Energy in the Magnetic field

It requires energy to start current flowing in a circuit; that is, it requires energy to overcome the "back-Emf," to get things started (the self-inductance just mentioned). This is not lost; it is recoverable. It comes back when you turn I off, representing energy stored in the field. (also LC circuit oscillates, E is transferred to and from $\vec{B}$ as I oscillates)

We showed

$$P = IV \text{ earlier}$$

as the work delivered to a system, but now we want the work done (power) delivered to overcome the back-Emf.

overcome $\mathcal{E}$

$$\Rightarrow \frac{dW}{dt} = I(-\mathcal{E}) = IL\frac{dI}{dt}$$

charge flows at this rate

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

$$\Rightarrow \boxed{W = \frac{1}{2}LI^2}$$

✓

Let's expand on this a little, ~~of that~~ but consider that, we have

$$\boxed{P = \int \vec{J} \cdot \vec{E}\, d\tau \equiv \text{Rate at which work is performed on charges in some volume } V \text{ by EM fields}}$$

so that

$$P = \int \left(\frac{\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}{\mu_0} \right) \cdot \vec{E}\, d\tau$$

ID #6

$$\vec{E} \cdot (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \cdot (\vec{E} \times \vec{B}) + \vec{B} \cdot (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow P = \frac{1}{\mu_0} \int \left\{ \vec{\nabla} \cdot (\vec{E} \times \vec{B}) + \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} \right\} d\tau$$

$$= \frac{1}{\mu_0} \int \left\{ \vec{\nabla} \cdot (\vec{E} \times \vec{B}) + \vec{B} \cdot \left[-\frac{\partial \vec{B}}{\partial t} \right] - \epsilon_0 \mu_0 \frac{\partial \vec{E} \cdot \vec{E}}{2 \partial t} \right\} d\tau$$

$$= \frac{1}{\mu_0} \int \left\{ \vec{\nabla} \cdot (\vec{E} \times \vec{B}) - \frac{1}{2} \frac{\partial}{\partial t} \left[\vec{B} \cdot \vec{B} + \epsilon_0 \mu_0 \vec{E} \cdot \vec{E} \right] \right\} d\tau$$

$$= \underbrace{\frac{1}{\mu_0} \oint (\vec{E} \times \vec{B}) \cdot d\vec{S}} - \frac{\partial}{\partial t} \left[\frac{1}{2\mu_0} \int \vec{B} \cdot \vec{B}\, d\tau + \frac{\epsilon_0}{2} \int \vec{E} \cdot \vec{E}\, d\tau \right]$$

let boundary surface $\to \infty$

and $\int \vec{E} \times \vec{B} \cdot d\vec{s} \to 0$

note: $\boxed{\frac{1}{\mu_0} \int (\vec{E} \times \vec{B}) \cdot d\vec{S}}$ Poynting flux (Poynting Vector) $\left(\frac{\vec{E} \times \vec{B}}{\mu_0} \right)$

$$\boxed{P = -\frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{S} - \frac{\partial}{\partial t}\left[\frac{1}{2\mu_0}\int\vec{B}\cdot\vec{B}\,d\tau + \frac{\epsilon_0}{2}\int\vec{E}\cdot\vec{E}\,d\tau\right]}$$

This is known as Poynting's Theorem; it is the expression of the conservation of energy in the EM field.

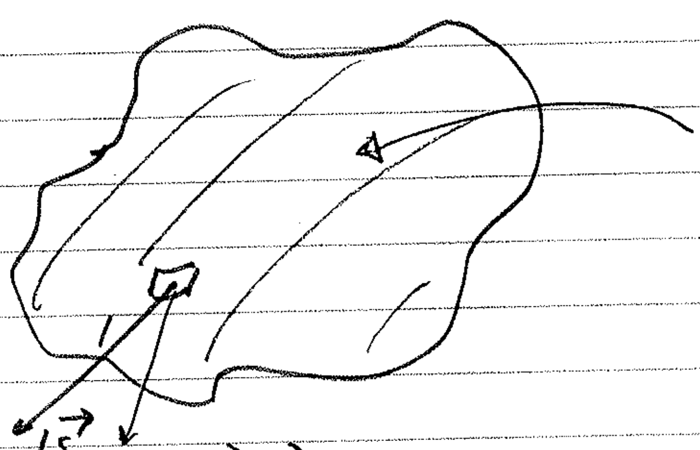
note: we see (define) $U_{EM} = \frac{1}{2\mu_0}\int\vec{B}\cdot\vec{B}\,d\tau + \frac{\epsilon_0}{2}\int\vec{E}\cdot\vec{E}\,d\tau$

$$\Rightarrow P = -\frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{S} - \frac{\partial}{\partial t}U_{EM}$$

this is the work on charges moving through an EM field $\Rightarrow$ mechanical energy of particles

and

we see that $\frac{1}{\mu_0}(\vec{E}\times\vec{B})$ is the energy flux carried by the EM field across surface $\vec{S}$.



$U_{em} = \int\frac{B^2}{2\mu_0}\,d\tau + \int\frac{\epsilon_0 E^2}{2}\,d\tau$

$\frac{1}{\mu_0}\vec{E}\times\vec{B}$ = Poynting flux, Poynting vector

$$P = -\frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{S} - \frac{\partial}{\partial t}U_{em}$$

work on a charge as it moves through the EM field. Work goes into changing the kinetic energy of the particles

$$\Rightarrow P = \frac{\partial}{\partial t}\int u_{mech}\, d\tau = \frac{\partial}{\partial t}U_{mech}$$

and so,

Energy equation

$$\boxed{\frac{\partial}{\partial t}\Big[\underbrace{U_{mech} + U_{em}}_{\text{field total energy + mechanical energy of charges}}\Big] = -\frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{S}} = -\frac{1}{\mu_0}\int \vec{\nabla}\cdot\left[\vec{E}\times\vec{B}\right] d\tau$$

$$\frac{\partial}{\partial t}\left[\int\left(u_{mech} + u_{em}\right)d\tau\right] = -\frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d\tau$$

$$\int\left\{\frac{\partial}{\partial t}\left(u_{mech} + u_{em}\right) + \vec{\nabla}\cdot\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)\right\}d\tau = 0$$

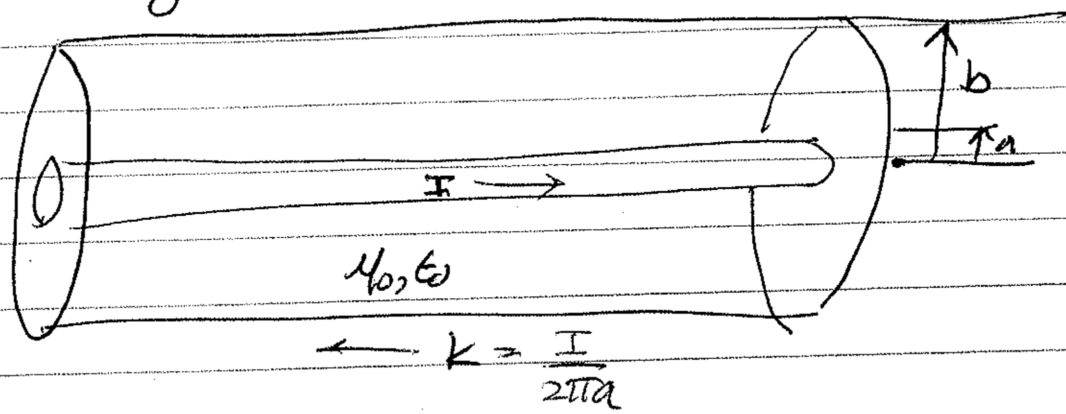
$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(u_{mech} + u_{em}\right) + \vec{\nabla}\cdot\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right) = 0}$$

recall the continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

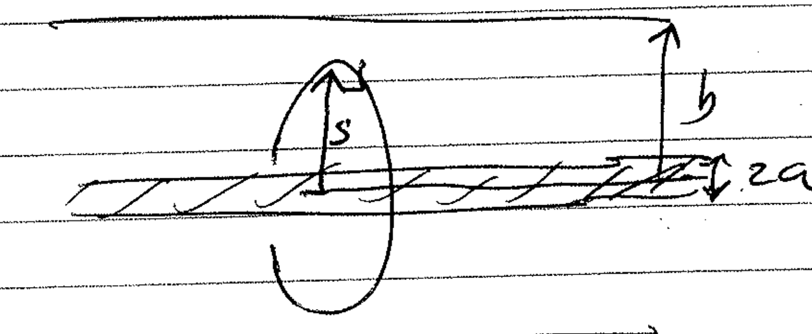
Example

a long coaxial cable carries current I



a) Find $\vec{B}$ everywhere

$a < s < b$



$B_\phi 2\pi s = I\mu_0$

$$\Rightarrow \quad B_\phi = \begin{cases} 0 & 0 < s < a \\ \mu_0 I / 2\pi s & a < s < b \\ 0 & b < s \end{cases}$$

b) find U_{em}:

$$U_{em} = \frac{1}{2\mu_0} \int B_\phi^2 \, ds\, dz\, s\, d\phi$$

$$= \frac{1}{2\mu_0} \left(\frac{\mu_0 I}{2\pi} \right)^2 \int \frac{1}{s^2} s\, ds\, d\phi\, dz$$

$$= \frac{\mu_0 I^2}{4\pi} dz \ln s \Big|_a^b$$

$$= \frac{1}{2} L I^2 \Rightarrow \frac{L}{dz} = \frac{\mu_0 \ln \frac{b}{a}}{2\pi}$$

c) $Ld\,\Phi_B = \int_a^b \vec{B}\cdot d\vec{S}$

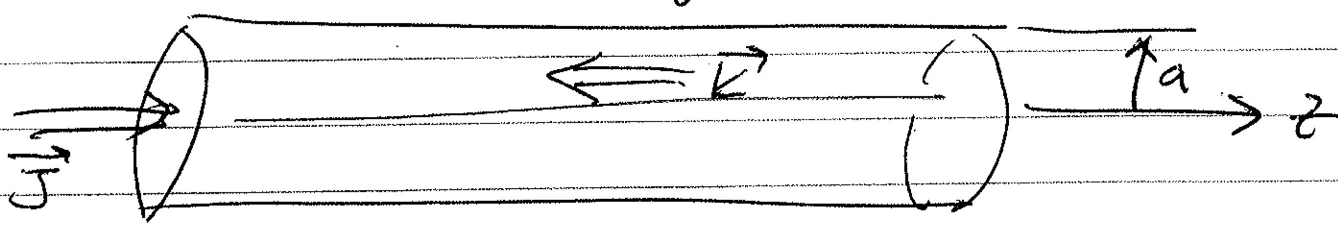
$$= \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{s}\,ds\,dz$$

$$= \frac{\mu_0 I}{2\pi}\,dz\,\ln\left(\frac{b}{a}\right)$$

$$= LI \Rightarrow \frac{L}{dz} = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$$

Prob 7.28

long cable of radius a



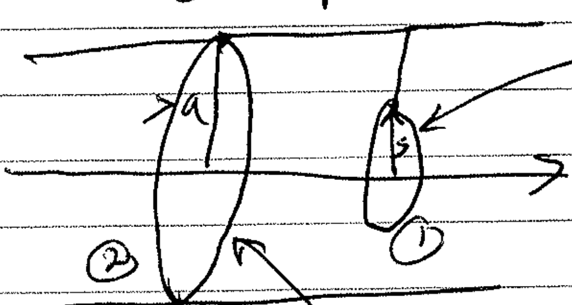
a) find L for the cable

(i) We will use $W = \frac{1}{2} L I^2$ to get L.

