Electric Conductivity

We have considered dielectrics (where $\vec{P} = \epsilon_0 \chi_e \vec{E}$) and magnetizable media (where $\vec{M} = \chi_m \vec{H}$). Now, let's consider another "electrical" property of matter,

"Conductivity"

(conductivity measured by σ or $\rho = 1/\sigma$). In a conductor there are free charges, and in the presence of a "force" ← $\vec{f}$, force per unit charge

charges move. That is, we measure

$$\vec{J} = \sigma\left(\frac{\vec{Force}}{q}\right)$$

← actually a "field", force per unit charge $\left(\frac{\vec{Force}}{q}\right)$.

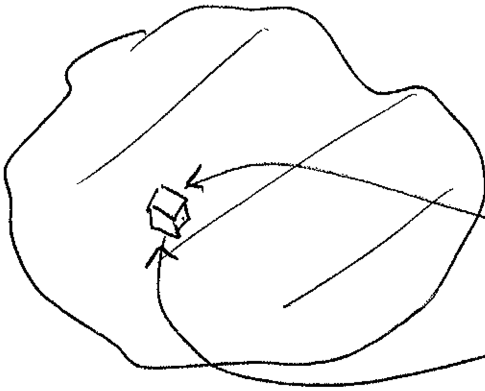
Here, we consider electrical forces and so,

$$\vec{J} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

or if $|\vec{v} \times \vec{B}| \ll |\vec{E}|$ then

$$\boxed{\vec{J} = \sigma \vec{E}, \quad \text{"Ohm's law"}}$$

Consider a conductor w/ σ where $\vec{E}=0$, initially.



Deposit some "excess" charge in the conductor

not "resistivity"

$dq = \rho\, d\tau$

charge density

Q: How long will this charge excess exist?

"Continuity Equation"

$J_x^i \qquad J_x^{out}$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\left\{\begin{array}{l} J_x^{out} \times A = \dfrac{\Delta Q^{out}}{s} \\ J_i^{out} \times A = \dfrac{\Delta Q^{in}}{s} \end{array}\right.$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\sigma\vec{E}) = 0$$

$$\frac{\partial \rho}{\partial t} + \sigma(\vec{\nabla}\cdot\vec{E}) = 0 \qquad \longrightarrow J_i^{out} - J_i^{in} = \frac{\Delta Q}{A\cdot s}$$

$$\longrightarrow \frac{\Delta(JA)}{A\Delta x} = \frac{\Delta Q}{(\Delta x A)}$$

$$\frac{\partial \rho}{\partial t} + \sigma\frac{\rho}{\epsilon_0} = 0 \longrightarrow \frac{\partial \ln\rho}{\partial t} = -\frac{\sigma}{\epsilon_0}$$

$$\boxed{\rho = \rho_0 \exp\left(-\frac{\sigma t}{\epsilon_0}\right)}$$

The charge dissipates exponentially. It falls to 1/exp of its size after a time

$$\tau_e = \epsilon_0/\sigma \quad (\text{e-folding time}).$$

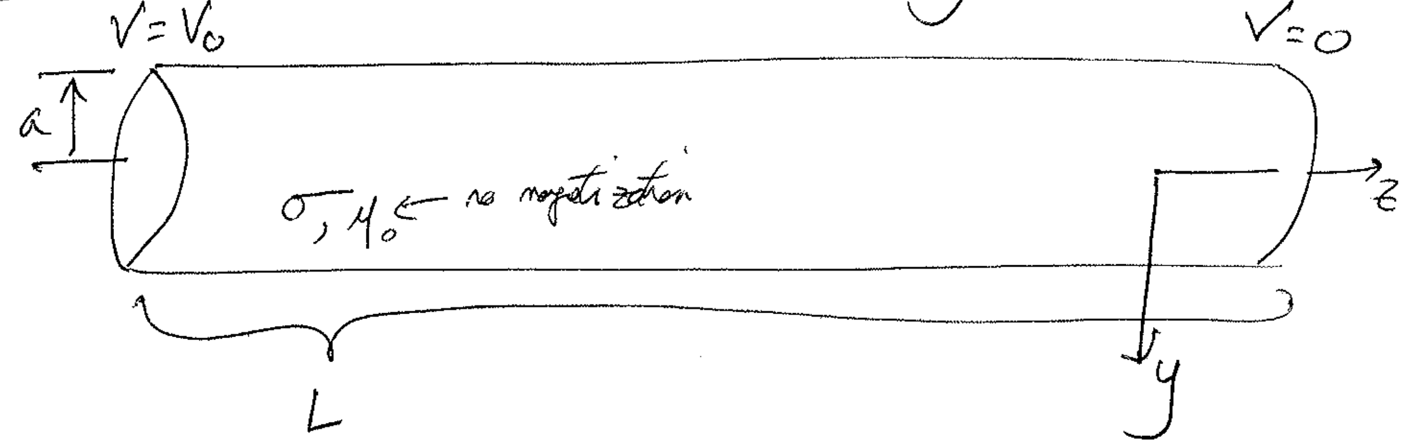
Materials	$\sigma\left(\frac{A}{mV}, (\Omega m)^{-1}\right)$	τ_e
Copper	5.8×10^{7}	1.5×10^{-14} s
Germanium	2.2	4×10^{-12} s
Glass	$\approx 10^{-12}$	9 s
Air	3×10^{-14}	~5 minutes

for metals, $\tau_e \ll 1\,s \Rightarrow \vec{E}=0$ is rapidly achieved (as we inferred / assumed last quarter).

a) Conductors can very quickly go to equilibrium, $\vec{E}=0$.

b) No "excess" charge in conductors. (can still have currents, however)

Consider a straight wire w/ conductivity σ



What are $\vec{J}$ and $\vec{B}$ for this piece of wire?

Because $\vec{J} = \sigma\vec{E}$, we need $\vec{E}$ to find $\vec{J}$. However, is it as simple as saying

$$\vec{E} = -\frac{\Delta V}{L}\hat{z} = \frac{V_0}{L}\hat{z}\,?$$

Let's see. We have

steady flow

(i) $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\vec{J} = 0 = \vec{\nabla}\cdot(\sigma\vec{E})$

(ii) steady-state $\Longrightarrow \vec{\nabla}\times\vec{E} = 0 \Longrightarrow \vec{E} = -\vec{\nabla}V$

combine (i) and (ii)

$$\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot(\sigma\vec{E}) = -\sigma\nabla^2 V = 0$$

$$\Longrightarrow \boxed{\nabla^2 V = 0 \text{ holds, Laplace eqn. holds}}$$

and the answer maybe is as simple as $\vec{E} = \frac{V_0}{L}\hat{z}$!

ad $V(z) = A' + B'z$

(b) at $z=0,\ V=V_0$; $z=L,\ V=0$

$$\Rightarrow A' = V_0 \quad , B' = -\frac{V_0}{L}$$

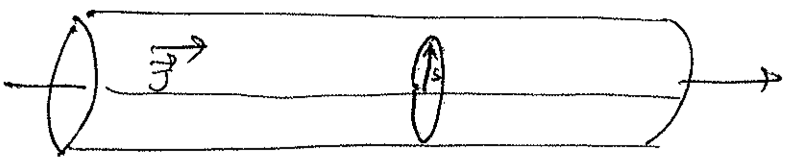
so $\boxed{V(z) = V_0 - \frac{V_0}{L} z} \Rightarrow \boxed{\vec{E} = \hat{z}\frac{V_0}{L}}$ ✓

as guessed ; hoped for.

The current is then

$$\boxed{\vec{J} = \hat{z}\,\sigma\frac{V_0}{L}}$$

$\vec{B}$ can be found from Ampere's law (if $L >> a$)



$$B_\phi 2\pi s = \mu_0 \int \vec{J}\cdot d\vec{s} = \mu_0 \frac{V_0}{L}\sigma \pi s^2 \quad \text{(inside wire)}$$

$$\Rightarrow \boxed{\vec{B}_\phi = \frac{\mu_0 \sigma V_0 s}{2L}\hat{\phi}}$$

Comment:

$$d\vec{F}_B = \vec{J}\,\cancel{dx}\,d^3x \times \vec{B} = -\hat{s}\left[\mu_0 \frac{\sigma^2 V_0}{2L^2} s\right] \cancel{ds}\, d^3x$$

$$= -\hat{s}\left[\mu_0 \frac{\sigma^2 V_0}{2L^2} s\right] s\,ds\,d\phi\,L$$

$$\left\{\vec{F}_B / L\right\} = -\hat{s}\left(\mu_0 \frac{\pi \sigma^2 V_0}{3L^2} a^3\right)$$; why doesn't wire collapse?

Ohm's Law

Let's return to $\vec{J} = \sigma \vec{E}$ and re-write this in a more familiar way

a) $\vec{J} = \sigma \vec{E}$

$\hat{z}\left(\frac{I}{A}\right)$ $\qquad$ $\frac{V_0}{L}\hat{z}$

$$\Longrightarrow V_0 = \left(\frac{L}{\sigma A}\right) I$$

$R \equiv$ resistance $\Longrightarrow$ $\boxed{V = IR}$ "Ohm's" law

$\equiv \left[\frac{L}{\sigma A}\right]$

geometric quantities $+ \sigma$

Ohmic losses, Joule Heating, Power

Recall: $\vec{J} = \sigma \vec{E}$ = constant for steady current,

however, $m\ddot{\hat{z}} = q(\vec{E}\cdot\hat{z}) \Rightarrow \ddot{\hat{z}} \neq 0$

$\rightarrow$ constant acceleration?

Recall, on a volume, we have the force $\boxed{\text{micro vs macro vs } \delta f}$

$$\Delta\vec{F} = \int_{\Delta V} \left[\rho_{micro} \vec{E}_{micro} + \vec{J}_{micro} \times \vec{B}_{micro} \right] dV$$

where
$$\rho_{micro} = \rho_{average} + \delta\rho$$
$$\vec{E}_{micro} = \vec{E}_{average} + \delta\vec{E}$$
$$\vec{J}_{micro} = \vec{J}_{average} + \delta\vec{J}$$
$$\vec{B}_{micro} = \vec{B}_{average} + \delta\vec{B}$$

(i) "average" are macroscopic properties

(ii) "δ" are fluctuations

1st order terms average to 0 over time and spatial integrates

$$\Rightarrow \Delta\vec{F} = \int_{\Delta V} \left[\rho_{ave}\vec{E}_{ave} + \vec{J}_{ave} \times \vec{B}_{ave} \right] dV + \int_{\Delta V} \left[\delta\rho\, \delta\vec{E} + \delta\vec{J} \times \delta\vec{B} \right] dV$$

$$= \underbrace{\left[\rho_{ave}\vec{E}_{ave} + \vec{J}_{ave} \times \vec{B}_{ave} \right] \Delta V}_{\text{power delivered to the charges by "battery"} \Rightarrow \text{acceleration}} + \underbrace{\int_{\Delta V} \left[\delta\rho\, \delta\vec{E} + \delta\vec{J} \times \delta\vec{B} \right] dV}_{\text{collisions between particles dissipating "energy" delivered} \Rightarrow \text{drag}}$$

Let's return to ohmic losses (consider Work)

← the work done on a charged particle, q_i

$$dW_i = q_i \vec{E}_{micro} \cdot d\vec{r}_i$$

For a system of charges, the rate at which work is performed is

← single particle (charge q_i)

$$\frac{dW_i}{dt} = q_i \vec{E}_{micro} \cdot \frac{d\vec{r}_i}{dt} = \text{Power} = P$$

$$\Rightarrow \boxed{\frac{dW_i}{dt} = \sum_i q_i \vec{E}_{micro} \cdot \vec{V}_i}$$

← work on a collection of $\vec{J}_i$ charges

$q_i = \rho_i d^3x_i \Rightarrow \vec{E}_{micro} \cdot (\rho_i d^3x_i \vec{V}_i)$

Consider a collection of charges that can be treated as averages

$$\cancel{\rho_{micro} = \sum \frac{q_i(t)}{d^3x_i} \Rightarrow \rho_{micro} = \rho_{ave} + \delta\rho}$$

~~and~~

$$\cancel{q_i \vec{V}_i = \left[\rho_{ave} + \delta\rho\right]\left[\vec{V}_i + \delta\vec{V}\right] d^3x_i}$$

$$\cancel{= \left[\vec{J}_{ave,i} + \underbrace{\delta\rho\,\delta\vec{V}}_{\delta J}\right] d^3x_i}$$

$$\Rightarrow \frac{dW_i}{dt} = \int \underbrace{(\vec{J}_{ave} + \delta\vec{J})}_{\vec{J}_{micro}} \cdot \underbrace{(\vec{E}_{ave} + \delta\vec{E})}_{\vec{E}_{micro}} d^3x$$

1st order terms integrate to 0 over time & space

$$\frac{dW_i}{dt} = \int \left[\vec{J}_{ave} \cdot \vec{E}_{ave} + \delta\vec{J} \cdot \delta\vec{E} \right] d^3x = P$$

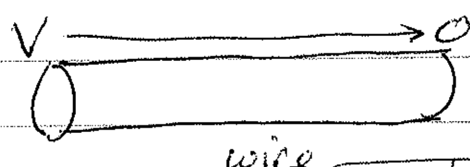
integrate over a volume ΔV

$$\Rightarrow \Delta P = \underbrace{\vec{J}_{ave} \cdot \vec{E}_{ave} \, \Delta V}_{\text{power delivered by battery to charges}} + \underbrace{\int_{\Delta V} \delta\vec{J} \cdot \delta\vec{E} \, d^3x}_{\text{power absorbed in collisions between accelerated charge carriers } (e^-\text{s}) \text{ and background ions}}$$

$$\boxed{\Delta P = \vec{J}_{ave} \cdot \vec{E}_{ave} \, \Delta V}$$

for a uniform wire $\boxed{\vec{J}_{ave} \, dV = I \, d\vec{l}}$

$$\Rightarrow \Delta P = \int_{\Delta V} I \, d\vec{l} \cdot \vec{E}_{ave} = I \underbrace{\int \vec{E}_{ave} \cdot d\vec{l}}_{} = -I \, \Delta V$$

note:  , $\Delta V = 0 - V$

potential drop along wire

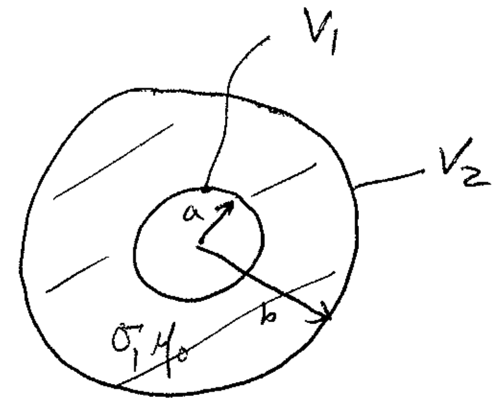
and $\boxed{\Delta P = I V}$

(HW)

Example, Prob. 7.1

2 concentric spheres and the region between is filled w/ a material w/ conductivity σ.

