l + H_{\phi,\mu_0}\, \delta = NDI$$

$$\boxed{\frac{B_{\phi,\mu}}{\mu}\, l + \frac{B_{\phi,\mu_0}}{\mu_0}\, \delta = NDI} \quad (b)$$

(a) & (b) $\leftarrow$ replace $\boxed{B_{\phi,\mu} \text{ w/ } B_{\phi,\mu_0}}$

$$\Longrightarrow \frac{B_{\phi,\mu_0}}{\mu}\, l + \frac{B_{\phi,\mu_0}}{\mu_0}\, \delta = NDI$$

$$B_{\phi,\mu_0} = \frac{NDI}{\frac{1}{\mu}\, l + \frac{\delta}{\mu_0}}$$

$$B_{\phi,\mu_0} = \frac{\mu}{l}\, \frac{NDI}{\left(\frac{\mu}{\mu_0}\frac{\delta}{l} + 1\right)}$$

If we take $\frac{\mu}{\mu_0}\frac{\delta}{l} \ll 1$

$$\boxed{B_{\phi,\mu_0} \simeq \frac{\mu}{l} NDI}$$

Significance

For an open core solenoid $B^{\text{no Fe}}_{\phi,\mu_0} = \mu_0 N I \frac{D}{l}$

$$\Longrightarrow \text{enhancement} = \frac{B_{\phi,\mu_0}}{B^{\text{no Fe}}_{\phi,\mu_0}} = \frac{\mu}{\mu_0} \gg 1$$

Suppose $\mu \approx \mu_0$

$$\Rightarrow \boxed{B_{\phi,\mu_0} \simeq \frac{\mu_0}{l} N D I}$$

Compare this to the case $\mu \gg \mu_0$

$$\boxed{\frac{B_\phi(\mu \gg \mu_0)}{B_\phi(\mu \simeq \mu_0)} \simeq \frac{\frac{\mu}{l} N D I}{\frac{\mu_0}{l} N D I} = \frac{\mu}{\mu_0} \gg 1}$$

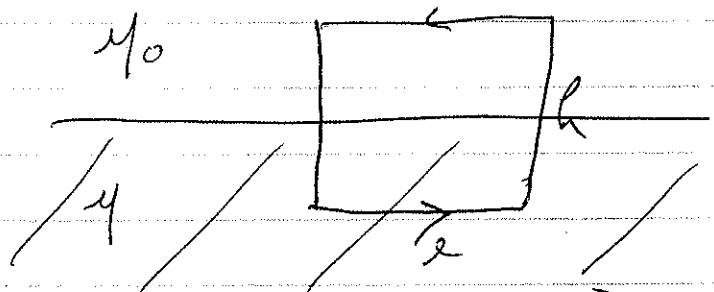
a huge jump in field

What about "Leakage"?

Let's look at the jump Conditions for magnetic material

a) $\vec{\nabla} \circ \vec{B} = 0 \Longrightarrow \Delta B_n = 0 \longrightarrow \mu_0 H_n^{\mu_0} = \mu H_n^{\mu}$

b) $\vec{\nabla} \times \vec{H} = \vec{J}_f \longrightarrow \oint \vec{H} \cdot d\vec{l} = \int \vec{J}_f \cdot d\vec{S}$



(i) $\vec{H}_T \perp \vec{J}_f$ ($\vec{J}_f$ comes out in the paper)
(goes)

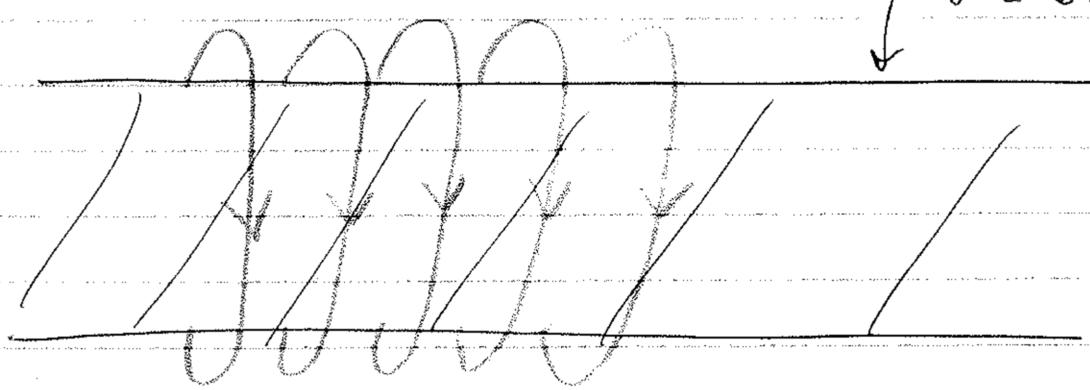
$$H_{T,\mu}\, l - H_{T,\mu_0}\, l = K_\perp l \rightarrow \boxed{\Delta H_T = K_\perp}$$

(ii) $\vec{H}_T \,//\, \vec{J}_f$ ($\vec{J}_f$ in the plane of the paper)