(ii) $\vec{J} = \frac{I}{\pi a^2}\hat{z}$

$\vec{K} = -\frac{I}{2\pi a}\hat{z}$

(iii) Using Ampere's law



$B_\phi 2\pi s = \mu_0 \frac{I}{\pi a^2}\pi s^2$

$\vec{B}_\phi = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}, \; s<a$

$$B_\phi = \frac{\mu_0\left[\frac{I}{\pi a^2}\underbrace{\pi a^2}_{\text{volume}} - \frac{I}{2\pi a}\underbrace{2\pi a}_{\text{surface}}\right]}{2\pi s}$$

$\vec{B}_\phi = 0 \quad s>a$

find $U_{em} = \frac{1}{2\mu_0}\int B^2 d\tau$

$$= \frac{1}{2\mu_0}\int\left(\frac{\mu_0 I}{2\pi a^2}\right)^2 s^2 s\,d\phi\,ds\,dz$$

$$= \frac{1}{\cancel{2}\mu_0}\left[\frac{\mu_0^2 I^2}{4\pi^2 a^4}\cancel{2}\pi\right]\int s^3 ds\,dz$$

$$= \frac{\mu_0 I^2}{4\pi \cancel{a^4}}\left(\frac{\cancel{a^4}}{4}\right)\int dz$$

$$\boxed{U_{em} = \frac{\mu_0 I^2}{16\pi}\int dz}$$

$$W = U_{em} = \frac{1}{2}LI^2$$

$$\Rightarrow \boxed{L = \frac{2U_{em}}{I^2} = \frac{\mu_0 \int dz}{8\pi}}$$

depends on geometry, like C

b) What is the Poynting Vector?

$$\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = \frac{I}{\sigma \pi a^2} \hat{z}$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = -\frac{1}{\cancel{\mu_0}} \cancel{\mu_0} \frac{I s}{2\pi a^2} \frac{I}{\sigma \pi a^2} \hat{s}$$

$$\vec{S} = -\frac{I^2}{2\pi^2 a^4 \sigma} \vec{s}$$

5/29/00

Transformer

led to dominance of AC over DC because

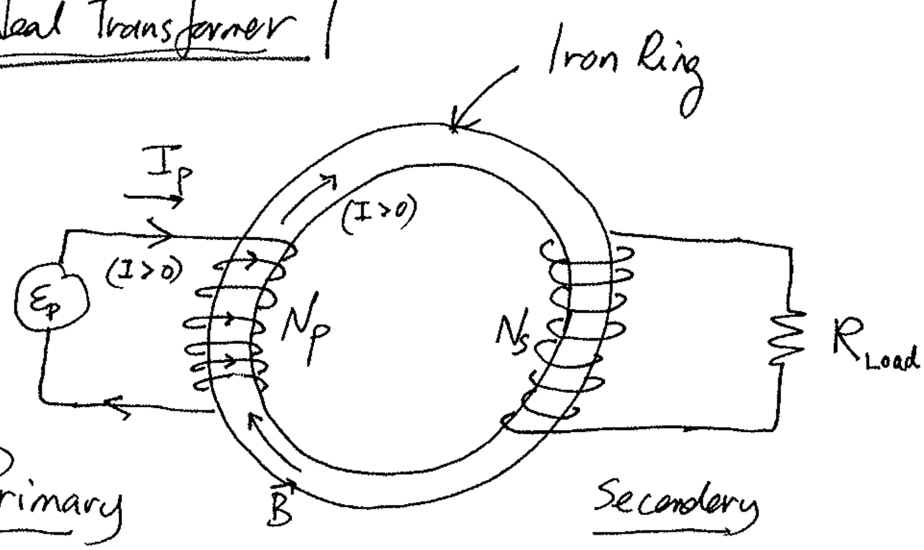
$P = IV \Rightarrow$ if $V\uparrow$ $I\downarrow$ to maintain P but then

Joule Heating $= I^2 R_{line} \to$ Joule Heating $\downarrow$ — AC transformers are easy to make

↳ easy to make step-down step-up transformers

(Generators) few 10^3 V → Transformer (step-up) → send 10^5 V → Transported → Transformer (step-down) → few 10^3 V → Homes (actually local transformers step voltage down to ~110 V)

Ideal Transformer



We make to find out how I_p induces $\mathcal{E}_s \to I_s$

→ we need

$$\Phi_s = M_{sp} I_p$$

"No leakage"

~~from our solenoid example~~

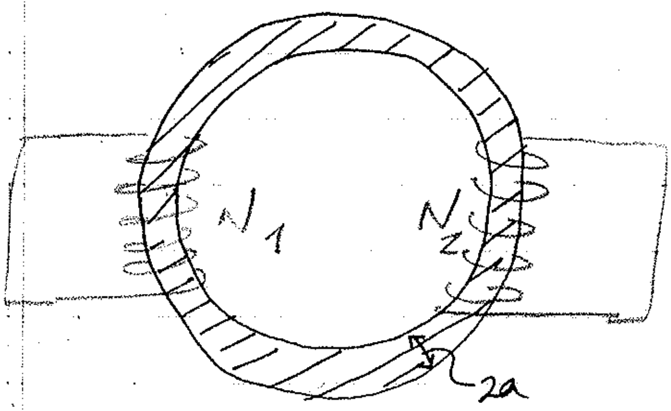
~~inductance per turn~~

$$\cancel{L_p = L_0 N_p^2, \quad L_{sec} = L_0 N_s^2}$$

$$\cancel{M = L_0 N_p N_s}$$

~~Show this~~

Mutual Inductance & Self Inductance



find Mutual Inductance

a) for a "long" solenoid, $\vec{B}_1 = \mu_0 N_1 I_1 \hat{\phi} \longrightarrow \Phi_2 = \underbrace{\mu_0 N_1 I_1 \pi a^2}_{\vec{B}_1 \cdot d\vec{S}_2} \underbrace{N_2}_{\#\text{ of loops}}$

$$\Rightarrow \boxed{M_{21} = \underbrace{[\mu_0 N_1 N_2][\pi a^2]}_{L_0 N_1 N_2} = M_{12} \equiv M}$$

b) what is self-induction?

$$\Phi_1 = \mu_0 N_1^2 I_1 \pi a^2 \Rightarrow \boxed{L = (\mu_0 N_1^2 \pi a^2) = L_0 N_1^2}$$

~~c) let's form~~ $\cancel{L_1 L_2 = (\mu_0 \pi a^2)^2 N_1^2 N_2^2}$

$\cancel{\rightarrow \boxed{M^2 = L_1 L_2}}$

~~multiply~~ $\cancel{L_p \times L_s = L_0^2 N_p^2 N_s^2}$

~~and~~

$\cancel{L_0^2 N_p^2 N_s^2 = M^2 \Longrightarrow M^2 = L_p L_s}$

Now, the $\mathcal{E}$ drops around each circuit must vanish,

Circuit 1 (Primary)

$$\mathcal{E}_p(t) = \overbrace{L_p \frac{dI_p}{dt}}^{\text{self-induction}} - \overbrace{M \frac{dI_s}{dt}}^{\text{induction from sec. coil}} \qquad (1)$$

Circuit 2 (Secondary)

$$0 = L_s \frac{dI_s}{dt} - M \frac{dI_p}{dt} + \overbrace{I_s R}^{\text{"load"}} \qquad (2)$$

(sign determined by the way I drew the coils)

$\boxed{V_p(t) \propto \cos(\omega t + \phi_p)\text{; but let's solve the corresponding complex problem}}$

Drive Circuit 1 w/ $\mathcal{E}_p(t) = \mathcal{E}_p e^{-i\omega t}$ (an AC current)

$$\Longrightarrow I_p(t) = I_p \underbrace{e^{-i\omega t}} \quad \text{and} \quad I_s(t) = I_s \underbrace{e^{-i\omega t}}$$

same time dependence

and so,

coeff. of (1) & (2) have no explicit dependence on time t

$$\begin{cases} \mathcal{E}_p = -i\omega L_p I_p + i\omega M I_s & (1') \\ 0 = -i\omega L_s I_s + i\omega M I_p + R I_s & (2') \end{cases}$$

2nd Eqn. $\Longrightarrow$ $I_p = \dfrac{i\omega L_s - R}{i\omega M} I_s$ (3)

Plug into $\mathcal{E}_p$ Equation (Equation 1')

$$\mathcal{E}_p = -i\omega L_p \left(\frac{i\omega L_s + R}{i\omega M}\right) I_s + i\omega M I_s$$

$$= \left(-i\omega M + \frac{L_p}{M} R + i\omega M\right) I_s$$

$$= + \frac{L_p}{M} \underbrace{I_s R}$$

$$= \frac{L_p}{M} \mathcal{E}_s \longleftarrow \boxed{\mathcal{E}_s = I_s R}$$

$$= \frac{L_0 N_p^2}{L_0 N_p N_s} \mathcal{E}_s \Longrightarrow \boxed{\frac{\mathcal{E}_p}{\mathcal{E}_s} = \frac{N_p}{N_s}}$$

if $N_p > N_s \Rightarrow \mathcal{E}_s \downarrow$ (step-down transformer)

Recall: $$\frac{I_p}{I_s} = \frac{i\omega L_s + R}{i\omega M} = \frac{L_s}{M} + i\frac{R}{\omega M}$$

if $R \ll \omega M$ (small resistance in secondary coil),

$$\boxed{\frac{I_p}{I_s} \simeq \frac{L_s}{M} = \frac{N_s}{N_p}}$$

"Maxwell's Equations"

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad\qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Are we done? No, why?

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

(i) take divergence of

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0$$ as it should!

(ii) take divergence of $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot\vec{J} \neq 0$$!!! unless $\frac{\partial\rho}{\partial t} = 0$

How do we fix this up?

$$\mu_0\vec{\nabla}\cdot\vec{J} = -\mu_0\frac{\partial\rho}{\partial t} = -\mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}$$

$$\longrightarrow \vec{\nabla}\cdot\left[\mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right] = 0$$

So, we can fix this up by adding $\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ to the Ampere Law, and

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \underbrace{\epsilon_0\frac{\partial\vec{E}}{\partial t}}_{\text{displacement current}}\right]}$$

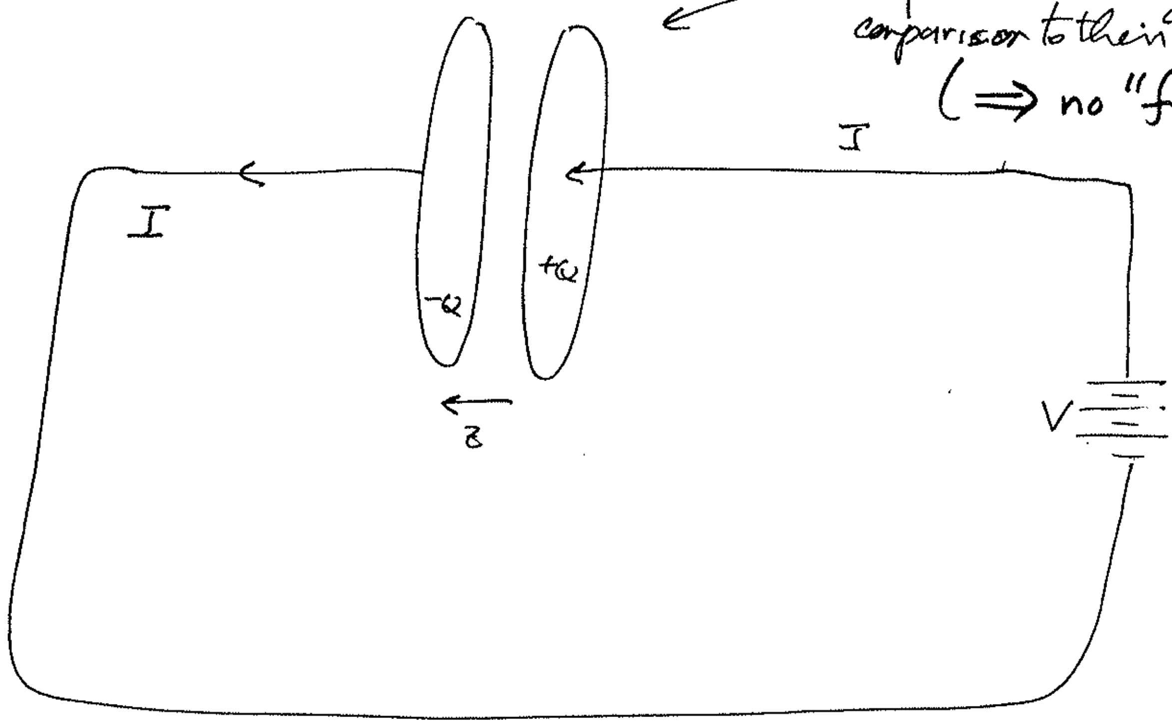
and so (finally),

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0\dfrac{\partial\vec{E}}{\partial t}\right] \end{array}}$$

What is the role of the Displacement Current?