$\Delta V = V_0$. What are I and R?



Sol^{n} Steady-state & uniform $\sigma \Longrightarrow \nabla^2 V = 0$

$$\Longrightarrow \boxed{V = D - C/r}$$ as solution of Laplace eqn. in spherical polar coordinates

a) Apply BC's

$$\begin{cases} r = a \Longrightarrow V_1 = D - C/a \\ r = b \Longrightarrow V_2 = D - C/b \end{cases}$$

$$\Longrightarrow V_1 + \frac{C}{a} = V_2 + \frac{C}{b} \Longrightarrow \boxed{C = \frac{V_2 - V_1}{\frac{1}{a} - \frac{1}{b}}}$$

and

$$\boxed{D = V_1 + \frac{b}{b-a}(V_2 - V_1)}$$

b) $V = D - C/r$

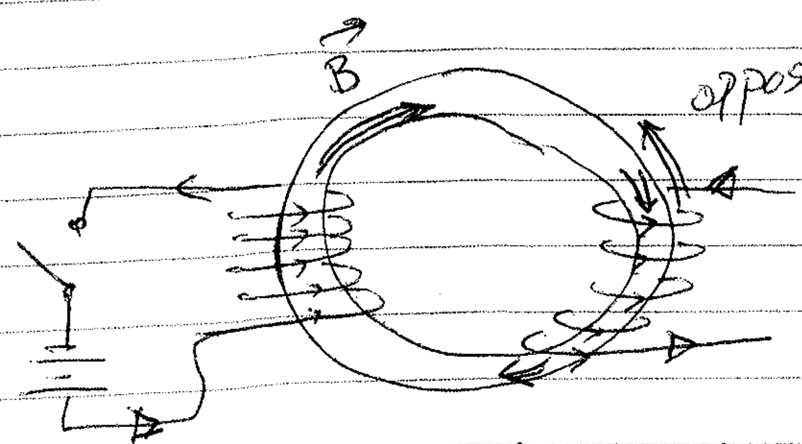
$$\vec{E} = \hat{r}(C/r^2) = \frac{ab}{b-a}\underbrace{(V_2 - V_1)}_{\Delta V = V_0}\frac{\hat{r}}{r^2}$$

$$\Longrightarrow \boxed{\vec{J} = \sigma \vec{E_r} = \sigma V_0 \frac{ab}{b-a}\frac{\hat{r}}{r^2}}$$

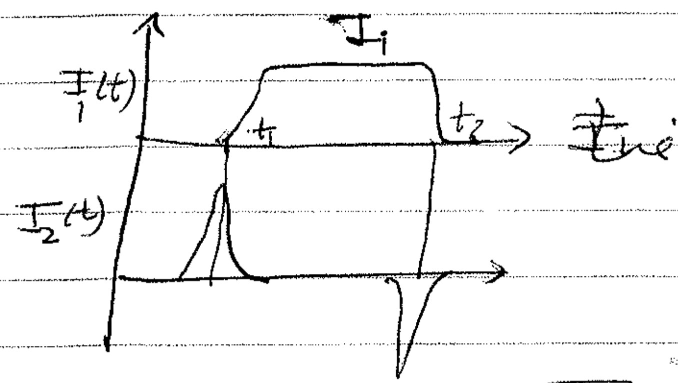
E-fields from Time-Varying $\vec{B}$ fields

$1820 \Rightarrow I \rightarrow \vec{B}$

$1831 \rightarrow$ Faraday (& Henry) found that $\vec{B} \Rightarrow \vec{E}$



close switch
$\rightarrow$ I flows
as shown



Current flows to make $\vec{B} \ni \Phi_B = \int \vec{B} \cdot d\vec{S} = 0$ during experiment

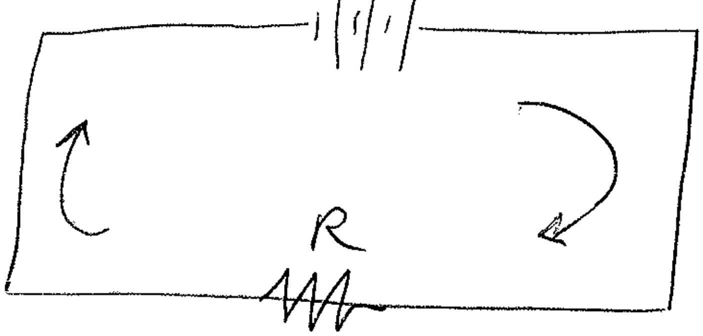
Lenz's Rule: the sense of the induced current (induced EMF) is to oppose the change in magnetic flux, Φ
(1833)

Faraday found that $\mathcal{E} = IR = -\frac{d}{dt}\Phi_B$

Electromotive force ($\mathcal{E}$, emf)

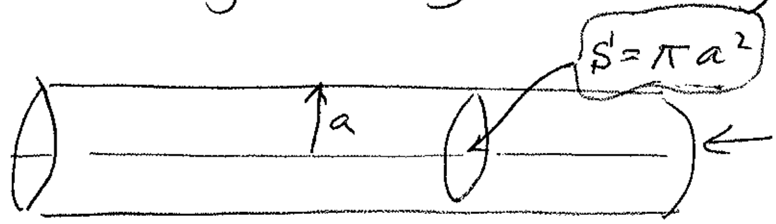
Voltage difference across some element of a "circuit"!

Consider V, potential across terminals



① Battery supplies a "push" right around itself. However, lights go essentially right after you "flick" the switch. Can we expect that this is due to e^-'s being around by the battery?

Ⓘ



← Copper wire w/ radius $\approx 10^{-3}$ m

We have $\vec{J} = \underbrace{(n_e e)}_{\text{charge density}} v_e$

$$\Rightarrow \vec{J}\, d\tau = I\, d\vec{l} \Rightarrow I\, d\vec{l} = e n_e \vec{v}_e\, d\tau$$

and

$$v_e = \frac{I\, dl}{e n_e\, d\tau} = \frac{I}{e n_e S}$$

I ← 1 Amp

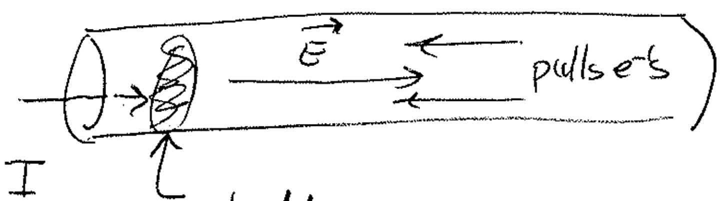
S = area

n_e ← $8.5 \times 10^{28}\, m^{-3}$

$$\boxed{v_e \approx 10^{-4}\, \frac{m}{s}}$$

$\Rightarrow$ takes too long for e^-'s to supply push directly

(4)



charge builds-up
$\Rightarrow$ E gets set up essentially instantaneously (c = signal speed)
$\Rightarrow$ and $\vec{E}$ gets erased on the light-travel time (essentially).

$\Longrightarrow$ two "forces" act on wire, battery + $\vec{E}$ (induced)

<u>fields</u>; $\vec{f} = \vec{f}_s + \delta\vec{E}$ (induced + fluctuations)

$\vec{f}_s$: battery (macroscopic applied field)

$\delta\vec{E}$: microscopic in origin, not the macroscopic applied field (but is really fluctuation field + induced field)

= In our language

"The emf ($\mathcal{E}$) is the work done for each unit of charge transferred around the circuit."

$$\Rightarrow \mathcal{E} = \oint \vec{f} \cdot d\vec{r} \; \boxed{= \underbrace{-\frac{d}{dt}\Phi_B}_{\text{Faraday's Observation}}}$$

$\longrightarrow$ "formal" definition

Onto Maxwell's Equations

Consider some $\mathcal{E}$'s (emf's) -- "Microscopic $\mathcal{E}$'s" fluctuations

(A) Resistive emf (microscopic)

$$\mathcal{E} = \int \delta \vec{E} \cdot d\vec{r}$$

fluctuating part of $\vec{E}_{micro}$

$$= \int_a^b \left(\vec{E}_{micro} - \vec{E}\right) \cdot d\vec{r}$$

Because charges do not accelerate, across $d\vec{r}$, on average $\underbrace{\int_a^b \vec{E}_{micro} \cdot d\vec{r}}_{\text{no net work performed}} = 0$

$$\mathcal{E} = -\int_a^b \vec{E} \cdot d\vec{r}$$

Ohm's law

Resistive emf,

$$\mathcal{E}_R = -\int_a^b \frac{\vec{J}}{\sigma} \cdot d\vec{r} = -\frac{I}{\sigma A}(b-a)$$

$$= -\frac{I}{\sigma A} l$$

recall: this is R

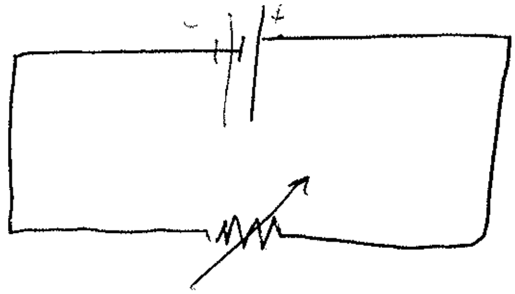
$$\boxed{\mathcal{E}_R = -RI}$$

(B) Battery emf

$$\mathcal{E}_{battery} = V \text{ for } I = 0 \quad (f = 0 \text{ because } \sigma \to \infty \;\rightarrow\; \delta E = -f_s)$$

$\boxed{\text{Prob. 7.5 Example}}$

V_b, r ← not an ideal battery



voltage across the terminals of the battery

$$V = \overbrace{(\mathcal{E} - Ir)} = IR$$

variable resistor

To deliver the maximum possible "Load" to the resistor, what should R be?

(i) "Load" = Power = $IV = I^2R$ should be maximized

$$\Rightarrow 0 = \underbrace{\mathcal{E} - \left(\frac{P}{R}\right)^{1/2} r}_{(\mathcal{E} - Ir)} - \underbrace{\left(\frac{P}{R}\right)^{1/2} R}_{IR}$$

$$P^{\frac{1}{2}}R + P^{\frac{1}{2}}r + (\mathcal{E} \quad)R^{1/2} = 0$$

$$P^{1/2} = \frac{+(\mathcal{E} \quad)}{(r+R)} R^{1/2}$$

$$P = \frac{(\mathcal{E} \quad)^2 R}{(r+R)^2}$$

$$\frac{dP}{dR} = \frac{(\mathcal{E} \quad)^2}{(r+R)^2} - \frac{2(\mathcal{E} \quad)^2}{(r+R)^3} R$$

$= 0$, for extrema

$$\Rightarrow (\mathcal{E} \quad)^2 (r+R) = 2(\mathcal{E} \quad)^2 R$$

$$\Rightarrow \boxed{R = \quad r}$$

batteries, and so on...