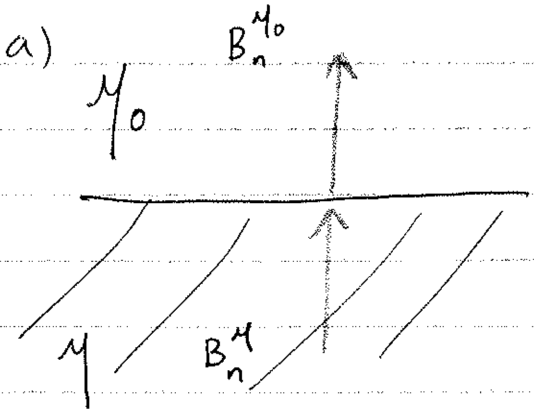
$$H_{T,\mu}\, l - H_{T,\mu_0}\, l = 0 \rightarrow \boxed{\Delta H_T = 0}$$

Okay, so we have

Consider the region outside the loops → $\vec{J}_f = 0$



a)

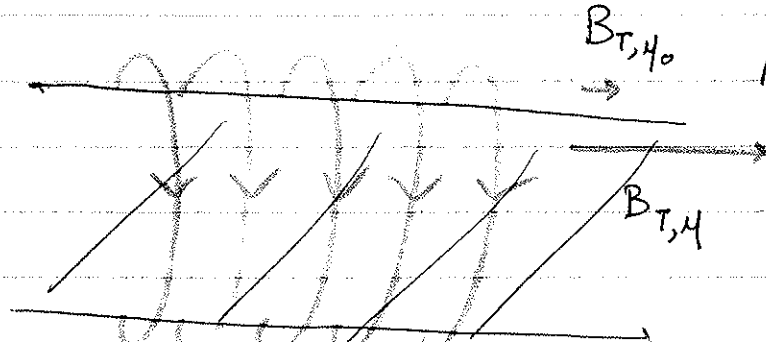


(i) $B_{n,\mu} = B_{n,\mu_0}$

$$\longrightarrow \mu H_{n,\mu} = \mu_0 H_{n,\mu_0}$$

and $$\boxed{\frac{H_{n,\mu_0}}{H_{n,\mu}} = \frac{\mu}{\mu_0} \gg 1}$$

b) $$H_{T,\mu} = H_{T,\mu_0} \longrightarrow \frac{B_{T,\mu}}{\mu} = \frac{B_{T,\mu_0}}{\mu_0} \longrightarrow \boxed{\frac{B_{T,\mu_0}}{B_{T,\mu}} = \frac{\mu_0}{\mu} \ll 1}$$



Near the coil, $\vec{H}$ is nearly tangential in the ferromagnetic material

$$\longrightarrow H_{n,\mu_0} \approx 0 \to B_{n,\mu} \approx 0$$

and $B_{n,\mu_0} \approx 0$

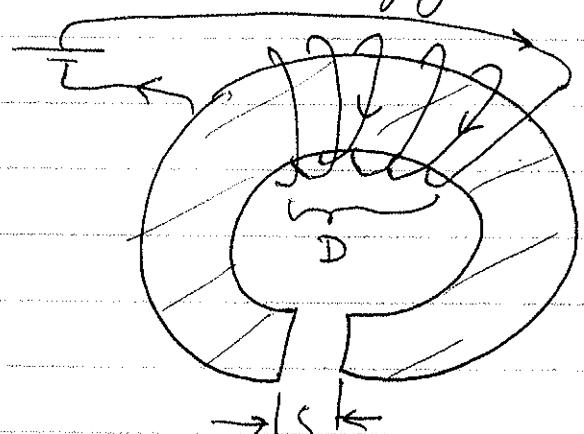
Consequently, we then have (also have that is),

$$B_{T,\mu_0} = \frac{\mu_0}{\mu} B_{T,\mu} << B_{T,\mu}$$

$\Longrightarrow$ little $\vec{B}$ leaks out

Electromagnets (ferromagnetic materials)

Consider a ferromagnetic material w/ loops of current carrying wire wrapped around it:



I ← N loops per unit length
→ $N \times D \equiv$ total # of loops

the circumference of the torus is

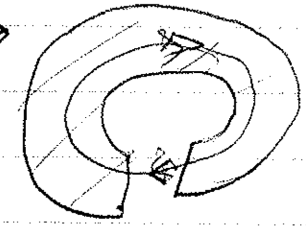
$$C = l + \delta$$

air gap of width δ

Find the field in the gap

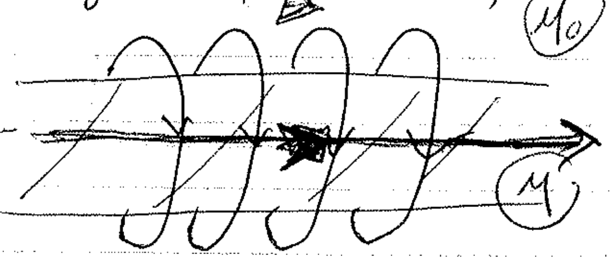
> Let's ignore a couple of things that we will address later -- "leakage"

(A) Ampère's law $\quad \oint \vec{H}\cdot d\vec{l} = \int \vec{J}_f \cdot d\vec{S}$



Apply Ampère's law for loops as shown to the left

In the region w/ the coils,



ϕ defined by circumference of torus

$$\underbrace{H_\phi^{\mu}\, l + H_\phi^{\mu_0}\,\delta}_{\oint \vec{H}\cdot d\vec{l}} = (NDI) \equiv \text{total current through surface}$$

μ_0 ← gap