Consider a // plate capacitor charged by a constant current I, as shown below

the plates are large in comparison to their separation ($\Rightarrow$ no "fringing")



Between the plates, we have

$$\vec{E} \approx \frac{Q}{\epsilon_0 A}\hat{z} \quad \left(\text{comes from } \oint \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_0} \right)$$

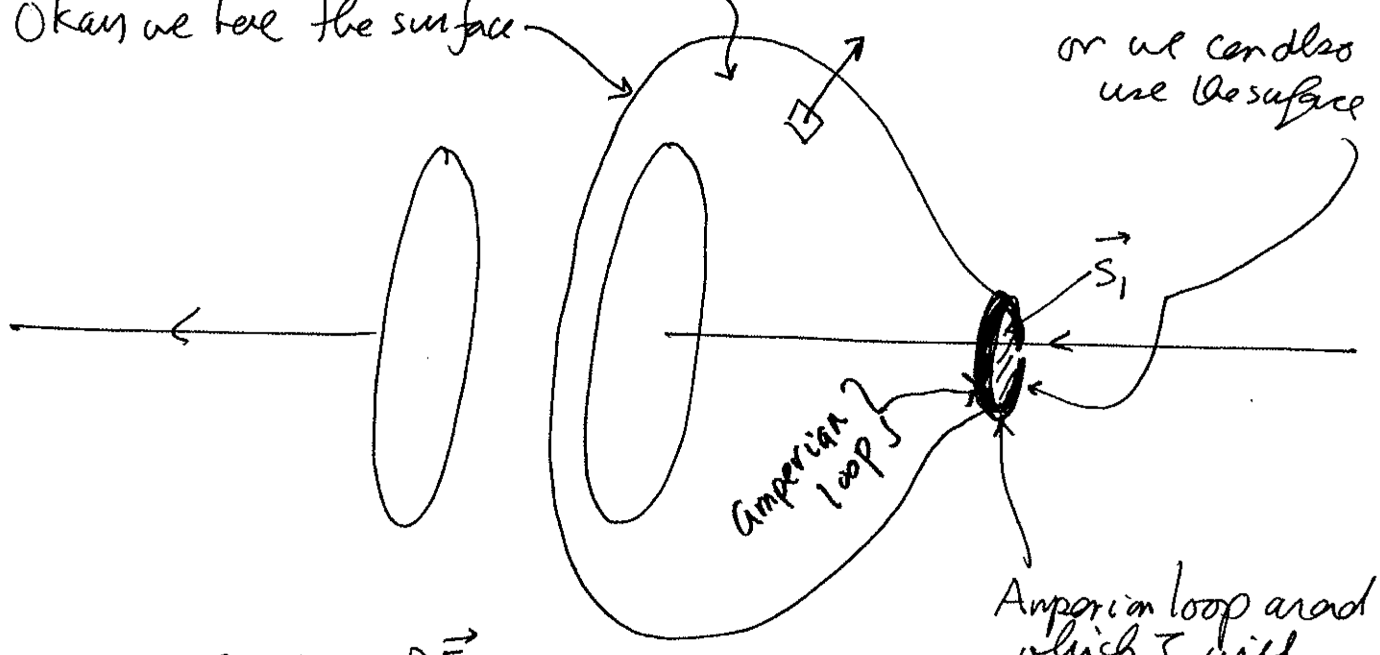
Okay,

$$\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

$$\rightarrow \int(\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \mu_0\int\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{S}$$

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0\int\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{S}}$$

Okay we have the surface



a) $\mu_0 \int_{S_1}\left(\vec{J}+\epsilon_0\underbrace{\frac{\partial\vec{E}}{\partial t}}_{\to\, 0 \text{ for steady } I \text{ in wire}}\right)\cdot d\vec{S} = \mu_0 I$

b) $\mu_0 \int_{S_2}\left(\underbrace{\vec{J}}_{\to\, 0 \text{ in between plates!}}+\epsilon_0\underbrace{\frac{\partial\vec{E}}{\partial t}}_{\to\, \neq 0 \text{ between capacitor plates because } \sigma(t) \neq \text{constant}}\right)\cdot d\vec{S} = \mu_0\int\epsilon_0\frac{\partial\vec{E}}{\partial t}\cdot d\vec{S}_2$

Dirac Quantization Condition (1931)

if monopoles exist, (comes from angular momentum quantization)

$$\underset{\text{charge}}{q}\,\underset{\text{magnetic charge}}{g} = \underset{\text{integer}}{n}\frac{h}{\mu_0} \Longrightarrow \vec{B} = \frac{\mu_0}{4\pi}\frac{g\hat{r}}{r^2}$$

now, we know that $q = |e|$ except for quarks

$$\Rightarrow g = n\left(\frac{h}{\mu_0 e}\right) = 3.29\times10^{-9}\ \text{A-m}$$

Maxwell's Equations become

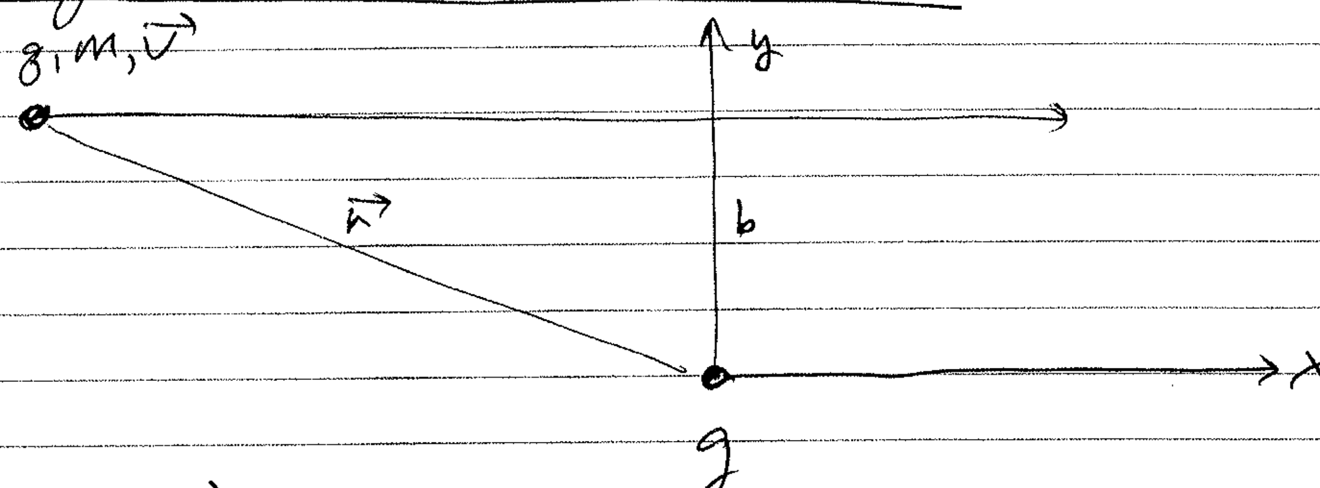
$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = \mu_0\rho_m$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \mu_0\vec{J}_m \qquad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\cdot\vec{B}}_{\mu_0\rho_m} - \mu_0\vec{\nabla}\cdot\vec{J}_m = 0$$

$$\Rightarrow \mu_0\left[\frac{\partial}{\partial t}\rho_m + \vec{\nabla}\cdot\vec{J}_m\right] = 0 \ \checkmark$$

Angular Momentum & Charge Quantization



force: $\vec{F}_L = q\vec{v} \times \left(\frac{\mu_0}{4\pi} \frac{g\hat{r}}{r^2} \right)$

① assume g is stationary, and q is nearly undeflected

$$\Rightarrow \vec{F}_L \approx q v_x \left(\frac{\mu_0}{4\pi} \frac{g}{(b^2 + [x - v_x t]^2)} \right) \hat{z} \quad \text{(out of paper}$$

"weak interaction" $\Rightarrow$ the force acts only for a short time around $t=0$, (closest approach),

$$\Delta t_{\text{interaction}} = \Delta T \approx 2 \frac{b}{v_x}$$

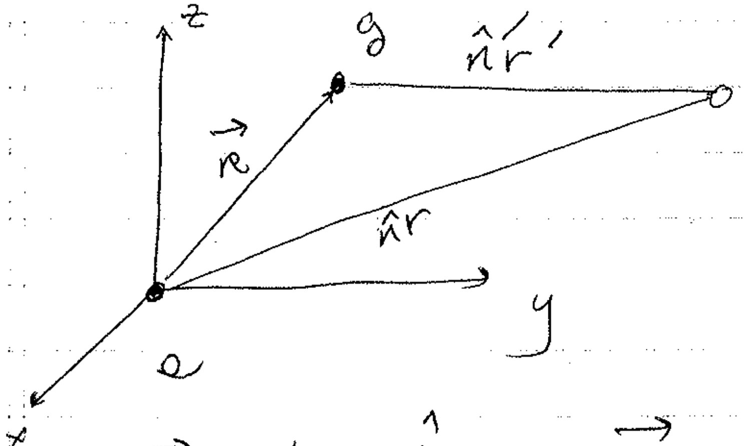
$$\Rightarrow \boxed{\Delta P_z \approx q \cancel{v_x} \left(\frac{\mu_0}{4\pi} \right) \frac{g}{(b^2)} 2 \frac{b}{\cancel{v_x}} = \frac{\mu_0}{2\pi} \left[\frac{qg}{b} \right]}$$

② this imparts an angular momentum

$$\Delta \vec{L} = \vec{r} \times \Delta P_z \hat{z} = \cancel{b} \left(\frac{\mu_0}{2\pi} \frac{qg}{\cancel{b}} \right) \quad \left(\begin{matrix} \text{around} \\ y\text{-axis} \end{matrix} \right)$$

$$\boxed{\Delta \vec{L} = \frac{\mu_0}{2\pi} q g \hat{y}}$$

Consider the $\overset{\text{angular}}{}$ Momentum in the EM field



Evaluate angular momentum at some point P.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\hat{n}}{r^2} \quad , \quad \vec{B} = \frac{\mu_0}{4\pi}\frac{\hat{n}'}{r'^2}$$

and the angular momentum density is given by

$$\hat{n}r \times \epsilon_0(\vec{E}\times\vec{B})$$

$$\vec{L}_{em} = \int \hat{n}r \times \epsilon_0(\vec{E}\times\vec{B})\,d^3x'$$

$$= \int\left[\hat{n}r \times \frac{1}{4\pi}\frac{\hat{n}e}{r^2}\times\vec{B}\right]d^3x'$$

$$= \frac{e}{4\pi}\int \hat{n}\times\hat{n}\times\vec{B}\left(\frac{d^3x'}{r}\right)$$

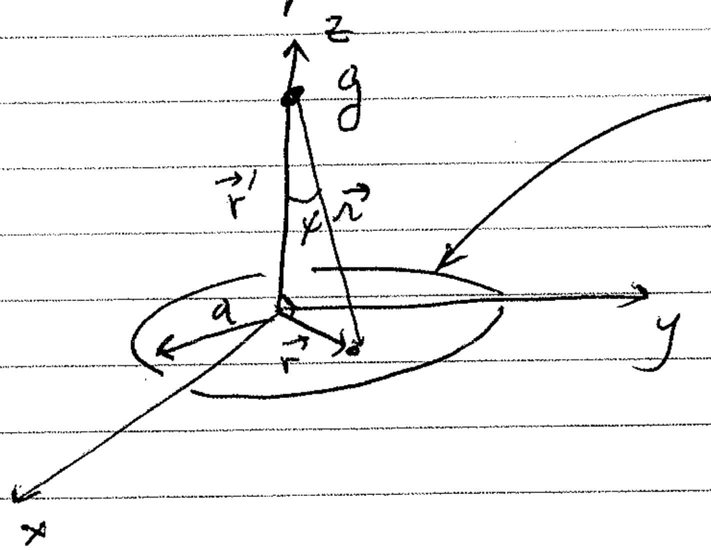
$$= \frac{e}{4\pi}\int\frac{d^3x'}{r}\left[\underbrace{\hat{n}(\hat{n}\cdot\vec{B}) - \vec{B}}\right]$$

note: $\underbrace{(\vec{a}\cdot\vec{\nabla})\vec{n}f(r) = \frac{f(r)}{r}\left[\vec{a}-\vec{n}(\vec{a}\cdot\vec{n})\right] + \vec{n}(\vec{a}\cdot\vec{n})\frac{\partial f}{\partial r}}$

from front of book. If $f(r) = 1 \Longrightarrow \frac{\partial f}{\partial r} = 0$ and

$$= \frac{e}{4\pi}\int d^3x'\left[-(\vec{B}\cdot\vec{\nabla})\hat{n}\right]$$

Cabrera experiment (Cabrera 1982, PRL, 48, 1378)