$\vec f = \vec f_s + (\delta\vec E, \delta\vec B)$

Induction $\mathcal{E}$ (emf) $\leftarrow$ Macroscopic forces

a) $\mathcal{E} = \oint \vec f \cdot d\vec r = \oint \left(\vec E + \vec V_e \times \vec B \right) \cdot d\vec r$

line integral along wire

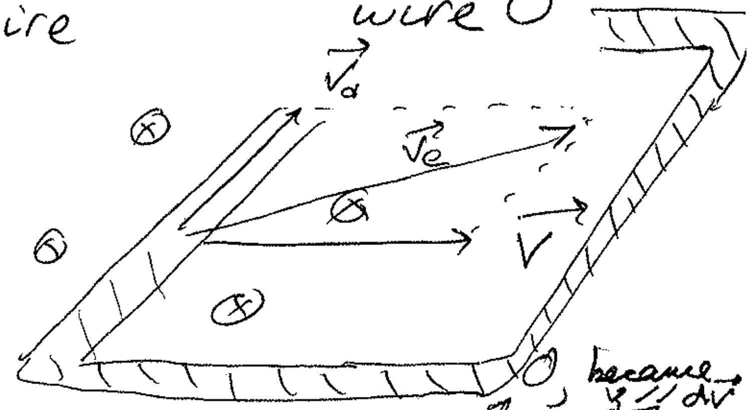
$d\vec r$ along wire

$\vec V_e \equiv$ Electron $\vec V \equiv \vec V + \vec V_d$

motion of wire

drift along wire

$$\mathcal{E} = \oint \Big[\vec E \cdot d\vec r + \vec V \times \vec B \cdot d\vec r + \vec V_d \times \vec B \cdot d\vec r \Big]$$



because $\vec V_d \parallel d\vec r$

$$= \oint \Big[\vec E \cdot d\vec r + \vec V \times \vec B \cdot d\vec r + \vec B \cdot \left(\vec V_d \times d\vec r \right) \Big]$$

recall, $d\vec r$ along wire

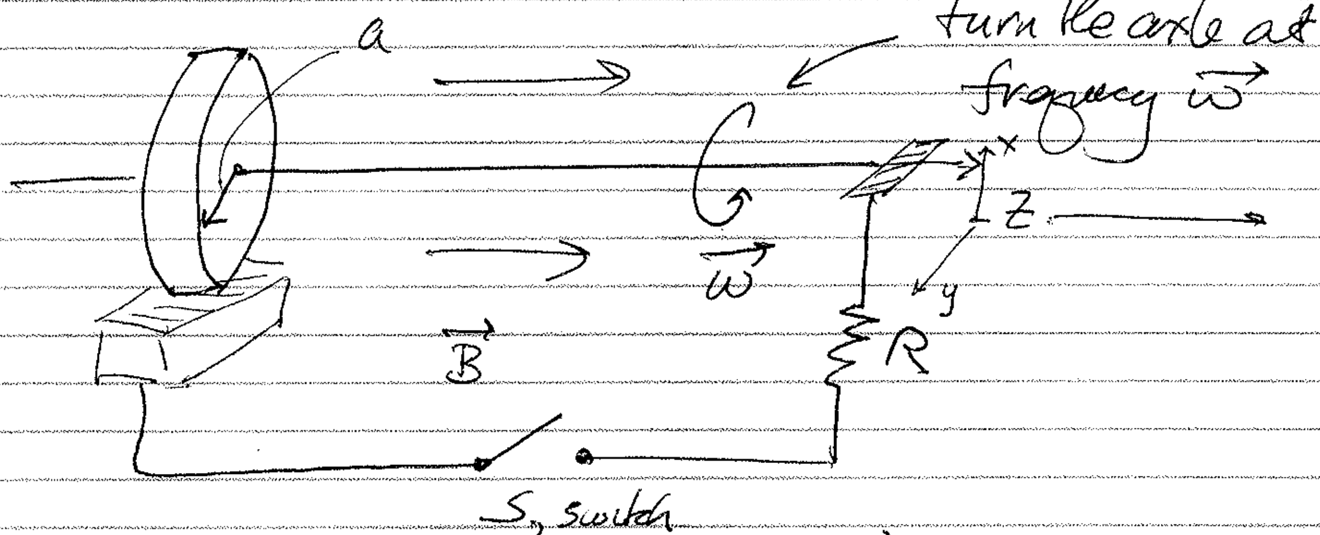
$$\boxed{\mathcal{E} = \oint \Big[\vec E \cdot d\vec r + \vec V \times \vec B \cdot d\vec r \Big]}$$

$\vec V$ of wire!

Empirically, Faraday found that

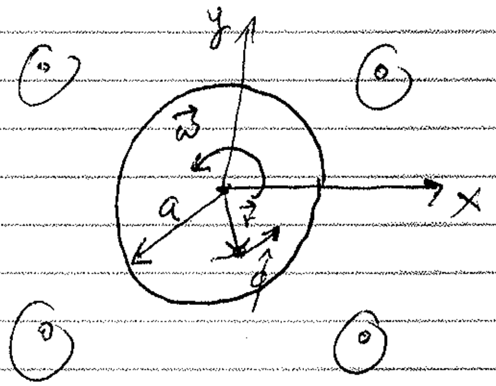
$$\mathcal{E} = -\frac{d}{dt}\Phi_B = \underbrace{-\frac{d}{dt}\int \vec B \cdot d\vec S}_{\text{recall (for later)}}$$

Homopolar Generator / Motor



a) If S is open $\leftarrow \left(\frac{\vec{F}}{q}\right)$

$$\mathcal{E} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$



$$\vec{v} = \vec{\omega} \times \vec{r} = \omega r \hat{\phi} \Rightarrow \boxed{\vec{\omega} \times \vec{B} = \omega r B_0 \hat{r}}$$

$$\mathcal{E} = \int_0^a (\omega r \hat{\phi} \times B_0 \hat{z}) \cdot d\vec{r}$$

$$= \frac{\omega r^2 B_0}{2} \Big|_0^a \text{ outward}$$

$$\boxed{\mathcal{E} = \frac{1}{2} \omega B_0 a^2}$$

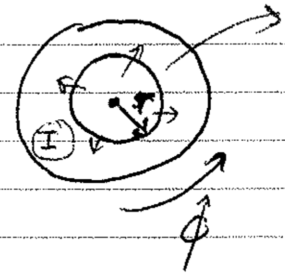
b) Close S $\Rightarrow \mathcal{E} = IR \rightarrow \frac{1}{2} \omega B_0 a^2 = IR$

$$\boxed{I = \frac{\omega B_0 a^2}{2R}}$$

(c) The current I back-reacts on the wheel. Find the induced torque and Power needed to power the generator.

(i) $\vec{I} = \frac{\omega B_0 a^2}{2R}\hat{r} \Rightarrow$ force

$$I\,d\vec{r} = \vec{K}\,dS \qquad dS = r\,d\phi\,dr$$



$$\vec{K} = \left(\frac{I}{2\pi r}\right)\hat{r} \Rightarrow d\vec{F} = \vec{K}\,dS \times \vec{B} = \frac{IB_0}{2\pi r}(r\,d\phi\,dr)(-\hat{\phi})$$

(ii) Torque, $d\vec{N} = \vec{r} \times \underbrace{d\vec{F}}_{-\hat{\phi}} = \frac{IB_0}{2\pi}\,d\phi\,dr\,(-\hat{z})$

$$\Rightarrow \vec{N} = \int \frac{IB_0}{2\pi}\,r\,d\phi\,dr\,(-\hat{z})$$

$$= \int \left(\frac{\omega B_0 a^2}{2R}\right)\left(\frac{B_0}{2\pi}\right)\left[r\,d\phi\,dr\right](-\hat{z})$$

$$= \frac{\omega B_0^2 a^2}{2R}\left(\cancel{\frac{2\pi}{2\pi}}\right)\left(\frac{r^2}{2}\right)\Big|_0^a(-\hat{z})$$

opposes mechanical input

$$\boxed{\vec{N} = \frac{\omega B_0^2 a^4}{4R}(-\hat{z})}$$

"we" must supply this mechanical effort to maintain $\vec{\omega}$..

opposes $\vec{\omega}$

(iii) $W = \int_0^{\phi} \vec{N} \cdot d\phi \hat{z}$ to turn from $\phi = 0 \to \phi$

$$= \frac{\omega B_0^2 a^4}{4R}\phi$$

Power is the rate of work $\Rightarrow \phi = \dot{\phi}\Delta t = \omega \Delta t$

$$\Rightarrow \boxed{\frac{\Delta W}{\Delta t} = P = \frac{\omega^2 B_0^2 a^4}{4R}}$$

$\leftarrow$ We supply this Power to keep the wheel spinning at constant ω

find Ohmic losses (load)

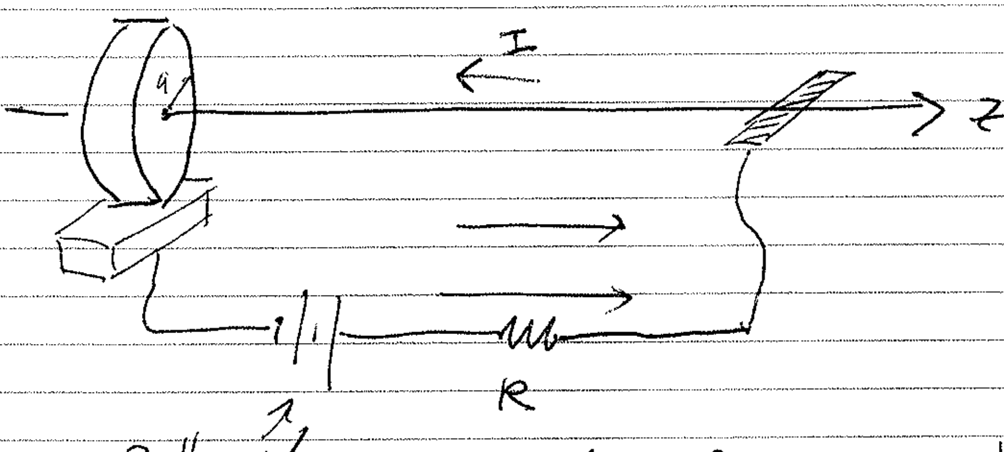
$$I = \frac{\omega B_0 a^2}{2R} \Rightarrow I^2 = \frac{\omega^2 B_0^2 a^4}{4R^2} = \frac{P}{R}$$

$$\Rightarrow \boxed{P_L = I^2 R}$$

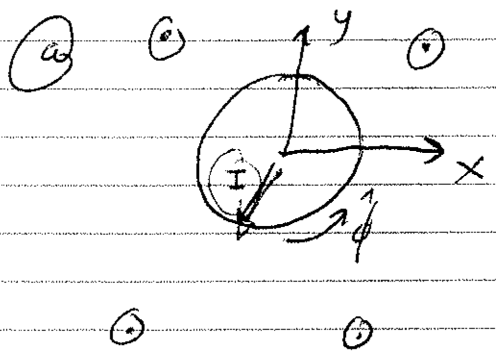
Same as input power

Motor

$\vec{B} = B_0 \hat{z}$



$\Rightarrow V = IR \rightarrow I = V/R$



$d\vec{F} = \left(\frac{I}{2\pi r}\right)\hat{r}\,(r\,d\phi\,dr) \times B_0 \hat{z}$

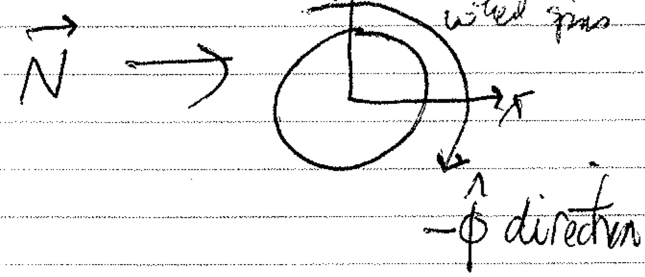
$\underbrace{\vec{F} = I r B_0 \int_0^a (-\hat{\phi})}$; total force

we are interested in torque

$d\vec{N} = \vec{r} \times d\vec{F} = r\left(\frac{I}{r}\right) r\,dr\,B_0(-\hat{z})$

$$\Rightarrow \boxed{\vec{N} = -\frac{I a^2 B_0}{2}\hat{z} = -\frac{V a^2 B_0}{2R}\hat{z}}$$

"locked" rotor torque (torque when $\vec{\omega} = 0$)



(b) as $\omega \uparrow \Rightarrow$ a back-reaction occurs $\Rightarrow$ "drag"

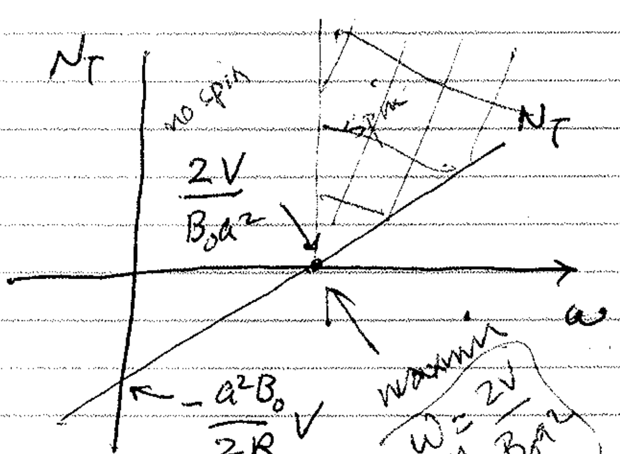
$$\omega \Rightarrow \vec{N}_b = \frac{\omega B_0^2 a^4}{4R}\hat{z} \quad \text{from ("generator")}$$

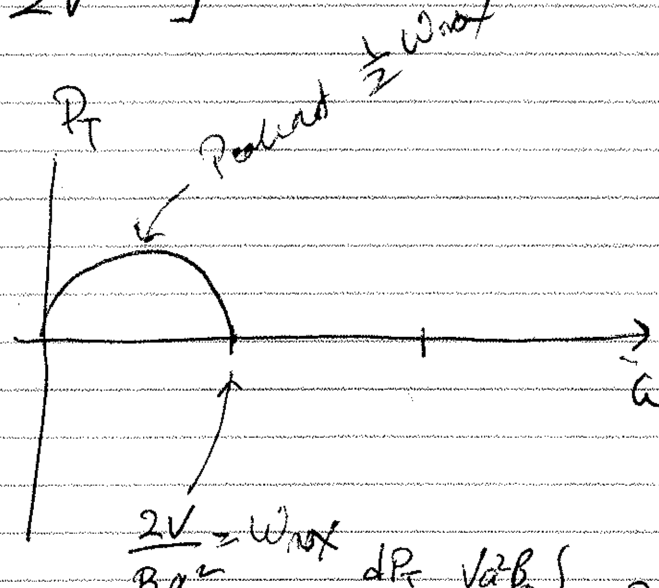
(c) Total torque as a function of ω

$$\vec{N}_T = \vec{N} + \vec{N}_b = \left[-\frac{Va^2B_0}{2R} + \frac{\omega B_0^2 a^4}{4R}\right]\hat{z}$$

$$= -\frac{Va^2B_0}{2R}\left[1 - \frac{B_0 a^2}{2V}\omega\right]\hat{z}$$

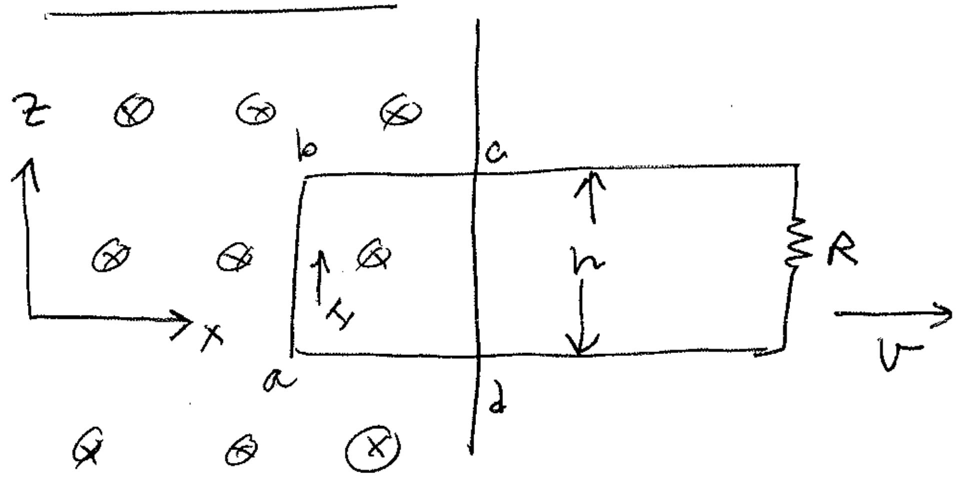
$$P_T = \frac{Va^2B_0}{2R}\left[1 - \frac{B_0 a^2}{2V}\omega\right]\omega$$





$$\frac{dP_T}{d\omega} = \frac{Va^2B_0}{2R}\left\{1 - \frac{B_0a^2}{2V}\omega\right\}$$

Motional Emf



What is the force on the moving loop?

$$\vec{F} = q\vec{v}\times\vec{B} = qvB\,\hat{z}$$

What is the induced emf?