$\vec{\mathscr{r}} = (x, y, -z')$

b/c $\vec{r}' = (0, 0, z')$

$\vec{r} = (x, y, 0)$

(a) suppose g comes down z-axis from above. Find Φ_B due to the monopole, g.

$$\Phi_B = \oint \vec{B} \cdot d\vec{S} = \frac{g\mu_0}{4\pi} \int_0^{2\pi}\int_0^a \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot (s\,d\phi\,ds)\,\hat{z}$$

$$= \frac{g\mu_0}{4\pi} \int_0^{2\pi}\int_0^a \frac{(-z')\,s\,d\phi\,ds}{(x^2+y^2+z'^2)^{3/2}}$$

note: $s = \sqrt{x^2+y^2}$

$$\Phi_B = \frac{\mu_0 g}{2} \int \frac{-z' s\,ds}{(s^2+z'^2)^{3/2}}$$

$$= \frac{-\mu_0 g z'}{2} \times \frac{1}{2} \times \frac{(-2)}{(s^2+z'^2)^{1/2}} \Big|_0^a$$

$$= \frac{\mu_0 g z'}{2} \left[\frac{1}{\sqrt{a^2+z'^2}} - \frac{1}{|z'|} \right]$$

The flux through the loop is then

$$\Phi_T = \Phi_B - LI$$

$$= \frac{1}{2}\mu_0 g \left[\frac{z'}{\sqrt{a^2+z'^2}} - \frac{z'}{|z'|}\right] - LI$$

at $z' \to -\infty$, $\vec{B} \to 0 \Rightarrow \Phi_T \to 0$

(b) Okay, now what is I?

from Maxwell's Equations (Faraday's law)

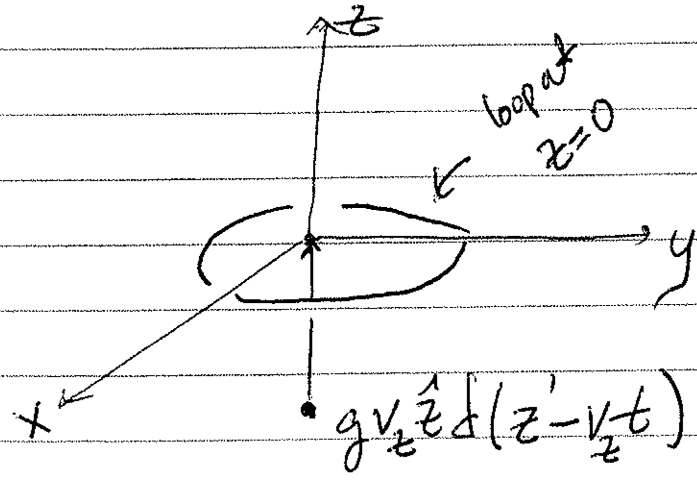
$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{J}_m$$

$$\int(\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\int \frac{\partial \vec{B}\cdot d\vec{S}}{\partial t} - \mu_0 \int \vec{J}_m \cdot d\vec{S}$$

stokes' theo

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\Phi_T - \mu_0 \int \vec{J}_m \cdot d\vec{S}$$

for a point charge $\vec{J}_m = g\vec{v}\,\delta^3(\vec{r}\,' - \vec{v}t)$



$$\Rightarrow -\mu_0 \int \vec{J}_m \cdot \underbrace{d\vec{S}}_{\hat{z}\, s\, ds\, d\phi}$$

$$= \cancel{-\mu_0 \int g v_z \hat{z}\cdot\delta^3(\ldots)\,\hat{z}\,ds}$$

$$\cancel{\hat{z}\; \mu_0 g v_z \int \delta(\ldots)\,ds}$$

$$= -\mu_0 g \int \vec{v}\, \delta^3(\vec{r}\,' - \vec{v}t) \cdot d\vec{S}$$

$\boxed{\text{loops sits in at } z = z' = 0 \Rightarrow \delta \text{ is only nonzero when } z - z' = 0 \text{ and } t = 0}$

$$\Rightarrow \int \delta^3(\vec{r}\,' - \vec{v}t)\, \vec{v} \cdot d\vec{S} = v_z\, \delta(-v_z t)$$

$$\Rightarrow \boxed{-\mu_0 \int \vec{J}_m \cdot d\vec{S} = -\mu_0 g v_z \delta(-v_z t)}$$

Return to Ampere's law

$$\underbrace{\oint \vec{E} \cdot d\vec{l}}_{\vec{E} = 0 \text{ in a SC}} = -\frac{\partial \Phi_T}{\partial t} - \mu_0 g v_z \delta(-v_z t)$$

$$0 = -\frac{\partial \Phi_T}{\partial t} - \mu_0 g v_z \delta(-v_z t)$$

Integrate from $t = -\infty$ to t

$$0 = -\int_{-\infty}^{t} \frac{\partial \Phi_T}{\partial t} dt - \mu_0 g v_z \int_{-\infty}^{t} \delta(-v_z t) dt$$

$$0 = -\Phi_{B}(t) + \Phi_{BT}(-\infty) - \mu_0 g v_z \int_{-\infty}^{t} \delta(-v_z t)\,dt$$

look at $\int_{-\infty}^{t} \delta(-v_z t)\,dt$

let $U = -v_z t$, $dU = v_z\,dt$

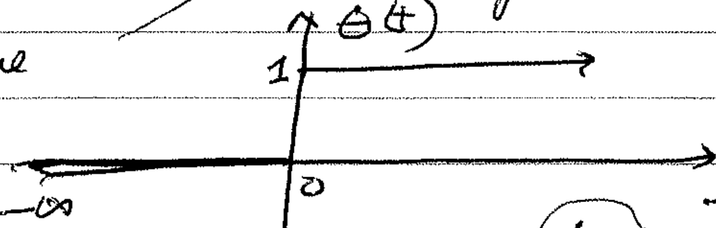
$$= \int_{+\infty}^{-U/v_z} \delta(U)\left(\frac{dU}{v_z}\right) = +\frac{1}{v_z}$$

$0, \vec{B} \to 0$ at $z' \to -\infty$

step function integral of $\delta(x)$

$$\Rightarrow 0 = -\Phi_{BT}(t) + \Phi_B(-\infty) - \mu_0 g\,\theta(t)$$

where



$$0 = -\Phi_{BT}(t) - \mu_0 g\,\theta(z)$$

switch from $t \to z$ as independent variable $z = v_z t$

$$\Rightarrow \boxed{\Phi_{BT}(t) = -\mu_0 g\,\theta(z)}$$

from Maxwell

from Φ & induction from point charge g

$$\Rightarrow -\mu_0 g\,\theta(z) = \Phi_{mB} - LI$$

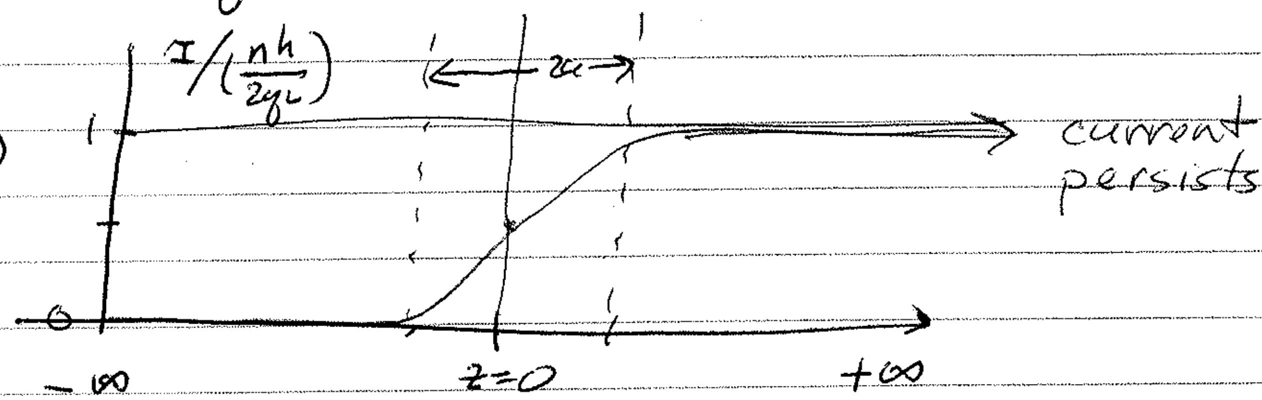
$$\Rightarrow LI = \Phi_{mB} + \mu_0 g\,\theta(z)$$

$$n\left(\frac{h}{\mu_0 q}\right)$$

$$LI = \frac{\mu_0 q}{2}\left[1+\frac{z}{\sqrt{a^2+z^2}}\right]$$

$$I = \frac{nh}{2qL}\left[1+\frac{z}{\sqrt{a^2+z^2}}\right]$$

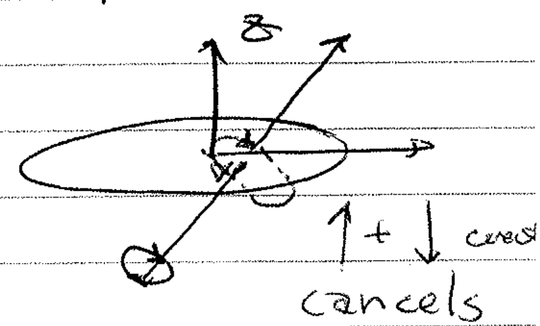
$\Rightarrow$

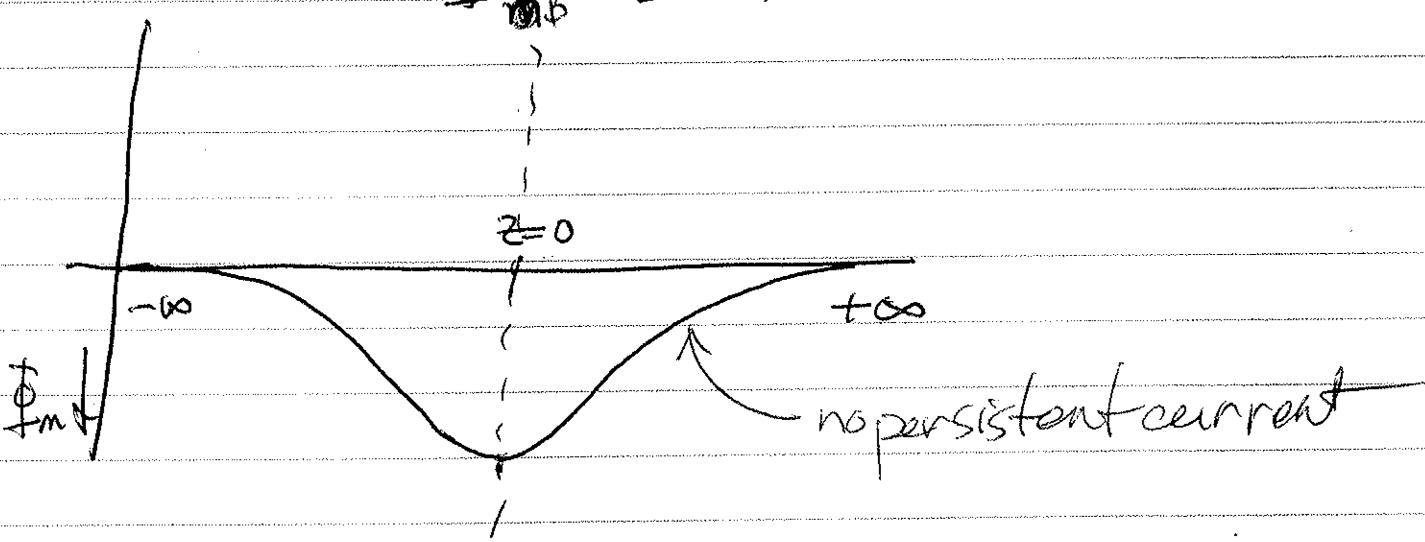


Dipole

$$\Phi_{mB} - LI = 0$$

$$\Rightarrow \Phi_{mB} = LI$$

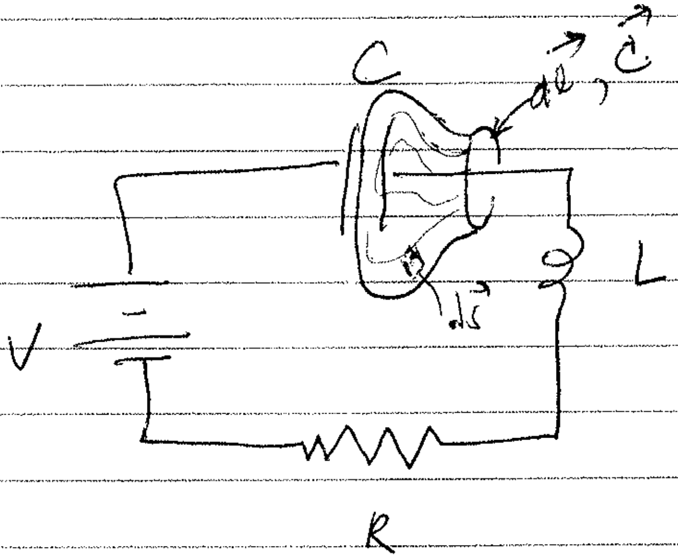




Maxwell Equations

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0 \underbrace{\frac{\partial \vec{E}}{\partial t}}_{\text{displacement current}}\right]$$



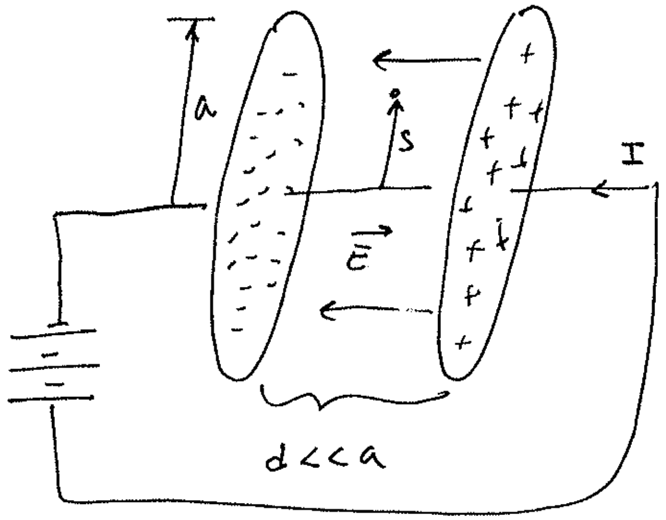
LRC Circuit

for Circuit $\vec{C}$, $\underbrace{\oint \vec{B}\cdot d\vec{l}}_{\vec{B}\text{ is clearly not zero}} = \mu_0 \underbrace{\int \vec{J}\cdot d\vec{S}}_{\text{no }\vec{J}\text{ through }d\vec{S}} = 0,$

Displacement Current

Example

A parallel plate capacitor is connected to an DC voltage and the charge on the positive plate is $\sigma(t)$. Ignore "fringing."



Determine the field between the plates for

$$s \gtrless a$$

← can't ignore displacement current

Use,

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \Big(\underbrace{\vec{J}}_{\to 0 \text{ between the plates}} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Big) \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S}$$

$\underline{s < a}$

← parallel plate problem ignoring fringing

$$B_\phi 2\pi s = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\sigma(t)}{\epsilon_0} \pi s^2 \right)$$

← $\sigma(t)$

$$B_\phi = \mu_0 \epsilon_0 \frac{s}{2\epsilon_0} \frac{\partial}{\partial t} \left(\frac{Q(t)}{\pi a^2} \right)$$

← plate is charge $\Rightarrow I > 0$ $\to Q(t) \uparrow$

$$= \mu_0 \epsilon_0 \frac{s I}{2\pi \epsilon_0 a^2}$$

$$\boxed{B_\phi = \mu_0 \frac{s I}{2\pi a^2}}$$

$\underline{s > a}$

no "fringing"

$$B_\phi 2\pi s = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\sigma(t)}{\epsilon_0} \pi a^2 \right)$$

$$B_\phi = \mu_0 \frac{a^2}{2s}\frac{\partial}{\partial t}\left(\frac{q(t)}{\pi a^2}\right)$$

$$\boxed{B_\phi = \frac{\mu_0 I}{2\pi s}}$$

<u>Energy flux</u>

Poynting Vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$= -\frac{1}{\mu_0}\left(\frac{\sigma}{\epsilon_0}\right)\times\left(\mu_0\frac{sI}{2\pi a^2}\right)\hat{s} \qquad s<a$$

Energy flows into the capacitor

Integrate $\vec{S}$ over the cylinder w/ $s=a$

$$\int \vec{S}\cdot d\vec{S} = -\frac{\sigma}{\epsilon_0}\frac{Ia}{2\pi a^2}2\pi a d$$

$$= -\frac{\sigma}{\epsilon_0} Id$$

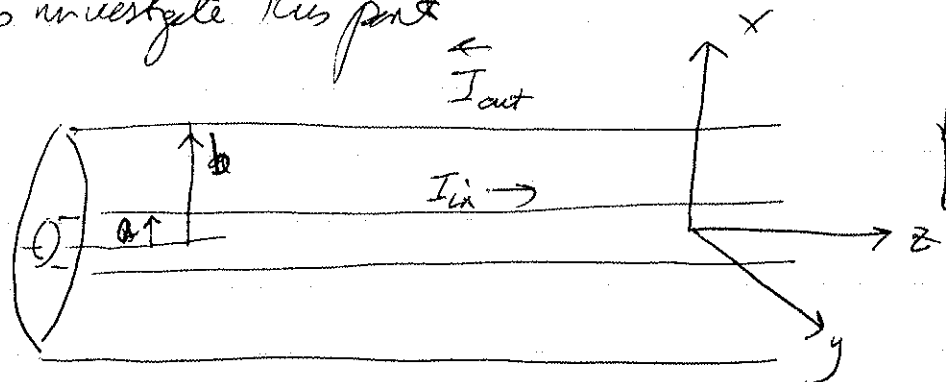
$$= -IV_0$$

negative of the power delivered by the battery to the circuit.

$\Longrightarrow$ "Battery" energy goes into building up the fields

Q: Why was $\vec{J}_D$ the last "bit" of Maxwell's equations to be discovered?

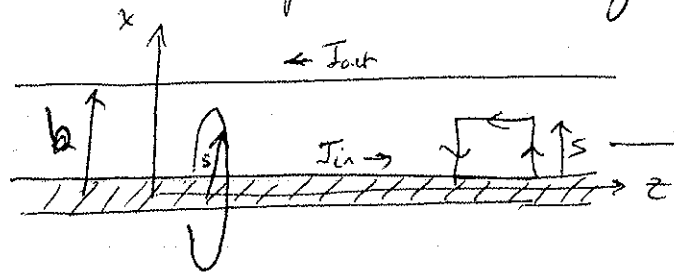
A: Let's investigate this point



Prob 7.66, 7.33

Here, $I(t) = I_0 \cos \omega t$

a) The $\vec{I}_{in}$ and $\vec{I}_{out}$ produce a magnetic field $\vec{B}(t)$



$\longrightarrow \oint \vec{B} \cdot d\vec{l} = 0$

$\rightarrow B_z = 0$

and $B_z = 0$ as $s \to \infty$

$\longrightarrow B_z = 0$

$\vec{B}$ due to inner core

$$\oint \vec{B} \cdot d\vec{l} = I_{in} \mu_0 \quad , s < b$$

$$= 0 \quad , s > b$$

$$\rightarrow \vec{B}_\phi = \frac{\mu_0 I_{in}}{2\pi s} \hat{\phi} \quad s < b$$

$$= 0 \quad s > b$$

b) $\vec{B}$ induced

$$\Phi_B = \oint \frac{\mu_0 I_{in}}{2\pi s} (dz\, ds)$$

$$= \frac{\mu_0 I_{in}}{2\pi} (dz) \ln\left(\frac{s}{a}\right) \rightarrow \frac{d\Phi_B}{dt} = - \cancel{\frac{\mu_0 I_0 \sin \omega t}{2\pi} \omega \ln\left(\frac{s}{a}\right)}$$

$\dot{I}_{in}$

$$\frac{d\Phi_B}{dt} = - \frac{\mu_0}{2\pi}\left[\omega I_0 \sin \omega t\right] dz \ln \frac{s}{a}$$

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l}$$

$$\Rightarrow \cancel{\mathcal{E}_{loop}} = -\frac{\mu_0 \omega I_0 \sin\omega t}{2\pi} z \ln\left(\frac{s}{a}\right)$$

$$\boxed{\mathcal{E}_{loop} = \frac{\mu_0 \omega I_0 \sin\omega t}{2\pi} \ln\left(\frac{s}{a}\right)}$$

flows in CW sense

The current density is then (displacement current)

$$\vec{J}_D = \epsilon_0 \frac{\partial}{\partial t}\vec{E}$$

$$\boxed{\vec{J}_D = \frac{\epsilon_0 \mu_0 I_0}{2\pi} \omega^2 \cos\omega t \ln\left(\frac{s}{a}\right)\hat{z}}$$

and

$$I_D = \int \vec{J}_D \cdot d\vec{S}$$

$$= \frac{\epsilon_0 \mu_0 I_0}{2\pi} \omega^2 \cos\omega t \int \left[\ln s - \ln a\right] s\, ds\, d\phi$$

$$= \epsilon_0 \mu_0 I_0 \omega^2 \cos\omega t \int \left[\ln s - \ln a\right] s\, ds$$

$$\boxed{I_D = \frac{\mu_0 \epsilon_0}{2} \omega^2 I_0 \cos(\omega t) \frac{b^2}{2}}$$

note: $$\frac{I_D}{I_0 \cos\omega t} = \frac{\mu_0 \epsilon_0}{4} \frac{\omega^2 b^2 I_0 \cos\omega t}{I_0 \cos\omega t}$$

$$= \frac{1}{4}\mu_0 \epsilon_0 \omega^2 b^2 = \frac{1}{4}\left(\frac{\omega b}{c}\right)^2$$

EM Waves in Vacuum

Take $\vec{\nabla} \times \left[\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \right]$

$$\vec{\nabla}\left[\vec{\nabla}\cdot\vec{E}\right] - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B} = -\frac{\partial}{\partial t}\left[\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$$

In vacuum

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = 0 \quad \& \quad \frac{\partial}{\partial t}\mu_0 \vec{J} = 0$$

$$\rightarrow \boxed{\nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$, wave equation

and

$$\mu_0 \epsilon_0 = \frac{1}{c^2} \; !$$

Take $\vec{\nabla} \times \left[\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \right]$

$\vec{J} \rightarrow 0$, in vacuum

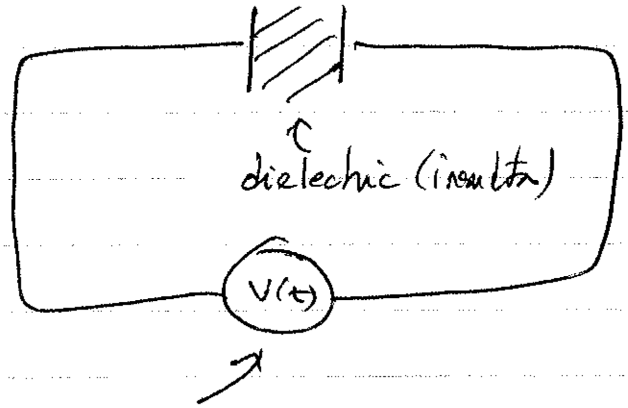
$$\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{B}})^{0} - \nabla^2 \vec{B} = \cancel{\mu_0 \vec{\nabla}\times\vec{J}} + \epsilon_0 \mu_0 \vec{\nabla} \times \frac{\partial}{\partial t}\vec{E}$$

$$= \epsilon_0 \mu_0 \frac{\partial}{\partial t}\left[-\frac{\partial \vec{B}}{\partial t}\right]$$

$$\rightarrow \boxed{\nabla^2 \vec{B} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0}$$, wave equation