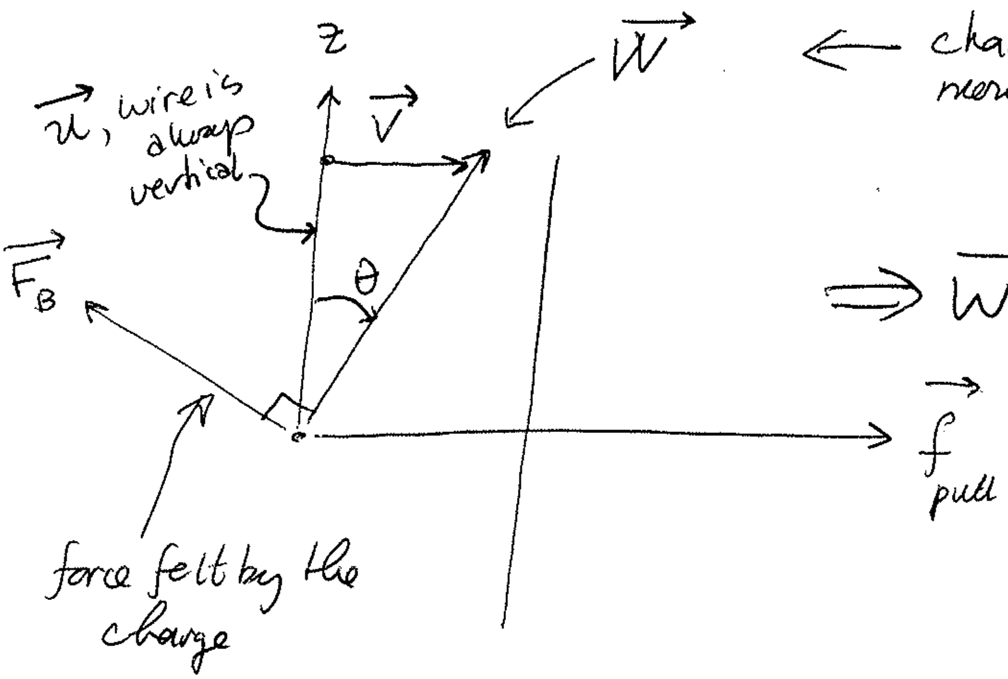
$$\text{Emf} = \oint \vec{f}\cdot d\vec{l} = \oint\left(\frac{\vec{F}}{q}\right)\cdot d\vec{l} = \oint(\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$= Bv\int \hat{z}\cdot d\vec{l} + \cdots$$

$$\boxed{\text{emf} = Bvh}$$

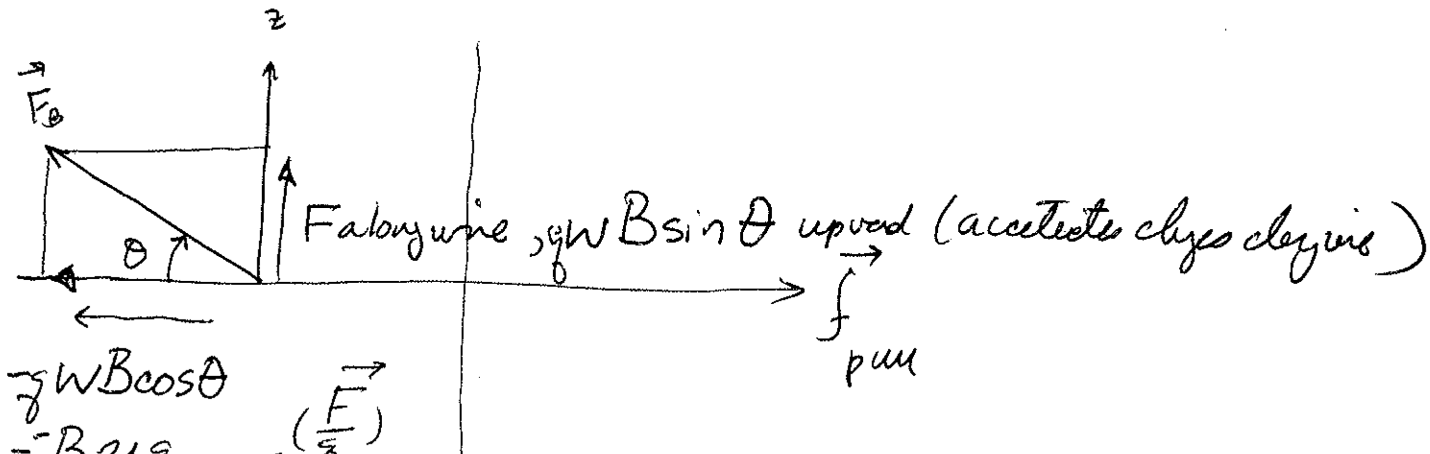
Question: What is the origin of this energy?
$\vec{B}$ fields do no work.

(See page 295-296 in text)



← charges in the wire actually move in this sense.

$$\Longrightarrow \vec{W} = \vec{u} + \vec{v}$$



$\left(\frac{\vec{F}}{q}\right)$ and $f_{pull} = Bu$ or else the wire slows down!

<u>What is the "emf" due to this force?</u>

moves at an $\measuredangle\,\theta$

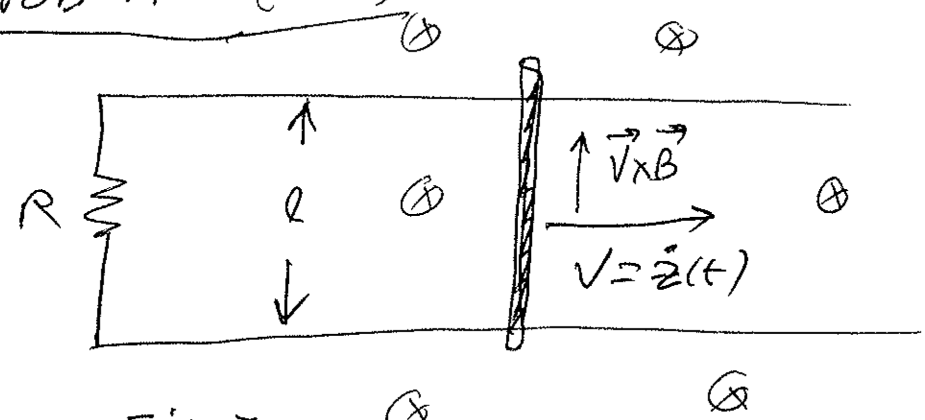
$$\int \vec{f}_{pull} \cdot d\vec{l} = \sin\theta\, Bu\left(\frac{h}{\cos\theta}\right) = \underbrace{u\tan\theta}\, Bh$$

$$= vBh$$

$$\equiv \text{Motional emf}$$

$\Longrightarrow$ whatever pulls the wire, does the work! (it is not the B-field)

Prob 7.7 (HW)



A metal bar of mass m slides on two // rails a distance l apart. A resistor R is connected across the rails and a uniform $\vec{B}$ points into the page. The bar moves to the right at speed v.

a) Find I

$$\mathcal{E}mf = \oint (\vec{V}\times\vec{B})\cdot d\vec{l} = VBl$$

$$\mathcal{E}mf = IR \longrightarrow \boxed{I = \frac{VBl}{R}\ ;\ \text{ccw}}$$

b) $\vec{F}_B = q(\vec{V}\times\vec{B}) \Uparrow$ if $q>0$

c) find $\dot{z}(t)$

$$m_b\ddot{z}(t) = \oint I\,d\vec{l}\times\vec{B} = -IlB$$

$$= -\frac{Bl}{R}\dot{z}\,lB$$

$$= -\frac{B^2l^2}{R}\dot{z}$$

$$\Longrightarrow \ddot{z} + \left(\frac{B^2l^2}{m_bR}\right)\dot{z} = 0$$

$$\boxed{\dot{z}(t) = -\left(\frac{B^2l^2}{m_bR}\right)z(t) + \dot{z}_0}$$

Faraday's Observation, $\mathcal{E} = -\frac{d}{dt}\Phi_B$

and so,

$$\underbrace{\mathcal{E} \equiv \oint \left[\vec{E} + \vec{v}\times\vec{B}\right]\cdot d\vec{r}}_{\text{Motional Emf}} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$$

def^{n} Faraday

time-vary $\vec{B}$

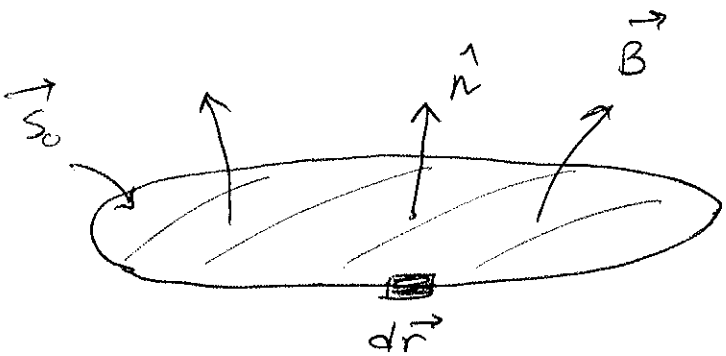
$$= -\left[\frac{d}{dt}\int d\vec{S}\right]\cdot\vec{B} - \int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$$

loop changes $\qquad$ $\vec{B}$ independent of time

$$= -\frac{d}{dt}\Phi_B^{\dot{C}} - \frac{d}{dt}\Phi_B^{\dot{B}}$$

look at Φ_B terms in more detail

(i) $\dot{B} = 0 \implies$ time-independent $\vec{B}$

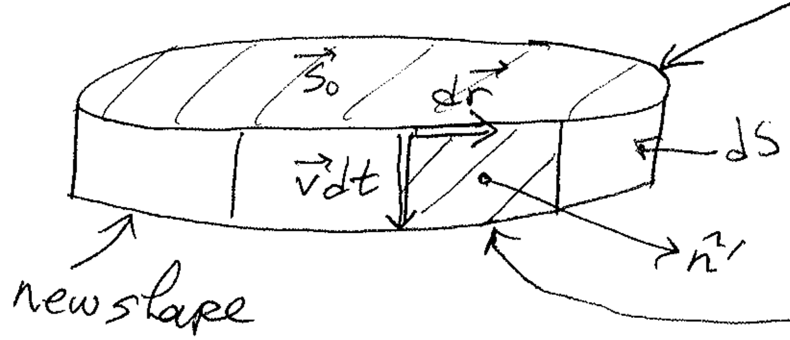


at time $t=0$

$\implies \Phi_B = \int \vec{B}\cdot d\vec{S}$

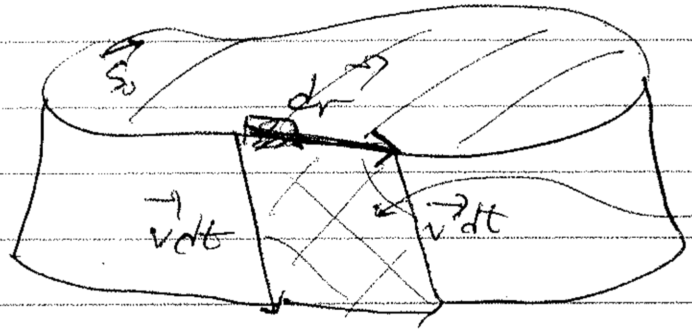
at time $t = t > 0$, if $\vec{v} \neq 0$

shape of circuit at $t=0$



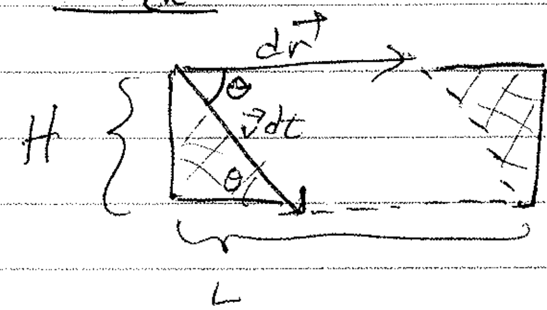
for small dt, new area is $\vec{S}_0 + d\vec{S}$

what is $d\vec{S}$?



we want to see how the area changes because of the motion of the circuit.