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \, !$$

Josephson Junction (Josephson 1962)



drive circuit with $V(t) = V_0 + V_1 \sin\omega t$

Josephson found that I would not flow across the insulator unless a resonance condition was met, namely, if

$$\boxed{\omega = \frac{eV_0}{\pi h}}$$

for $V_0 = 1\,mV$, $\omega = \frac{eV_0}{\pi h} \rightarrow V_0 = \frac{\pi h \omega}{e}$

$$= \frac{3.14 \times 6.6 \times 10^{-27}}{4.8 \times 10^{-10}} \omega$$

$$= 4.32 \times 10^{-18} \omega \text{ (c.g.s.)}$$

$$V_0 = 1.3 \times 10^{-14} \omega \text{ (mks)}$$

for Maves, $\sim 10^{11}$ R/s

$$\rightarrow \boxed{V_0 = 1.3\,mV}$$

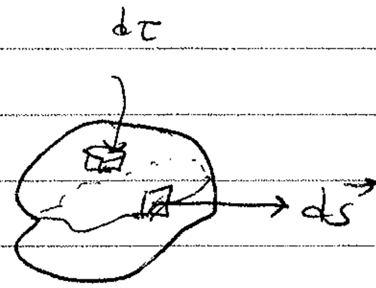
a very good way to measure voltages,

Conservation laws

ⓐ Conservation of charge, Continuity Equation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

Integrate over volume, $d\tau$



$$\underbrace{\frac{\partial \int \rho \, d\tau}{\partial t}}_{\frac{\partial}{\partial t} Q} = -\int (\vec{\nabla} \cdot \vec{J}) \, d\tau = \underbrace{-\oint \vec{J} \cdot d\vec{S}}_{\text{net flow of charge into volume}}$$

$\frac{\partial}{\partial t} Q \leftarrow$ net change of charge in volume

ⓑ Poynting's Theorem, Conservation of Energy

$$\boxed{\frac{\partial}{\partial t}\Big[u_{mech} + \underbrace{u_{em}}_{\frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\,d\tau + \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}\,d\tau}\Big] = -\vec{\nabla} \cdot \vec{S} \leftarrow \vec{S} = \text{Poynting Vector} = \frac{1}{\mu_0} \vec{E} \times \vec{B}}$$

Integrate over volume, $d\tau$ $\Rightarrow \frac{\partial}{\partial t} \int \overbrace{(u_{mech} + u_{em})}^{\text{total energy}} d\tau = \overbrace{-\oint \vec{S} \cdot d\vec{s}}^{\text{EM flux in/out}}$

Can we write down other conservation laws for matter & EM fields?

Maxwell Stress Tensor

$$\boxed{\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}}$$ ← force law for point charge, q

$$= \int \left[\vec{E} + \vec{v}\times\vec{B}\right]\rho\, d\tau \quad \text{, distributed charge, } \rho$$

$$= \int \left[\rho\vec{E} + \vec{J}\times\vec{B}\right] d\tau, \quad \vec{J} = \rho\vec{v}$$

eliminate source terms using Maxwell equations

$$= \int \left[\underbrace{(\epsilon_0 \vec{\nabla}\cdot\vec{E})}_{\text{Gauss's law}}\vec{E} + \underbrace{\left[\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial \vec{E}}{\partial t}\right]}_{\text{Ampere's law}}\times\vec{B}\right] d\tau$$

look at the $\epsilon_0 \frac{\partial \vec{E}}{\partial t}\times\vec{B}$ term

Consider, $$\frac{\partial}{\partial t}\left[\vec{E}\times\vec{B}\right] = \underbrace{\frac{\partial \vec{E}}{\partial t}\times\vec{B}}_{\text{relates}} + \vec{E}\times\frac{\partial \vec{B}}{\partial t} \rightarrow \vec{E}\times(-\vec{\nabla}\times\vec{E})$$

$$= \frac{\partial \vec{E}}{\partial t}\times\vec{B} - \vec{E}\times(\vec{\nabla}\times\vec{E})$$

Plug into $\vec{F}$

$$\vec{F} = \int \left\{(\epsilon_0\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{\partial}{\partial t}\epsilon_0(\vec{E}\times\vec{B})\right\} d\tau$$

$$-\frac{1}{\mu_0}\left[\frac{1}{2}\vec{\nabla}B^2 - (\vec{B}\cdot\vec{\nabla})\vec{B}\right] \qquad -\epsilon_0\left[\frac{1}{2}\vec{\nabla}E^2 - (\vec{E}\cdot\vec{\nabla})\vec{E}\right]$$

(A)

A $= \int \Big[\frac{1}{2\mu_0}\vec{\nabla}B^2 + \frac{\epsilon_0}{2}\vec{\nabla}E^2 + \epsilon_0 \Big\{ \underbrace{(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}}_{\vec{\nabla}} \Big\} + \frac{1}{\mu_0}\underbrace{(\vec{B}\cdot\vec{\nabla})\vec{B}}$

$- \frac{\partial}{\partial t}\epsilon_0(\vec{E}\times\vec{B}) \Big] d\tau$

add $(\vec{\nabla}\cdot\vec{B})$ ... value $\vec{B}$ similar $\vec{E}$ term

$$\vec{F} = -\int \Big(\frac{1}{2\mu_0}\vec{\nabla}B^2 + \frac{\epsilon_0}{2}\vec{\nabla}E^2 + \frac{\partial}{\partial t}\epsilon_0(\vec{E}\times\vec{B}) \Big) d\tau$$

$$+ \int \epsilon_0 \Big[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} \Big] d\tau + \int \frac{1}{\mu_0} \Big[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} \Big] d\tau$$

Rewrite this expression in another way

for example, $\vec{E} \Rightarrow E_i$, $\vec{B} \Rightarrow B_i$ and rewrite the $\vec{F}$ integral.

$\boxed{\text{Consider } F_i \text{ term}}$ ← look at the vector parts and consider the i-th component

$$F_i = -\int\left[\frac{1}{2\mu_0}\frac{\partial}{\partial x_i}(\vec{B}\cdot\vec{B}) + \frac{\epsilon_0}{2}\frac{\partial}{\partial x_i}(\vec{E}\cdot\vec{E}) + \frac{\partial}{\partial t}\epsilon_0 \underbrace{\left(E_2B_3 - E_3B_2\right)}_{\vec{E}\times\vec{B}|_i}\right]d\tau$$

$$+\epsilon_0\int\left[(\vec{\nabla}\cdot\vec{E})E_i + (\vec{E}\cdot\vec{\nabla})E_i\right]d\tau + \frac{1}{\mu_0}\int\left[(\vec{\nabla}\cdot\vec{B})B_i + (\vec{B}\cdot\vec{\nabla})B_i\right]d\tau$$

$\boxed{\text{To demonstrate what's happening, consider individual terms}}$

$$= -\frac{\epsilon_0}{2}\frac{\partial}{\partial x_i}\left(E_1E_1 + E_2E_2 + E_3E_3\right) + \epsilon_0\left(\frac{\partial}{\partial x_1}E_1 + \frac{\partial}{\partial x_2}E_2 + \frac{\partial}{\partial x_3}E_3\right)E_i$$

$$+\epsilon_0\left(E_1\frac{\partial}{\partial x_1} + E_2\frac{\partial}{\partial x_2} + E_3\frac{\partial}{\partial x_3}\right)E_i$$

combined last 2 terms

$$= -\frac{\epsilon_0}{2}\frac{\partial}{\partial x_i}(\vec{E}\cdot\vec{E}) + \epsilon_0\Big(\underbrace{\sum \frac{\partial}{\partial x_j}}_{\text{sum over } j} E_jE_i\Big) \qquad \epsilon_0\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}E_jE_i\right)$$

$$\Rightarrow -\frac{1}{2\mu_0}\frac{\partial}{\partial x_i}(\vec{B}\cdot\vec{B}) + \frac{1}{\mu_0}\left(\sum\frac{\partial}{\partial x_j}B_jB_i\right)$$ } b/c of symmetric form of T_{ij}

sum over j

let us rewrite (for illustrative purposes)

$$\underbrace{-\frac{1}{2\mu_0}\frac{\partial}{\partial x_i}\left(B_1^2 + B_2^2 + B_3^2\right)}_{\text{no } j\text{-term}} + \frac{1}{\mu_0}\left(\frac{\partial}{\partial x_1}B_1B_i + \frac{\partial}{\partial x_2}B_2B_i + \frac{\partial}{\partial x_3}B_3B_i\right)$$

$$= -\delta_{ij}\frac{1}{2\mu_0}\frac{\partial}{\partial x_j}(\vec{B}\cdot\vec{B}) + \frac{1}{\mu_0}\left(\frac{\partial}{\partial x_j}B_jB_i\right)$$

$$= \frac{\partial}{\partial x_j}\left[\frac{B_j B_i}{\mu_0} - \delta_{ij}\frac{B^2}{2\mu_0}\right]$$

So the $\vec{E}$ and $\vec{B}$ terms [excluding $\vec{S}$] become

$$= \frac{\partial}{\partial x_j}\left[\frac{B_j B_i}{\mu_0} - \delta_{ij}\frac{B^2}{2\mu_0}\right] + \frac{\partial}{\partial x_j}\left[\epsilon_0 E_j E_i - \delta_{ij}\frac{\epsilon_0}{2}E^2\right]$$

$$= \frac{\partial}{\partial x_j}\left[T_{ij}\right]$$ Maxwell Stress Tensor

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \delta_{ij}\frac{B^2}{2}\right) + \epsilon_0\left(E_i E_j - \delta_{ij}\frac{E^2}{2}\right)$$

<u>note:</u> $\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x_1}A_1 + \frac{\partial}{\partial x_2}A_2 + \frac{\partial}{\partial x_3}A_3 = \frac{\partial}{\partial x_j}A_j$ sum over j

<u>generalizing (expressing in "vector" notation)</u>

$$\frac{\partial}{\partial x_j}T_{ij} \Longrightarrow \vec{\nabla}\cdot\overrightarrow{\overrightarrow{T}} \text{ or } \vec{\nabla}\cdot\overleftrightarrow{T}$$

so that

$$\vec{F} = \int \vec{\nabla}\cdot\overleftrightarrow{T}\,d\tau - \frac{\partial}{\partial t}\int \epsilon_0 \overbrace{\vec{E}\times\vec{B}}^{\vec{E}\times\vec{B} = \mu_0\vec{S}}\,d\tau$$

the force $\vec{F}$ changes the momentum of the charges

$$\Rightarrow \vec{P}_{mech} = \int \vec{\wp}_{mech}\, d\tau$$

and

$$\vec{F} = \frac{\partial}{\partial t}\left[\int \left(\vec{\wp}_{mech} + \underbrace{\epsilon_0\mu_0\vec{S}}_{\text{EM momentum density}}\right) d\tau\right] = \vec{\nabla}\cdot\overleftrightarrow{T}$$

Comment: $\epsilon_0\mu_0\vec{S} = \underbrace{\epsilon_0\mu_0}_{1/c^2}\left[\frac{\text{Energy}}{\text{Area time}}\right] = \frac{1}{c}\left[\frac{\text{momentum}}{\text{Area time}}\right] = \left[\frac{\text{momentum}}{\text{Volume}}\right]$

$(c \times \text{time}) = \Delta x$ moved

momentum density $\times$ speed

= momentum density in EM field

$$\boxed{\vec{F}_{Total} = \frac{\partial}{\partial t}\int \left(\vec{\wp}_{mech} + \vec{\wp}_{em}\right) d\tau = \vec{\nabla}\cdot\overleftrightarrow{T}}$$

So, we see why $\overleftrightarrow{T}$ is called the

Maxwell Stress Tensor

$$\hat{x}_i\cdot\left(\vec{\nabla}\cdot\overleftrightarrow{T}\right) = \sum_{j=1}^{3}\frac{\partial}{\partial x_j}T_{ij} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}T_{ji} \equiv \left(\vec{\nabla}\cdot\overleftrightarrow{T}\right)_i$$

$\Rightarrow \overleftrightarrow{T}$ is symmetric

okay, we can now write

$$F_i = (\vec{\nabla}\cdot\overleftrightarrow{T})_i - \epsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B})_i$$

$$= \int (\vec{\nabla}\cdot\overleftrightarrow{T})_i \, d\tau - \int \epsilon_0 \frac{\partial}{\partial t} \underbrace{\mu_0 \vec{S}_i}_{} \, d\tau$$

↑ i-th component of Poynting Vector

$$\frac{\partial}{\partial t}\int (p_{mech,i} + p_{em,i})\, d\tau = \int (\vec{\nabla}\cdot\overleftrightarrow{T})_i \, d\tau$$

$$= \int (\vec{\nabla}\cdot\overleftrightarrow{T})_i \cdot d\vec{S}$$

$$\boxed{\frac{\partial}{\partial t}\int (p_{mech} + p_{em})_i = \int T_{ij}\, dS_j}$$

we can get the force on a volume of charge + mass through a surface integral

(3) **Momentum Conservation**

Lorentz force per unit volume $\vec{f}$ is

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} \Rightarrow \vec{F} = \int \vec{f}\, d\tau \equiv \text{Total Force}$$

Express this in terms of the fields $\vec{E}$ & $\vec{B}$

$$\vec{f} = \underbrace{\left(\epsilon_0 \vec{\nabla}\cdot\vec{E}\right)}_{\rho}\vec{E} + \underbrace{\left[\frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]}_{\vec{J}} \times \vec{B}$$

note: $\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial \vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial \vec{B}}{\partial t}$

$$= \epsilon_0 \vec{E}(\vec{\nabla}\cdot\vec{E}) + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\left[\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial \vec{B}}{\partial t}\right]$$

Maxwell's Eqn.

$$= \epsilon_0 \vec{E}(\vec{\nabla}\cdot\vec{E}) + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\left[\frac{\partial}{\partial t}\vec{E}\times\vec{B} + \vec{E}\times(\vec{\nabla}\times\vec{E})\right]$$

make the expression "symmetric" by adding

$\frac{1}{\mu_0}\vec{B}(\vec{\nabla}\cdot\vec{B})$ to the RHS

$$= \Big[\epsilon_0 \vec{E}(\vec{\nabla}\cdot\vec{E}) + \frac{1}{\mu_0}\vec{B}(\vec{\nabla}\cdot\vec{B}) - \frac{\vec{B}}{\mu_0}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})$$
$$- \epsilon_0\frac{\partial}{\partial t}\vec{E}\times\vec{B}\Big]$$

pretty much of a mess

Can we "simplify" the $\vec{f}$ expression?