Consider



a) make the parallelogram as shown
b) add the triangles to make it a rectangle

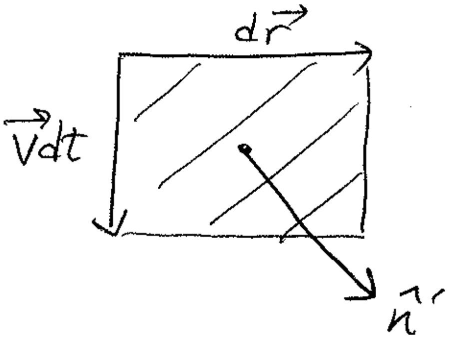
$$\text{Area of Rectangle} = LH = \underbrace{(dr + vdt\cos\theta)}_{L} \times (vdt\sin\theta)$$

= "  " $+ \underset{\uparrow\ 2\text{ triangles}}{2} \left[\underbrace{\tfrac{1}{2}(vdt\cos\theta)\, vdt\sin\theta}_{\text{area of triangle}} \right]$

$\Rightarrow$ "  " $= vdt\,dr\sin\theta$

$$\Rightarrow \boxed{d\vec{S} = \vec{v}dt \times d\vec{r}}$$

The change in the area $d\vec{S}$ is then



$$d\vec{S} = (\vec{v}\,dt \times d\vec{r})$$
$$\hat{n}'\,dS = \vec{v}\,dt \times d\vec{r}$$

The change $d\Phi_B^{\dot{c}}$ is then

$$d\Phi_B^{\dot{c}} = \vec{B}\cdot d\vec{S} = \vec{B}\cdot(\vec{v}\,dt \times d\vec{r})$$

The total change in $\Phi_B^{\dot{c}}$ is then

$$\frac{d\Phi_B^{\dot{c}}}{dt} = \oint \vec{B}\cdot(\vec{v}\times d\vec{r})$$

exchange "dot" and "cross"

$$= \oint (\vec{B}\times\vec{v})\cdot d\vec{r}$$

$$\boxed{\frac{d\Phi_B^{\dot{c}}}{dt} = -\oint (\vec{v}\times\vec{B})\cdot d\vec{r}}$$

$$\frac{\vec{F}_{Lorentz}}{q}$$

Okay, so the emf due to the motion of the circuit is the "work" done by the $(\vec{v}\times\vec{B})$ force!

cancel

Hmm, so this now that

$$\oint \left[\vec{E} + \vec{v}\times\vec{B}\right]\cdot d\vec{r} = -\frac{d\Phi_B^{\dot{c}}}{dt} - \frac{d\Phi_B^{\dot{B}}}{dt}$$

$$\Rightarrow \boxed{\oint \vec{E}\cdot d\vec{r} = -\frac{d\Phi_B^{\dot{B}}}{dt} = -\int\frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}}$$

a time-varying $\vec{B}$-field induces an electric field!

Use Stokes's theorem to convert the line integral to a surface integral,

$$\oint \vec{E}\cdot d\vec{r} = \int(\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\int\frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

$$\int\left[\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right]\cdot d\vec{S} = 0$$

for arbitrary $d\vec{S}$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Faraday's law in differential form for time varying fields

Currently Maxwell Equations

$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

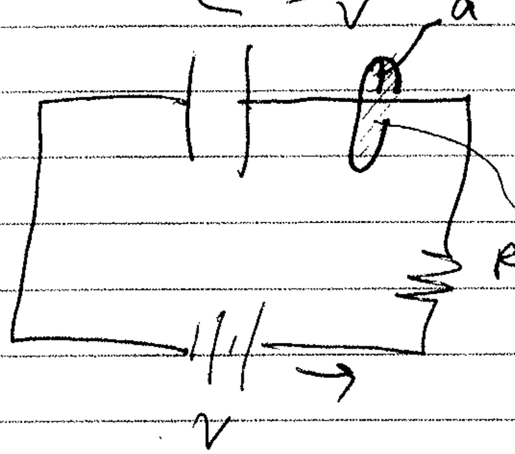
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Ampere's law law has issues as is easily seen from simple experiment and "Bonafide" (theory)

$\boxed{\text{Ampere's law: } \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}}$

Further

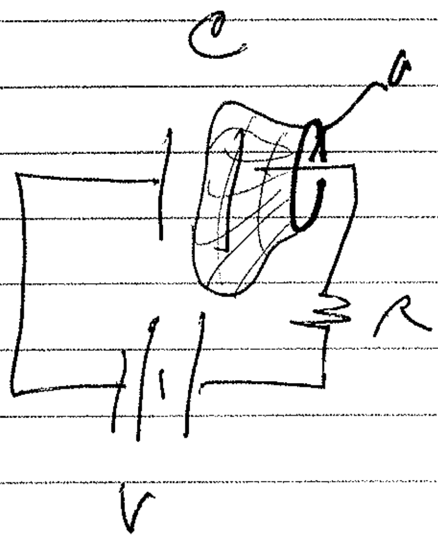
$C = \frac{Q}{V}$ $\Rightarrow V = \frac{Q}{C} + IR$



$$\mathcal{E}_b = \frac{Q}{C} + R\frac{dQ}{dt}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S} = \mu_0 I$$

for surface as shown



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{S} = 0 \;!$$

for surface as shown

$\Rightarrow$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ has issues

$$\Rightarrow \vec{\nabla}\cdot\left[\vec{\nabla}\times\vec{B}-\mu_0\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\right]=0$$

or $$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\underbrace{\epsilon_0\frac{\partial\vec{E}}{\partial t}}_{\text{displacement current}}$$

Maxwell Equations

$$\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Suppose $g \equiv$ magnetic charge exists

$$\Rightarrow \vec{\nabla}\cdot\vec{B}=\mu_0\rho_m$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})=-\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B})=0$$

b/c $\vec{\nabla}\perp(\vec{\nabla}\times\vec{E})$

but $\vec{\nabla}\cdot\vec{B}=\mu_0\rho_m \Rightarrow -\frac{\partial}{\partial t}\left[\mu_0\rho_m\right]=+\vec{\nabla}\cdot\vec{J}_m$

current of magnetic charge

magnetic charge conservation, $\frac{\partial}{\partial t}\rho_m+\vec{\nabla}\cdot\vec{J}_m=0$

to fix Faraday Law $\Rightarrow$ add $\boxed{\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}-\mu_0\vec{J}_m}$

Maxwell

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right) \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$0 = -\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B} = 0 \checkmark$$; Faraday's law is good

(b) $$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = \mu_0\,\vec{\nabla}\cdot\vec{J} = \mu_0\left(-\frac{\partial\rho}{\partial t}\right) \neq 0$$

if ρ varies w/time

$\Rightarrow$ a fix is needed. — we added this to $\mu_0\vec{\nabla}\cdot\vec{J}$

Suppose, $$\mu_0\left[\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}\right] = 0$$

Gauss's law

$$= \mu_0\left[\vec{\nabla}\cdot\vec{J} + \epsilon_0\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}\right] = 0$$

$$\mu_0\,\vec{\nabla}\cdot\left[\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

add to Faraday's law

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]}$$

displacement current

Correct?

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

take

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$= -\frac{\partial}{\partial t}\left[\mu_0\underbrace{\cancel{\vec{J}}}_{0,\text{ in free space}} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

$$\vec{\nabla}\big(\underbrace{\cancel{\vec{\nabla}\cdot\vec{E}}}_{\rho/\epsilon_0\text{ ; }0,\text{ in free space}}\big) - \nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

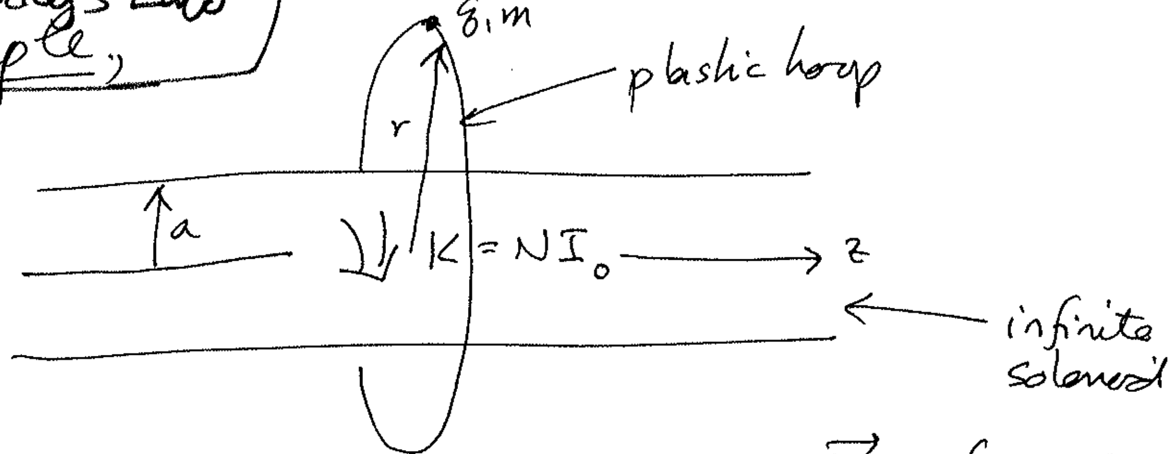
$$\Rightarrow \boxed{\nabla^2\vec{E} - \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0}$$

(and a similar wave equation for $\vec{B}$)

$$\mu_0\epsilon_0 = \frac{1}{c^2} \Rightarrow \text{EM radiation is "light"}$$

(or vice versa)

Faraday's Law Example,



$$\Phi_B = (\mu_0 N I_0 \pi a^2)$$

$$\vec{B} = \hat{z}\begin{cases} \mu_0 N I_0 & s < a \\ 0 & s > a \end{cases}$$

Turn-off current

(i) $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\mu_0 N \pi a^2 \frac{dI}{dt} = E_\phi 2\pi r$

Emf is induced by the changing I

(ii) The torque exert on q is

$$\vec{N} = \vec{r} \times \vec{F}$$

$$= r q E_\phi \hat{\phi} \quad \text{rotation around } z\text{-axis}$$

(i) & (ii) $\Longrightarrow$ $|\vec{N}| = rq\left[-\frac{\mu_0 N \pi a^2 \frac{dI}{dt}}{2\pi r}\right]$

$$N = \frac{dL}{dt} = -q\left(\frac{\mu_0 N a^2}{2}\right)\frac{dI}{dt}$$

integrate over time,

limits

$$L = -\frac{1}{2}(\mu_0 N a^2 q)\left[I(\infty) - I(0)\right]$$

$$= \frac{1}{2}\mu_0 N I_0 a^2 q$$

a) steady-state $\longrightarrow$ steady-state

b) int. states are lithical

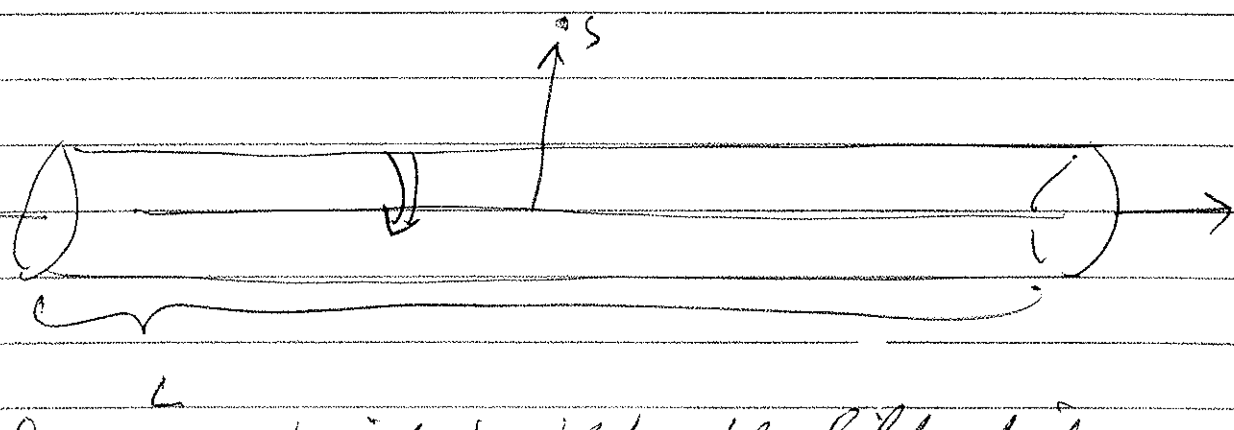
$$L = |\vec{r} \times \vec{p}| = m r v_\phi = \frac{1}{2}\mu_0 N I_0 a^2 q$$

$$\boxed{v_\phi = \frac{1}{2}\frac{\mu_0 N I_0 a^2 q}{m r}}$$

We made an assumption here. Took the idea that I changed instantaneously. This, strictly speaking, is not correct. This is referred to as the

"quasi-static approximation"

When does this break down?