Consider:

$$\epsilon_0 \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) \right] \Big|_i$$ ← i-th component

magic

$$= \epsilon_0 \frac{\partial}{\partial x_i}\left[E_i E_i - \frac{1}{2} E^2 \right] + \epsilon_0 \frac{\partial}{\partial x_j} E_i E_j + \epsilon_0 \frac{\partial}{\partial x_k}(E_i E_k)$$

hold i-fixed, sum over j, k ← second i

and, similarly, (for $\vec{B}$ term)

$$\frac{1}{\mu_0}\left[\frac{\partial}{\partial x_i}\left(B_i B_i - \frac{1}{2} E^2 \right) + \frac{\partial}{\partial x_j} B_i B_j + \frac{\partial}{\partial x_k} B_i B_k \right]$$

and the i-th component of $\vec{f}$ is

pick it up here →

$$f_i = \epsilon_0 \sum_{\ell=1}^{3} \frac{\partial}{\partial x_\ell}\left(E_i E_\ell - \frac{1}{2}\delta_{i\ell} E^2 \right) + \frac{1}{\mu_0}\sum_{\ell=1}^{3} \frac{\partial}{\partial x_\ell}\left(B_i B_\ell - \frac{1}{2}\delta_{i\ell} B^2 \right) - \epsilon_0 \frac{\partial}{\partial t}\left[(\vec{E}\times\vec{B})_i \right]$$

define:

$$T_{i\ell} \equiv \epsilon_0 E_i E_\ell + \frac{1}{\mu_0} B_i B_\ell - \frac{1}{2}\delta_{i\ell}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

$$\equiv T_{\ell i}$$

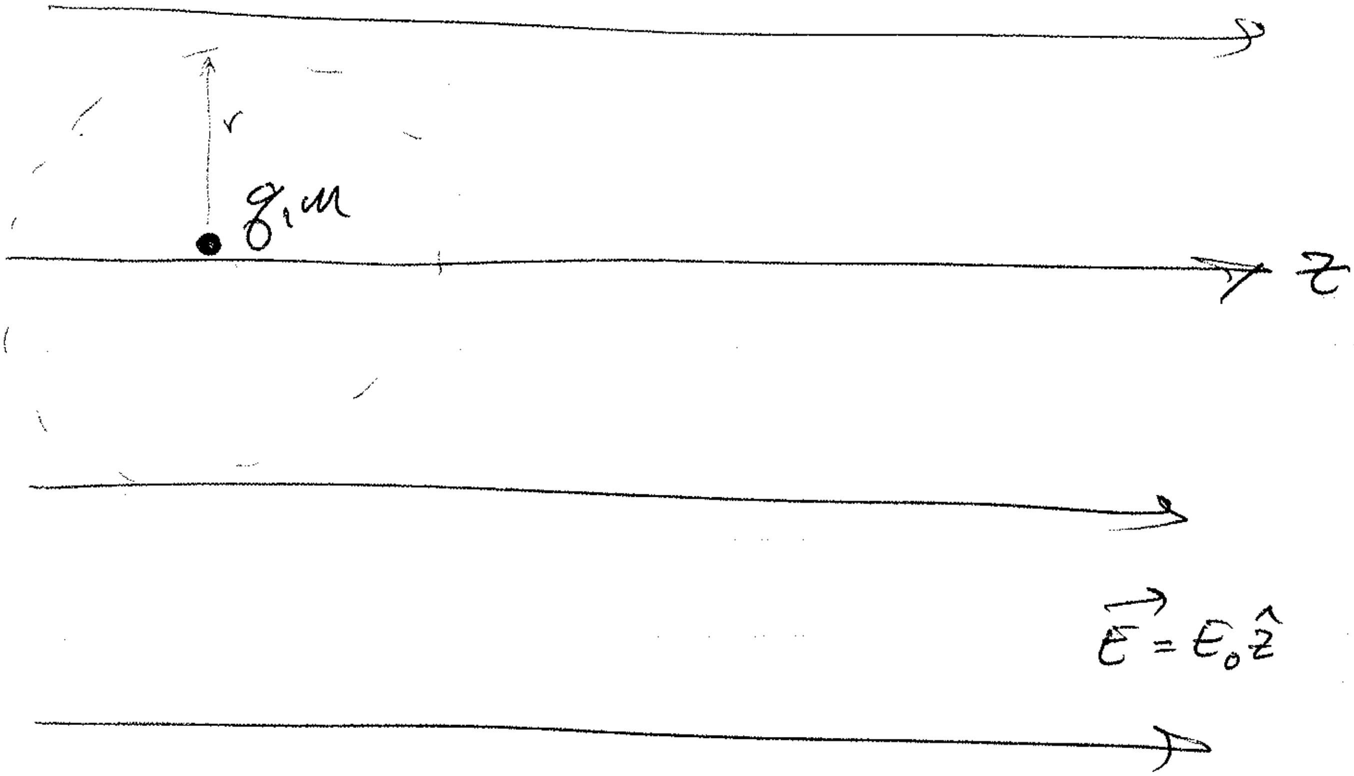
as the Maxwell Stress Tensor

and we can write

$$\sum_{\ell=1}^{3} \frac{\partial}{\partial x_\ell} T_{i\ell} = \sum_{\ell=1}^{3} \frac{\partial}{\partial x_\ell} T_{\ell i} \equiv (\vec{\nabla}\cdot\overleftrightarrow{T})_i$$

Example

Place a point charge in an external field; what force does the point charge feel?

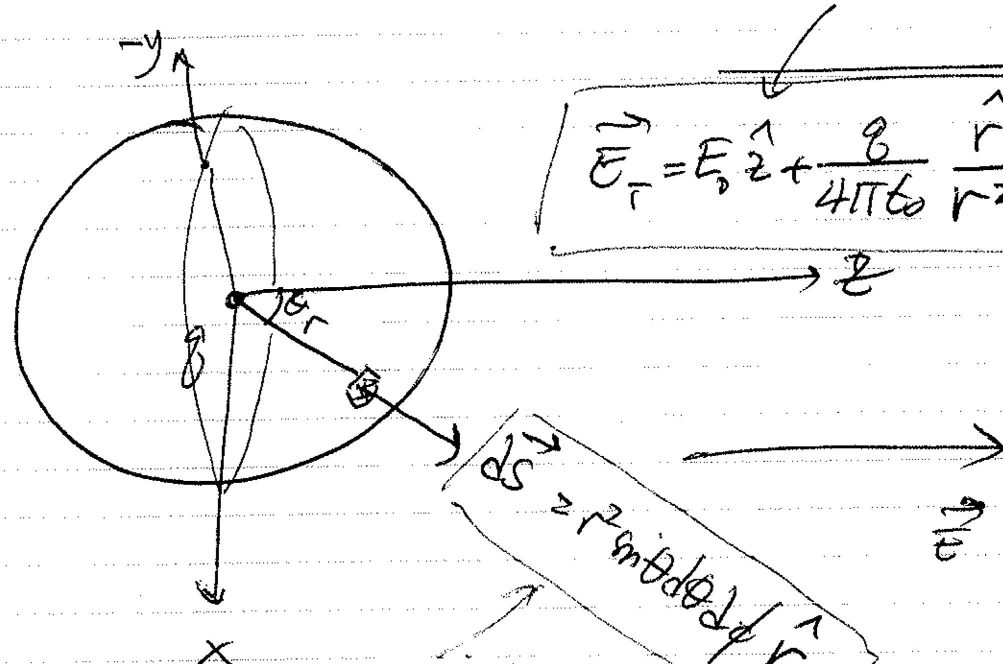


Hmm, $\boxed{\vec{F} = q\vec{E} = qE_0\hat{z}}$

okay, looks good. Let's redo this problem using the Maxwell Stress Tensor,

$$T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij}\left[\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right]$$

for our current problem



$$\hat{z} = (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$\boxed{\vec{E}_T = E_0\hat{z} + \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}}$$

$$\boxed{d\vec{S} = r^2\sin\theta\, d\theta\, d\phi\, \hat{r}}$$

area element is in $\hat{r}$ direction

$$\Rightarrow \quad T_{ij}\, dS_j = T_{ir}\, dS_r$$

$\boxed{\text{what are } T_{rr},\ T_{\theta r},\ T_{\phi r}\ ?}$

$$\overleftrightarrow{T} = \begin{pmatrix} \epsilon_0 E_r E_r - \frac{\epsilon_0}{2}E^2 & \epsilon_0 E_r E_\theta & \epsilon_0 \cancel{E_r E_\phi}^{\,0} \\ \epsilon_0 E_\theta E_r & \epsilon_0 E_\theta E_\theta - \frac{\epsilon_0}{2}E^2 & \epsilon_0 \cancel{E_\theta E_\phi}^{\,0} \\ \epsilon_0 \cancel{E_\phi E_r}^{\,0} & \epsilon_0 \cancel{E_\phi E_\theta}^{\,0} & \epsilon_0 \cancel{E_\phi E_\phi}^{\,0} - \frac{\epsilon_0}{2}E^2 \end{pmatrix}$$

as noted earlier, Maxwell Stress Tensor is symmetric needed

$$T_{rr} = \epsilon_0 \left[E_0 \cos\theta + \frac{q}{4\pi\epsilon_0 r^2} \right]^2 - \frac{\epsilon_0}{2} \left[\left(E_0 \cos\theta + \frac{q}{4\pi\epsilon_0 r^2} \right)^2 + E_0^2 \sin^2\theta \right]$$

$$= \frac{\epsilon_0}{2} \left[E_0 \cos\theta + \frac{q}{4\pi\epsilon_0 r^2} \right]^2 - \frac{\epsilon_0}{2} E_0^2 \sin^2\theta$$

$$= \frac{\epsilon_0}{2} \left[\underset{1}{E_0^2 \cos^2\theta - E_0^2 \sin^2\theta} + \underset{2}{2\frac{q E_0 \cos\theta}{4\pi\epsilon_0 r^2}} + \underset{3}{\frac{q^2}{(4\pi\epsilon_0)^2 r^4}} \right]$$

$$T_{r\theta} = \epsilon_0 \left[E_0 \cos\theta + \frac{q}{4\pi\epsilon_0 r^2} \right] \left[E_0 \sin\theta \right]$$

$$= \epsilon_0 \left[E_0^2 \sin\theta \cos\theta + \frac{q E_0 \sin\theta}{4\pi\epsilon_0 r^2} \right] = T_{\theta r}$$

not needed

$$T_{\theta\theta} = \epsilon_0 \left[E_0^2 \sin^2\theta \right] - \frac{\epsilon_0}{2} \left[E_0^2 \cos^2\theta + \frac{2 q E_0 \cos\theta}{4\pi\epsilon_0 r^2} + \frac{q^2}{(4\pi\epsilon_0)^2 r^4} + E_0^2 \sin^2\theta \right]$$

$$= \epsilon_0 \left[E_0^2 \sin^2\theta - \frac{1}{2} E_0^2 - \frac{q E_0 \cos\theta}{4\pi\epsilon_0 r^2} - \frac{2q^2}{(4\pi\epsilon_0)^2 r^4} \right]$$

$$T_{\phi\phi} = -\frac{\epsilon_0}{2} \left[E_0^2 + \frac{2 q E_0 \cos\theta}{4\pi\epsilon_0 r^2} + \frac{q^2}{(4\pi\epsilon_0)^2 r^4} \right]$$

$$T_{\phi r} = 0 = T_{r\phi}$$

for 1 component i of force, we integrate:

$$T_{ij}\, dS_j = T_{rr}\, dS_r + T_{r\theta}\, \cancel{dS_\theta} + T_{r\phi}\, \cancel{dS_\phi}\ .$$

~~At~~ each point on surface of sphere, we can find $\underline{f_r, f_\theta}$, but this is a problem because if we were to integrate over the sphere $\underline{\hat{r}, \hat{\theta}}$ change directions.