For a finite solenoid, length L, the field at its center on axis is

$$B_z^L = \frac{\mu_0}{2}\frac{N I L}{\sqrt{R^2 + L^2/4}}$$

For an infinite solenoid,

$$B_z^\infty = \mu_0 N I$$

so, to be arbitrary, let's say that if $L \geq L_0$,

$$B_z^L \geq 0.9\, B_z^\infty$$

$$\frac{B_z^L}{B_z^\infty} = \frac{\frac{\cancel{\mu_0}}{2}NIL}{\sqrt{R^2+L^2/4}}\,\frac{1}{\cancel{\mu_0 NI}} = \frac{L}{2\sqrt{R^2+\frac{L^2}{4}}} = 0.9$$

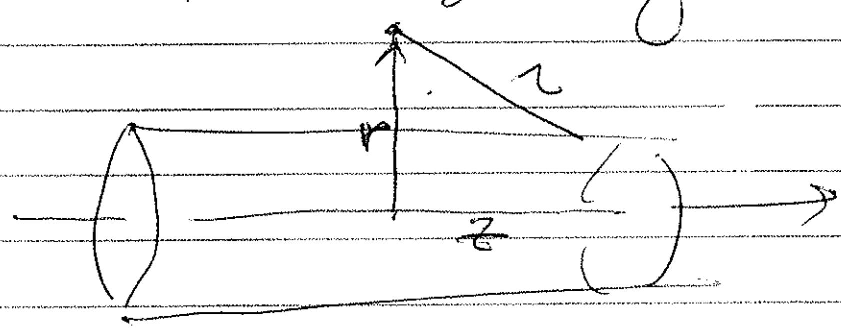
$$\Rightarrow \frac{L^2}{4} = 0.9^2\left(R^2+\frac{L^2}{4}\right)$$

$$0.19\left(\frac{L^2}{4}\right) = 0.81R^2$$

$$\Rightarrow \left(\frac{L}{R}\right) \simeq 4$$

if the length, L, is roughly, $4\times R$, then the field is ~90% of the infinite solenoid.

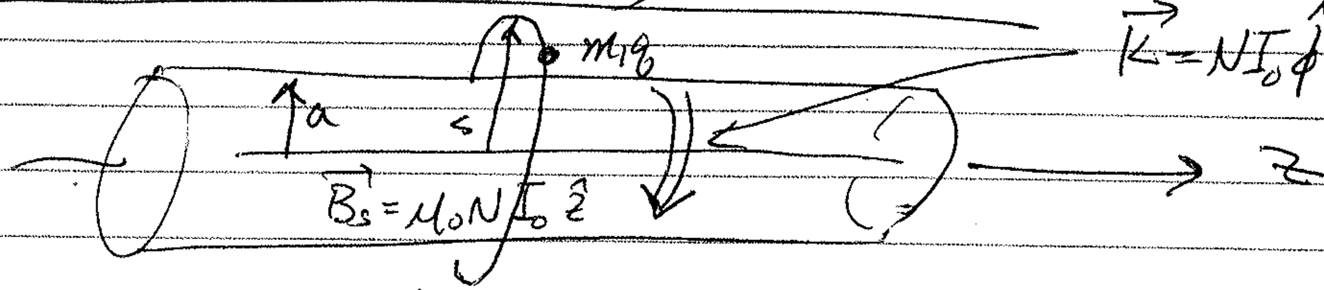
So, as long as the lag between center & $L \sim 4R$ is small, then change is quasi-static



$$\tau = \frac{\mathscr{r}}{c} - \frac{r}{c} \approx \frac{\sqrt{r^2+z^2}}{c} - \frac{r}{c} \sim \frac{z}{c} - \frac{r}{c} + \frac{r^2}{2cz}$$

if $\tau \ll$ time over which I changes, then we're ~~good~~

Consider the same problem, but use $\vec{A}$



a) what is $\vec{A}$ outside of the solenoid?

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \int \vec{B} \cdot d\vec{S} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$$

$$= \oint \vec{A} \cdot d\vec{l}$$

and so,

$$\mu_0 N I_0 \pi a^2 = A_\phi 2\pi s$$

$$\Rightarrow \boxed{\vec{A}_\phi = \frac{\mu_0 N I_0 a^2}{2s} \hat{\phi}}$$

b) Cut the current $\Rightarrow I \to 0 \Rightarrow A_\phi \to 0 \to \mathcal{E}$

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \Phi_B$$

$$= -\frac{d}{dt} \int \vec{B} \cdot d\vec{S}$$

but

$$= -\frac{d}{dt} \oint \vec{A} \cdot d\vec{l}$$

$$\Rightarrow E_\phi 2\pi s = -\frac{d}{dt}\left(A_\phi 2\pi s\right)$$

and so, $\boxed{E_\phi = -\frac{dA_\phi}{dt}}$ * (see next page

c) $m(s\ddot{\phi}) = qE_\phi$

$$= -q\frac{d}{dt}\left(\frac{\mu_0 N I a^2}{2s}\right)$$

Integrate over time

$$ms\dot{\phi}\Big|_0^{\dot{\phi}} = -\frac{\mu_0 N a^2 q}{2s}\left(\overset{0}{I(\infty)} - I_0\right)$$

$$mV_\phi = \frac{\mu_0 N a^2 q}{2s} I_0$$

$$\boxed{V_\phi = \frac{q}{m}\left(\frac{\mu_0 a^2 N I_0}{2s}\right)}$$ ✓

Comment

(i) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$; Faraday's law

if $\frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \vec{\nabla} \times \vec{E} = 0$ and

$\vec{E} = -\vec{\nabla} V$, but

$\vec{\nabla} \times \vec{E} \neq 0$, so what's up?

(ii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Well $\vec{\nabla} \cdot \vec{B} = 0$ (still)

$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ is still true

$$= -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right)$$

$\Rightarrow \vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0$ is true

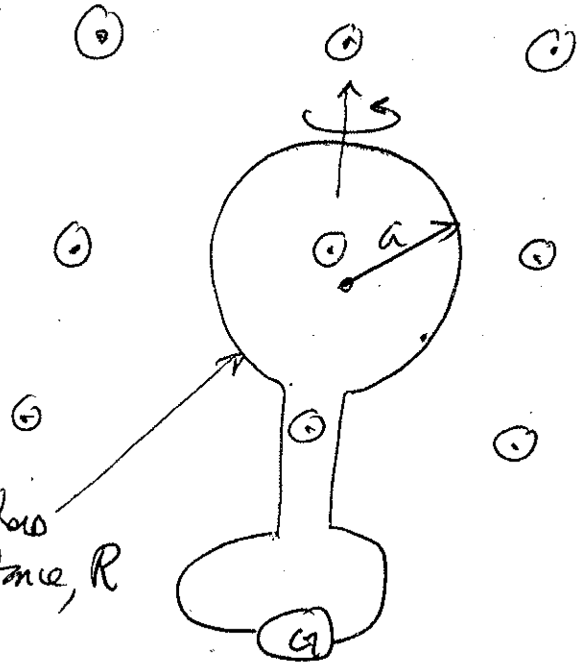
scalar function (again)

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

and $\boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}}$

Device to Measure $\vec{B}$

Flip Coil (measurement of $\vec{B}$)



loop has resistance, R

a) loop sits in plane of paper and $\vec{B}$ is normal to the paper.

b) coil is then rotated 180° reversing the flux through the loop

as Φ_B changes, we see an

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

at the galvanometer.

So,

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$= +IR$$ ← Voltage drop around loop of resistance R

$$= R\frac{dQ(t)}{dt}$$ ← charge (net charge) that has flowed through the Galvanometer at time t.

$$\Longrightarrow R\frac{dQ}{dt} = -\frac{d}{dt}\Phi_B$$

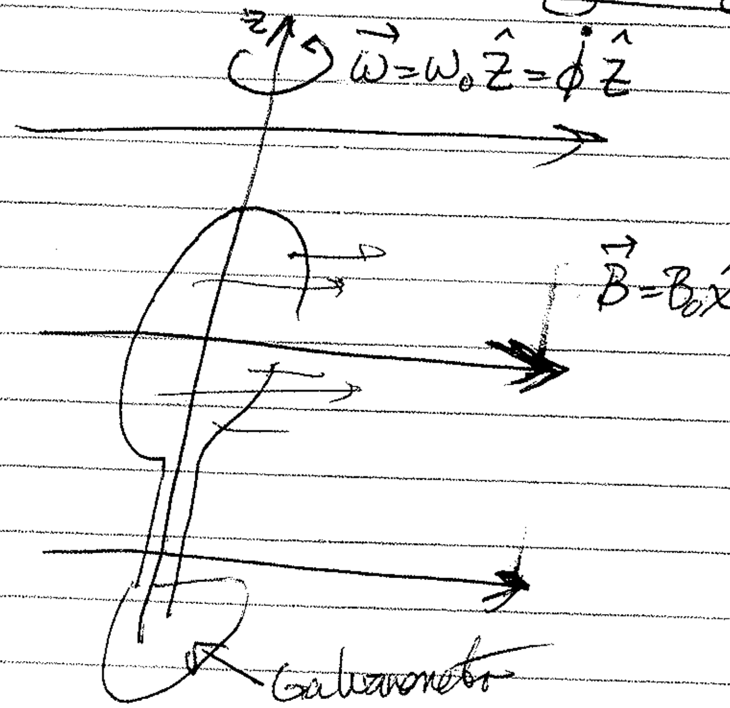
Integrate over time from $t = 0 \longrightarrow \infty$

$$R\left[Q(\infty) - Q(0)\right] = -\Phi_B(\infty) + \Phi_B(0)$$

$$RQ(\infty) = +\pi a^2 B + \pi a^2 B$$

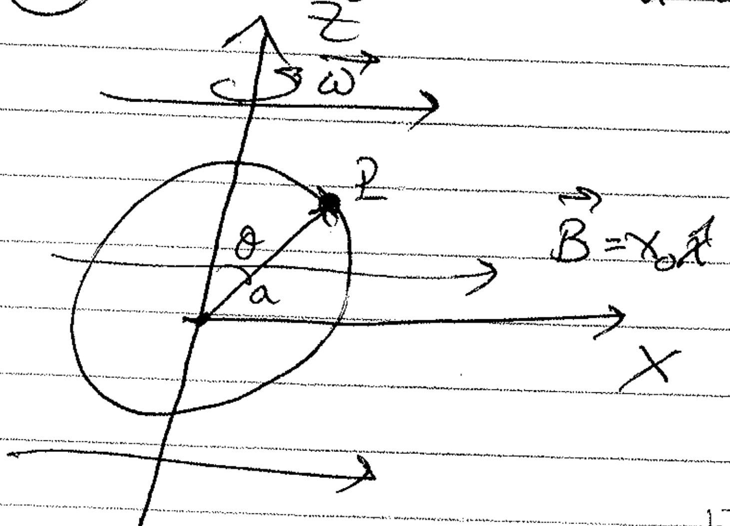
$$\Longrightarrow \boxed{B = \frac{RQ}{2\pi a^2}}$$ ← find $\vec{B}$ from measurement of Q

Redo the flip coil using Lorentz force law



We will solve
$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l}$$
around coil at a given time t ($\Rightarrow \phi(t)$) and then find $\int \mathcal{E}(t)\,dt$ to get flow of charge

(a) Side and let coil lie in x-z plane



Point P has coordinates

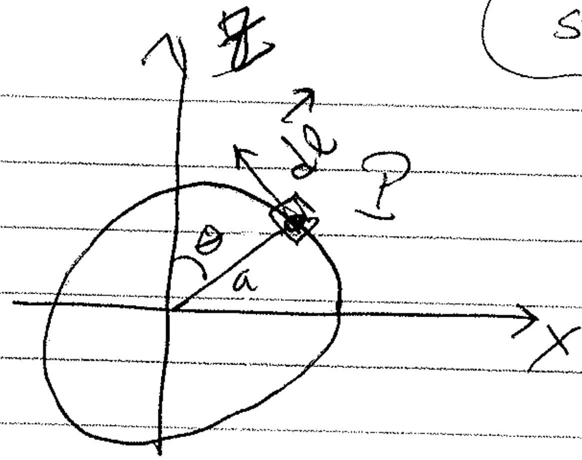
$$\vec{r}_p = (a \sin\theta \cos\phi,\ a \sin\theta \sin\phi,\ a\cos\theta)$$
$$\vec{\omega} = (0, 0, \omega_0)$$

$$\Rightarrow \boxed{\vec{v}_p = \vec{\omega} \times \vec{r}_p = (-\omega_0 a \sin\theta \sin\phi,\ \omega_0 a \sin\theta \cos\phi,\ 0)}$$

$$\Rightarrow \vec{f} = \vec{v}_p \times \vec{B} = (-\omega_0 a \sin\theta \sin\phi,\ \omega_0 a \sin\theta \cos\phi,\ 0) \times (B_0, 0, 0)$$

$$\vec{f} = (0, 0, -\omega_0 a \sin\theta \cos\phi\, B_0)$$

see next page



$\vec{dl} = a\,d\theta\,(-\hat{\theta})$

moves towards z-axis

$\hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$

$$\Rightarrow \boxed{\vec{dl} = -a\,d\theta\,(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)}$$

$$\Rightarrow \vec{f}\cdot\vec{dl} = (0,0,-\omega_0 a\sin\theta\cos\phi B_0)\cdot(-a\,d\theta)(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$= a\,d\theta\,\omega_0 a B_0\,(0,0,\sin^2\theta\cos\phi)$$

$$\mathcal{E} = \oint \vec{f}\cdot\vec{dl}$$

$$= -\omega_0 a^2 B_0 \oint \sin^2\theta\cos\phi\,d\theta$$

$$= -\omega_0 a^2 B_0 \pi\cos\phi$$

$$\boxed{\mathcal{E} = -(B_0\pi a^2)\,\dot{\phi}\cos\phi}$$

$$\dot{\phi} = \frac{d\phi}{dt}$$

now, $\mathcal{E} = IR = R\dfrac{dQ}{dt} = -B_0\pi a^2\dot{\phi}\cos\phi$

integrate from $t=0$ when $\phi = \frac{\pi}{2}$ to

$t=\infty$ when $\phi = \frac{3\pi}{2}$

$$\Rightarrow R[Q(\infty)-Q(0)] = -B_0\pi a^2\left[\sin\left(\tfrac{3\pi}{2}\right)-\sin\left(\tfrac{\pi}{2}\right)\right]$$

$$\Rightarrow RQ(\alpha) = B_0 \pi a^2 2$$

$$\boxed{B_0 = \frac{RQ(\alpha)}{2\pi a^2}}$$

as before

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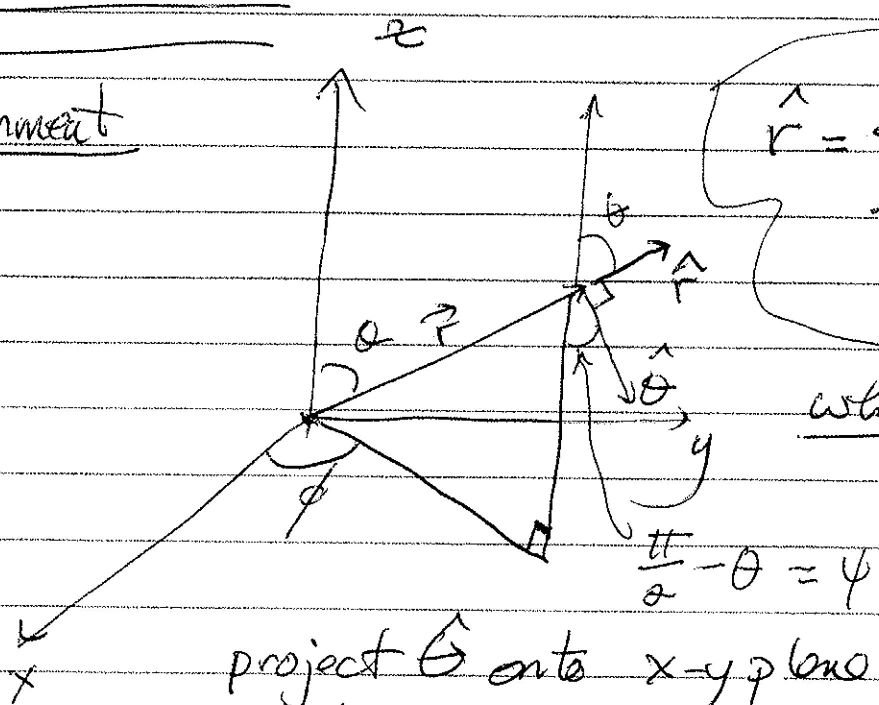
* Comment



$$\frac{\pi}{2} - \theta = \psi$$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

what is $\hat{\theta}$?

project $\hat{\theta}$ onto x-y plane and take $(\cos\phi, \sin\phi)$

$$\hat{\theta} = \sin\left(\frac{\pi}{2}-\theta\right)\cos\phi\,\hat{x} + \sin\left(\frac{\pi}{2}-\theta\right)\sin\phi\,\hat{y} + \cos\left(\frac{\pi}{2}-\theta\right)\hat{z}$$

$$\boxed{\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}}$$

Inductance

this is good,

Hmmm, but we ignored induced currents (and the induced fields). Is this okay?

Let's see, if we change I by $dI \Longrightarrow$ we get some dI

Suppose we write,

Inductance $\Rightarrow$ $\boxed{d\Phi_B^{ind} \propto dI = L\, dI}$

self-inductance { return to this later }

We then have,

$$d\Phi_B = d\Phi_B^{c} + d\Phi_B^{ind}$$

$$= d\Phi_B^{c} + L\,dI$$

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = \underbrace{-\frac{d\Phi_B^{c}}{dt} - L\frac{dI}{dt} = R\frac{dQ}{dt}}_{\text{integrate over time}}$$

$$-\left(\Phi_B(\infty) - \Phi_B(0)\right) - L\left(\cancel{I(\infty) - I(0)}^{\,0}\right)$$

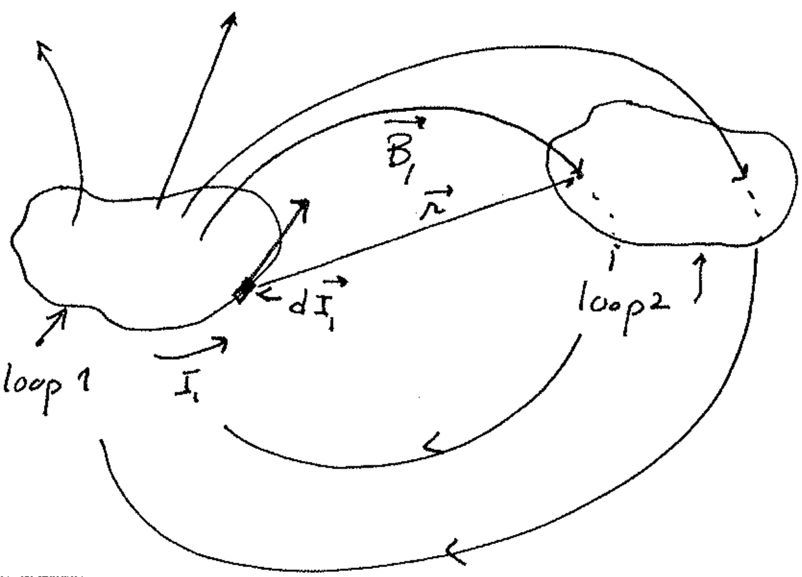
$$= R\left(Q[\infty] - Q[0]\right)$$

and we get the same result. Self-inductance affects how the system charges, but once we hit steady-state, it goes away.

Let's look at inductance in more detail

Inductance

When talking about generators, we had to consider back emf's, i.e., $\mathcal{E}$ drives current which generates a Φ_B opposite to the change in $\Phi_{B,\text{original}}$. We introduced inductance to account for back emf.



Mutual Inductance

(i) $\vec{B}_1 = \frac{\mu_0}{4\pi}\oint \frac{d\vec{I}_1 \times \hat{r}}{r^2} \Rightarrow \Phi_2 = \int \vec{B}_1 \cdot d\vec{S}_2$

$\vec{B}_1 \propto \vec{I}_1 \Rightarrow \Phi_2 \propto I_1$ and we define

$$\Phi_2 = M_{21} I_1$$

M_{21}: mutual inductance

(ii) M_{21} seems complex, but let's consider some manipulations of M_{21}.

Show that $M_{21} = M_{12}$

$$\Phi_2 = \int \vec{B}_1 \cdot d\vec{S}_2$$

$$= \int (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{S}_2$$

Stokes's theorem

$$\rightarrow \Phi_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$$

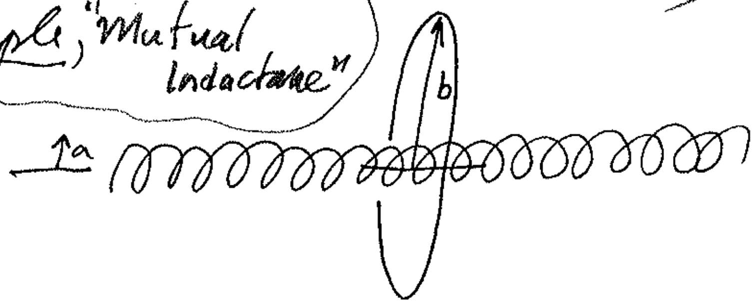
$$= \oint \left[\frac{\mu_0}{4\pi} \oint \frac{I_1 d\vec{l}_1}{r} \right] \cdot d\vec{l}_2$$

$$= \frac{\mu_0 I_1}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$$

$$\Longrightarrow M_{21} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$$ is symmetric in "1" and "2" ← Neumann formula

$$\Longrightarrow M_{21} = M_{12} \, !$$ ← Mutual Inductance

Example, "Mutual Inductance"



← Infinite solenoid; (N, I)

Current I flows in the loop, what is the Φ_B through the solenoid?

$$\Phi_{sol} = M_{sol,loop} \, I_{loop}$$

↑ exceedingly complex ⟨ find $\vec{B}_{loop}$ and then do integral $\oint \vec{B}_{loop} \cdot d\vec{S}_{sol}$ for every loop!

$\vec{B}_\ell$ is difficult and we need integrate flux each turn of the solenoid!

$$\Phi_s = \int \vec{B}_\ell \cdot d\vec{S}_s = M_{se} I_\ell$$

b) $\Phi_\ell = \int \vec{B}_s \cdot d\vec{S}_\ell$ ⟵ Use symmetry

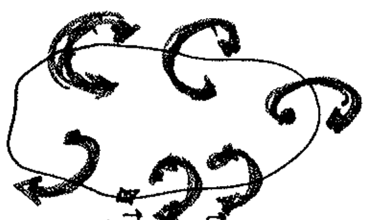
$$= \mu_0 N I_s \pi a^2 \Rightarrow \Phi_\ell = M_{\ell s} I_s(t)$$

$$\boxed{M_{\ell s} = \mu_0 N \pi a^2 = M_{\ell s}}$$

$$\Rightarrow \boxed{\Phi_{sol} = (\mu_0 N)\pi a^2 I_{loop}}$$

take advantage of the fact that $M_{21} = M_{12}$ and use the flux through loop to find $M_{sol,\,loop}$ $= \oint \vec{B}_s \cdot d\vec{S}_{loop}$ to get $M_{sol,\,loop}$ and then $M_{loop,\,sol}$

Self-Inductance



$d\vec{I} = I d\vec{\ell}$

Current makes $\vec{B}$ that threads loop.

If I varies $\Rightarrow \Phi_B$ through loop varies

$\Rightarrow \mathcal{E}$ opposing change in set-up

$\Rightarrow$ Self induction

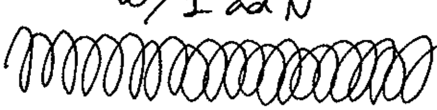
self-inductance

$$d\Phi = L\, dI$$

$$\Rightarrow \mathcal{E} = -L \frac{dI}{dt}$$

In practice, this is a difficult proposition.

infinite solenoid w/ I and N



I a z

and $\Phi_B = \int \underbrace{(\mu_0 N I \pi a^2)}_{\text{flux thru 1 loop}} \times \underbrace{N dz}_{\text{\# of loops in } dz}$

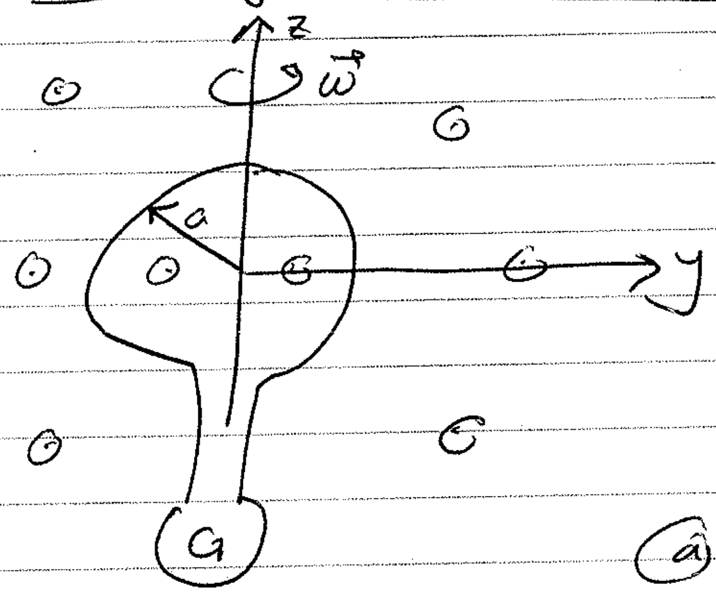
$$\Rightarrow \vec{B} = \mu_0 N I \hat{z}$$

$$\Phi_B = \mu_0 I \pi a^2 N^2 l_z = L I$$

some length along
z

$$\Longrightarrow \frac{L}{l_z} = \left(\mu_0 N^2 \pi a^2\right) \equiv \text{inductance per unit length}$$

Alternating Current (AC) Generator



(i) $\vec{\omega} = \omega_0 \hat{z}$

(ii) $\vec{B} = B_0 \hat{x}$

Set-up the problem

(a) $\mathcal{E}_s = ?$

$\boxed{\mathcal{E}}$

a) $\mathcal{E} = \oint \vec{f}\cdot d\vec{l} = -\dfrac{d\Phi}{dt}$

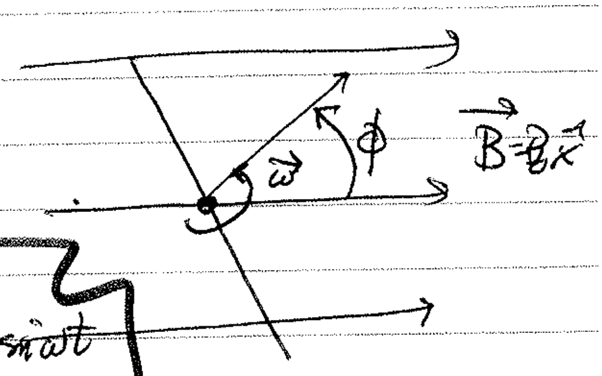
$$\mathcal{E}_F = -\frac{d\Phi_B^c}{dt}, \quad \mathcal{E}_L = -\left[L\frac{dI}{dt}\right], \quad \mathcal{E}_R = -IR$$

no driving voltage

$$\Rightarrow 0 = \mathcal{E}_F + \mathcal{E}_L + \mathcal{E}_R$$

find $\mathcal{E}_F = -\dfrac{d\Phi_B^c}{dt}$

Top



$$\Phi_B = (B_0 \pi a^2)\cos\phi$$

$$= \Phi_0 \cos(\omega t) \Rightarrow \boxed{\frac{d\Phi}{dt} = -\omega\Phi_0 \sin\omega t}$$

$$\text{Solve } \underbrace{L\frac{dI}{dt}}_{-\mathcal{E}_L} + \underbrace{RI}_{-\mathcal{E}_R} = \underbrace{\omega\Phi_0 \sin\omega t}_{\mathcal{E}_F} \quad \Big\} \text{ from } 0 = \mathcal{E}_F + \mathcal{E}_i$$