<u>So, in vector form</u>

$$\overleftrightarrow{T} \cdot d\vec{S} = \epsilon_0 \vec{E}_T \left(\vec{E}_T \cdot d\vec{S} \right) - \frac{\epsilon_0}{2} E_T^2 \left(r^2 \hat{r}\, d\Omega \right)$$

we only get $\hat{r}$ because of the $\underline{\delta_{ij}}$

To do this problem, we need to change the $\vec{E}_T$ part to Cartesian coordinates (fixed basis)

$$\boxed{\begin{aligned} \hat{r} &= \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z} \\ \hat{\theta} &= \cos\theta\cos\phi\, \hat{x} + \cos\theta\sin\phi\, \hat{y} - \sin\theta\, \hat{z} \end{aligned}}$$

This already shows us something good,

$\vec{E}_T \cdot d\vec{S}$ is independent of $\phi \Rightarrow \int \left\{ \begin{matrix} \cos\phi \\ \sin\phi \end{matrix} \right\} d\phi \to 0$

and we lose all $\hat{x}, \hat{y}$ components

$\Rightarrow$ only $\hat{z}$ component is left!

We can then find that

$$\overleftrightarrow{T}\cdot d\vec{S}\Big|_z = \hat{z}\,\frac{\epsilon_0}{2}\Bigg[\left(E_0\cos\theta + \frac{q}{4\pi\epsilon_0 r^2}\right)^2\cos\theta - E_0^2\sin^2\theta\cos\theta$$
$$+2E_0\sin^2\theta\left(E_0\cos\theta + \frac{q}{4\pi\epsilon_0 r^2}\right)r^2 d\Omega\Bigg]$$

and then

$$\int \overleftrightarrow{T}\cdot d\vec{S}\,\Big|_z = qE_0\hat{z} = q\vec{E}$$

as we arrived at in a much simple manner.

Maxwell Eqns. in Matter

Recall

① $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ ③ $\vec{\nabla}\cdot\vec{B} = 0$

② $\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ④ $\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0\dfrac{\partial \vec{E}}{\partial t}\right]$

$\vec{P} \Rightarrow -\sigma \; \sigma \Rightarrow$ chge sep^n $\Rightarrow$ dipole moment, $\vec{m}_e$ · if $\vec{P}\uparrow$ $\Rightarrow \vec{m}_e\uparrow \Rightarrow \Delta\sigma\uparrow \Rightarrow$ current

Dielectrics

$$\rho_p = -\vec{\nabla}\cdot\vec{P} \quad \text{and} \quad \vec{J}_p = \frac{\partial \vec{P}}{\partial t} \quad \left(\frac{\partial \rho_p}{\partial t} + \vec{\nabla}\cdot\vec{J}_p = 0\right)$$

$$\frac{\partial}{\partial t}\left[-\vec{\nabla}\cdot\vec{P}\right]$$

Magnetic Material

$$\vec{J}_M = \vec{\nabla}\times\vec{M}$$

$$① \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\epsilon_0} + \frac{\rho_p}{\epsilon_0} = \frac{1}{\epsilon_0}\left(\rho_f - \vec{\nabla}\cdot\vec{P}\right)$$

and
$$\vec{\nabla}\cdot\underbrace{\left[\left(\epsilon_0\vec{E} + \vec{P}\right)\right]} = \rho_f$$

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$

$$④ \quad \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J}_f + \epsilon_0\frac{\partial \vec{E}}{\partial t} + J_M + J_p\right]$$

$$= \mu_0\left[\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t} + \epsilon_0\frac{\partial \vec{E}}{\partial t}\right]$$

$$\vec{\nabla}\times\left[\mu_0^{-1}\vec{B} - \vec{M}\right] = \vec{J}_f + \frac{\partial}{\partial t}\left[\epsilon_0\vec{E} + \vec{P}\right]$$

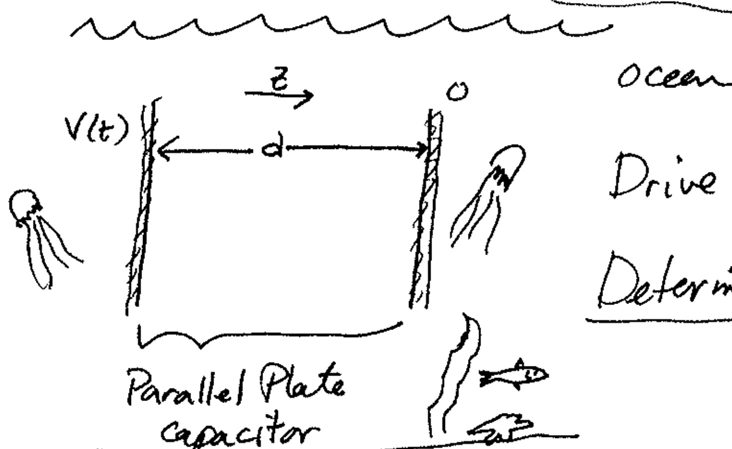
$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial}{\partial t}\vec{D}}$$

and

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_f & \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{H} &= \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \end{aligned}}$$

Example; Prob 7.37

Parallel Plate w/ sea water ($\epsilon \neq \epsilon_0$) between plates



Drive "capacitor" with $V = V_0 \cos(2\pi\nu t)$.

Determine I_c / I_D

Sea water: at $\nu = 4\times10^8\,Hz$, $\underbrace{\epsilon = 81\epsilon_0}_{\text{sea water}}$, $\underbrace{\mu \approx \mu_0}_{\text{not magnetic}}$, and $\underbrace{\rho = 0.23\ \Omega\text{-m}}_{\text{resistivity} = \frac{1}{\sigma}}$

a) $\vec{E} = \frac{V(t)}{d}\hat{z} = \frac{V_0}{d}\cos(2\pi\nu t)\hat{z}$ ← $\vec{E}$ depends on Voltage

$\rightarrow \vec{J}_c = \sigma\vec{E}(t) = \frac{\sigma V_0}{d}\cos(2\pi\nu t)\hat{z}$ ← Ohm's law

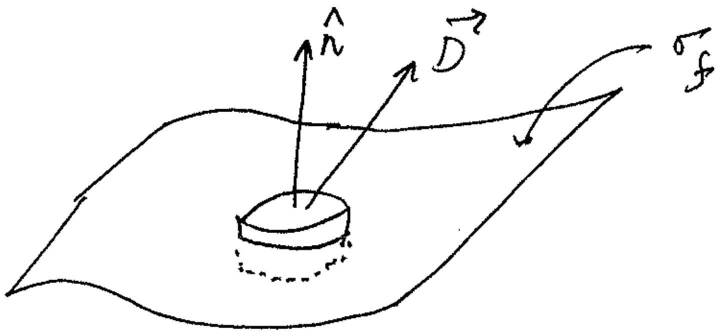
b) $\frac{\partial(\vec{E}\epsilon)}{\partial t} = -\frac{\epsilon V_0}{d} 2\pi\nu \sin(2\pi\nu t)\hat{z} = \vec{J}_D \leftarrow \vec{E} \Rightarrow \vec{J}_D$

$$\Rightarrow \frac{J_c}{J_D} = \frac{I_c}{I_D} = \frac{\frac{\sigma V_0}{d}\cos(2\pi\nu t)}{-\epsilon\frac{V_0}{d}2\pi\nu\sin(2\pi\nu t)} = -\frac{\sigma\cot(2\pi\nu t)}{2\pi\epsilon\nu}$$

$$\boxed{\frac{I_c}{I_D} = -\frac{1}{\underbrace{2\pi\epsilon\nu\rho}_{2.4}}\cot(2\pi\nu t)}$$

General Boundary Conditions

① $\vec{\nabla}\cdot\vec{D} = \rho_f \implies \int \vec{D}\cdot d\vec{S} = \int \rho_f \, d\tau$



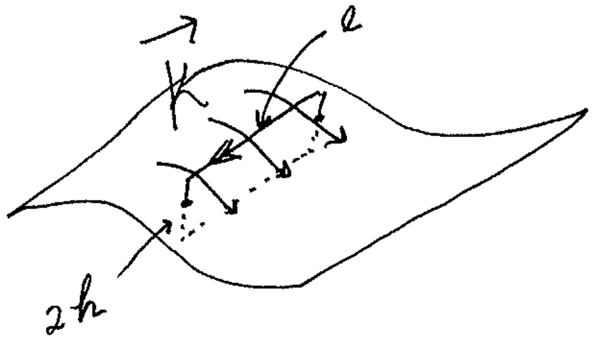
$$\implies \vec{D}_1\cdot\hat{n}A - \vec{D}_2\cdot\hat{n}A = \sigma_f A$$

$$\implies \boxed{\Delta D^{\perp} = \sigma_f}$$

② $\vec{\nabla}\cdot\vec{B} = 0 \implies \int \vec{B}\cdot d\vec{S} = 0$

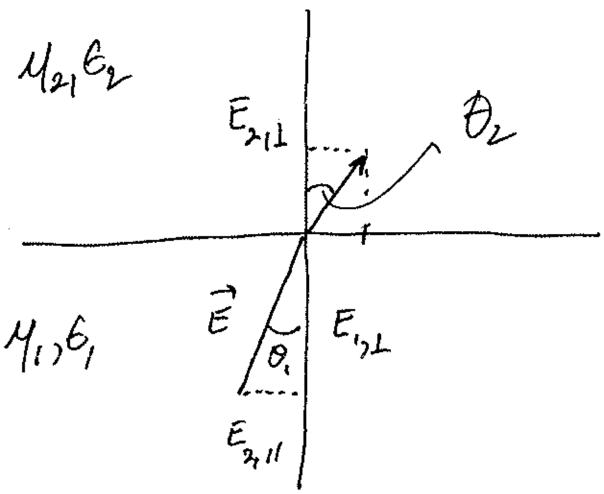
same geometry as for ① $\implies \boxed{\Delta B^{\perp} = 0}$

③ $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \implies \int \vec{E}\cdot d\vec{l} = -\dfrac{d}{dt}\int\vec{B}\cdot d\vec{S}$



$$\left(E_1^{\parallel} l - E_2^{\parallel} l\right) + \left(E_1^{\perp} h - E_1^{\perp} h\right) + \left(E_2^{\perp} h - E_2^{\perp} h\right) = -\frac{d}{dt}\int\vec{B}\cdot d\vec{S}$$

How does $\vec{E}$ refract?



(i) $\Delta D^{\perp} = 0 \rightarrow \epsilon_1 E_{1,\perp} = \epsilon_2 E_{2,\perp}$

or

$$E_{2,\perp} = \frac{\epsilon_1}{\epsilon_2} E_{1,\perp}$$

(ii) $\Delta E^{\parallel} = 0 \rightarrow E_{2,\parallel} = E_{1,\parallel}$

$$\Rightarrow \boxed{\tan\theta_2 = \frac{E_{2,\parallel}}{E_{2,\perp}} = \frac{E_{1,\parallel}}{E_{1,\perp}\frac{\epsilon_1}{\epsilon_2}} = \frac{\epsilon_2}{\epsilon_1}\tan\theta_1}$$

As an exercise, consider $\vec{B}$ and $\vec{H}$ at the interface of linear materials