Solve the homogeneous part ($=0$) and then the particular part (steady-state)

a) Homogeneous

$$L\frac{dI}{dt} + RI = 0 \Rightarrow \frac{1}{I}\frac{dI}{dt} = -\frac{R}{L}$$

$$\ln I \Big|_{I(0)}^{I} = -\frac{R}{L}(t-0)$$

$$\boxed{I = I_0 e^{-Rt/L}}$$

b) Particular (Steady-state)

$$L\frac{dI}{dt} + RI = \omega\Phi_0 \sin\omega t$$

Set $I = A\sin(\omega t + \varphi)$

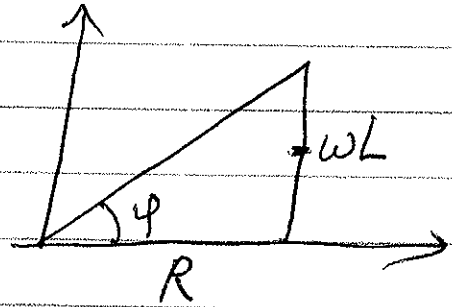
$$\omega L A\cos(\omega t+\varphi) + RA\sin(\omega t+\varphi) = \omega\Phi_0 \sin\omega t$$

find φ and A

at $t=0$, $\omega L A\cos(\varphi) + RA\sin(\varphi) = 0$

$$\Rightarrow \boxed{\tan\varphi = -\frac{\omega L}{R}}$$

at $\omega t = \pi/2$, $\omega L A \underbrace{\cos\left(\frac{\pi}{2}+\varphi\right)}_{-\sin\varphi} + RA\underbrace{\sin\left(\frac{\pi}{2}+\varphi\right)}_{\cos\varphi} = \omega$…



$$\cos\varphi = \frac{R}{\sqrt{\omega^2L^2+R^2}}, \quad \sin\varphi = -\frac{\omega L}{\sqrt{\omega^2L^2+R^2}}$$

$$\Rightarrow \omega L A\left(\frac{\omega L}{\sqrt{\omega^2L^2+R^2}}\right) + RA\left(\frac{R}{\sqrt{R^2+\omega^2L^2}}\right) = \omega\Phi_0$$

$$A\left[\sqrt{R^2+\omega^2L^2}\right] = \omega\Phi_0 \Rightarrow \boxed{A = \frac{\omega\Phi_0}{\sqrt{R^2+\omega^2L^2}}}$$

$$\Rightarrow \boxed{I(t) = I_0 e^{-Rt/L} + \frac{\omega\Phi_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t+\varphi)}$$

$\Uparrow$

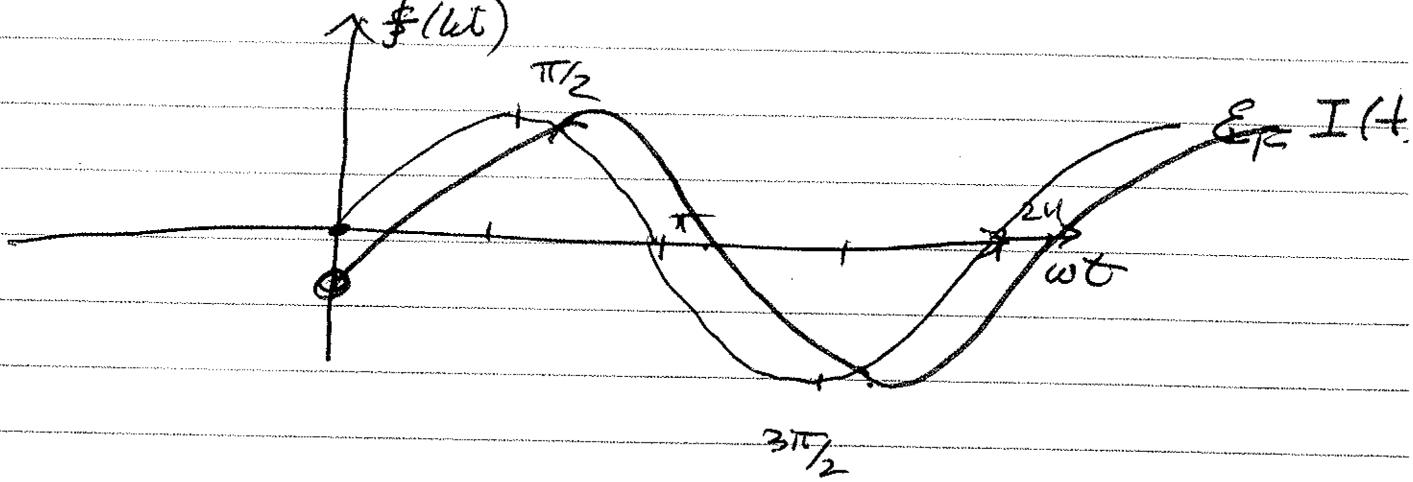
$\mathcal{E}_F$ and $I(t)$ are out of phase

$\sin(\omega t)$ $\quad$ $\sin(\omega t+\varphi)$

$\tan\varphi = -\frac{\omega L}{R}$

$\Rightarrow$ current lags the $\mathcal{E}_F$.

$\sin(\omega t)$ vs $\sin(\omega t - \varphi_0)$



$$\Phi_0 = B\pi a^2$$

$$\left\{\begin{aligned} \mathcal{E} &= \Phi_0 \omega \sin(\omega t) \\ I &= \frac{\omega \Phi_0}{R\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \sin\Big(\omega t - \underbrace{\tan^{-1}\frac{\omega L}{R}}_{\phi_\omega}\Big) \end{aligned}\right.$$

$\leftarrow$ In steady-state

Determine the power output

$$P = IV = I^2 R = \frac{\Phi_0^2 \omega^2}{R^2\left(1+\frac{\omega^2 L^2}{R^2}\right)} \sin^2(\omega t - \phi_\omega) \times R$$

$$\langle P \rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{\omega^2 B^2 \pi^2 a^4}{R^2\left(1+\frac{\omega^2 L^2}{R^2}\right)} \times R \sin^2(\omega t - \phi_\omega)\,dt$$

average over 1 cycle

$$= \frac{\omega^3 B^2 a^4 \pi}{2R\left(1+\frac{\omega^2 L^2}{R^2}\right)}\int_0^{2\pi/\omega} \sin^2(\omega t - \phi_\omega)\,dt$$

let $u = \omega t - \phi_\omega \Rightarrow du = \omega dt$

$$= \frac{\omega^2 B^2 a^4 \pi}{2R\left(1+\frac{\omega^2 L^2}{R^2}\right)}\int_{-\phi_\omega}^{2\pi - \phi_\omega} \sin^2 u\,du$$

$$\boxed{\langle P \rangle = \frac{(\pi \omega B a^2)^2}{2R\left(1+\frac{\omega^2 L^2}{R^2}\right)}}$$

recall: $\boxed{\frac{(\omega B a^2)^2}{4R} = P_{HG}}$

differs by $\frac{2\pi^2}{(1+\omega^2 L^2/R^2)}$

alternatively,

$$\boxed{L\frac{dI}{dt} + RI = \omega\Phi_0 \sin\omega t}$$

Consider $\tilde{I}(t) = \tilde{I}_0 e^{i(\omega t)}$, and driving force $\omega\Phi_0 e^{i\omega t}$

use only imaginary part of this result

$$L i\omega \tilde{I}_0 e^{i\omega t} + R\tilde{I}_0 e^{i\omega t} = \omega\Phi_0 e^{i\omega t}$$

(i) Solve homogeneous equation

$$\left[i\omega L\tilde{I}_0 + R\tilde{I}_0\right] = 0$$

$$\tilde{I}_0\left[i\omega L + R\right] \Rightarrow i\omega L = -R,\ \omega = -\frac{R}{iL} = i\frac{R}{L}$$

and $\tilde{I}_h(t) = \tilde{I}_0 e^{-Rt/L}$ $\longrightarrow$ damping transient

(ii) Solve particular solution

$$i\omega L\tilde{I}_0 + R\tilde{I}_0 = \omega\Phi_0$$

$$\tilde{I}_0 = \frac{\omega\Phi_0}{i\omega L + R} = \frac{\omega\Phi_0}{(\omega L)^2 + R^2}(R - i\omega L)$$

$$\Rightarrow I_{0,R} = \frac{\omega\Phi_0 R}{(\omega L)^2 + R^2},\ I_{0,I} = -\frac{\omega^2 L\Phi_0}{(\omega L)^2 + R^2}$$

$$\Rightarrow \tilde{I}_0 = I_{0,R} + i I_{0,I} = \sqrt{I_{0,R}^2 + I_{0,I}^2}\, e^{i \tan^{-1}\left(\frac{I_{0,I}}{I_{0,R}}\right)}$$

and our sol^n becomes

$$\tilde{I}(t) = \sqrt{I_{0,R}^2 + I_{0,I}^2}\, e^{i\left(\omega t + \tan^{-1}\frac{I_{0,I}}{I_{0,R}}\right)}$$

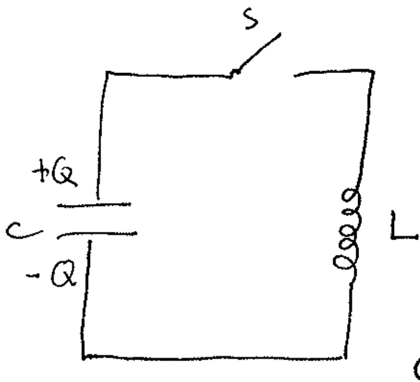
we want imaginary part of solution

$$\Rightarrow I(t) = \sqrt{\frac{(\omega \mathcal{E}_0 R)^2}{[(\omega L)^2 + R^2]^2} + \frac{(\omega^2 \mathcal{E}_0 L)^2}{[(\omega L)^2 + R^2]^2}}\, \sin\left(\omega t + \tan^{-1}\left(-\frac{\omega L}{R}\right)\right)$$

$$\boxed{I(t) = \frac{(\omega \mathcal{E}_0)}{\sqrt{(\omega L)^2 + R^2}} \sin\left(\omega t - \tan^{-1}\left[\frac{\omega L}{R}\right]\right)}$$

RLC Circuits

(HW)
Prob 7.25



C is charged to V and then S is closed connecting C to an inductor w/ L.

Determine $I(t)$

$$\mathcal{E}_L = -\underbrace{\frac{d}{dt}\Phi_B} = -\frac{d}{dt}(LI)$$

as $Q\downarrow, I\uparrow$

(i) $\boxed{C = Q/V \; , \; \mathcal{E}_L = -L\frac{dI}{dt} \;\text{ and }\; V + \mathcal{E}_L = 0}$

$$\Rightarrow -L\frac{dI}{dt} + \frac{Q}{C} = 0 \rightarrow L\frac{d^2Q}{dt} + \frac{Q}{C} = 0 \;, \; I = -\frac{dQ}{dt}$$

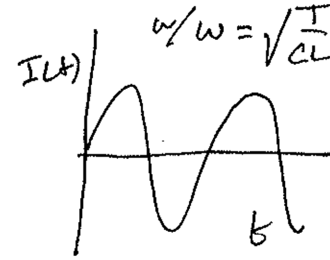
let $Q = Q_0 e^{at} \Rightarrow La^2 Q + \frac{1}{C}Q = 0 \Rightarrow a = \pm i\sqrt{\frac{1}{LC}}$

and $Q = Q_0 \exp\left(\pm i\sqrt{\frac{1}{LC}}\,t\right)$

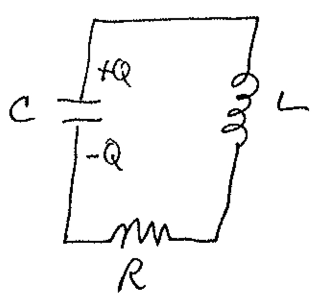
for finite charge

$$\Rightarrow \boxed{I(t) = +Q_0\sqrt{\frac{1}{CL}}\exp\left(\pm i\sqrt{\frac{1}{CL}}\,t\right)} \longrightarrow \text{current oscillates}$$

w/ $\omega = \sqrt{\frac{1}{CL}}$



Suppose we add a resistor, R



the voltage drop around the circuit must be 0

$$\Rightarrow \frac{Q}{C} - L\frac{dI}{dt} - IR = 0 \;, \; I = -\frac{dQ}{dt} \;, \text{ as } Q\downarrow, I\uparrow$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

let: $Q = Ae^{at}$

$$La^2Q + RaQ + \frac{1}{C}Q = 0$$

$$\Rightarrow a^2 + \frac{R}{L}a + \frac{1}{LC} = 0 \Rightarrow a = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= -\frac{R}{2L}\left(1 \mp \sqrt{1 - \frac{4L}{CR^2}}\right)$$

for physical solution, $a = -\frac{R}{2L} - \frac{R}{2L}\sqrt{1 - \frac{4L}{R^2C}}$

$$I(t) = -\frac{dQ}{dt}$$

$$= Q_0\frac{R}{2L}\left(1 + \sqrt{1 - \frac{4L}{R^2C}}\right)\exp\left(-\frac{R}{2L}\left[1 + \sqrt{1 - \frac{4L}{R^2C}}\right]t\right)$$