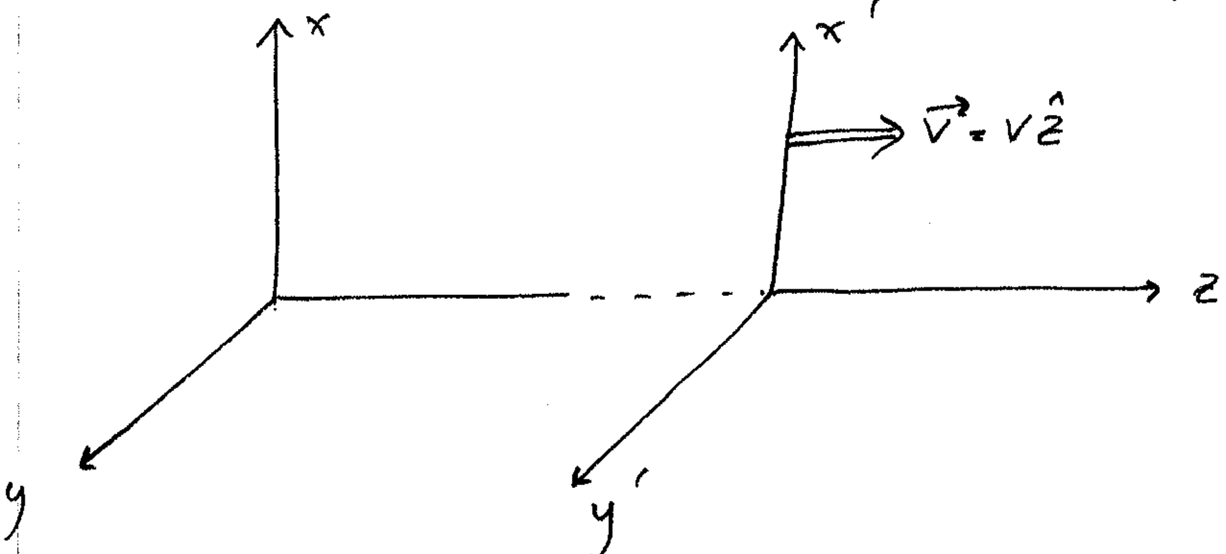
Special Relativity

Consider Galilean Transformations

The "Special Theory of Relativity" is concerned w/obs. made by two different observers, one of whom has a constant v/with respect to the other (no $\dot{v}$). We therefore use Cartesian frames to describe things.



In pre-relativistic physics, the frames are related by

$$\text{(A)} \quad x' = x, \; y' = y, \; z' = z + vt, \; t' = t$$

(where we assume the frames coincided at $t=0$). The above form the so-called

"Galilean Transformation"

(A) are obvious, but are they correct? What do I mean "correct"? Well, do the equations of motion have the same form for both ($\vec{x}$ and t) and ($\vec{x}'$ and t')

frames. That is, if I hold a system in a box, and translate the box at constant $\vec{v}$; does the motion internal to the box care?

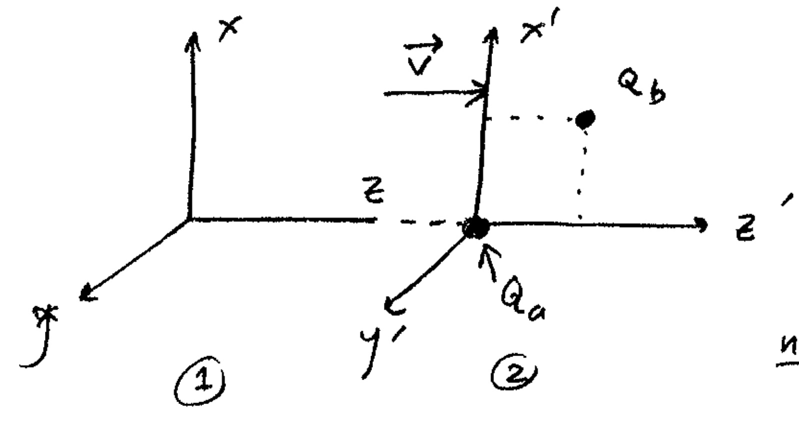
(A) breaks down as $v \to c$ ($\beta \to 1$)

Examples:

(a) Velocity Addition: $$\frac{dz'}{dt'} = \frac{dz}{dt}\frac{dt}{dt'} + v\frac{dt}{dt'} = V_0 + v$$

can be $> c$ if $V_0 + v > c + v$; light!

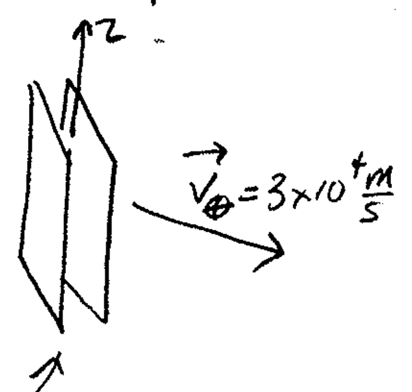
(b) EM phenomena: Trouton & Noble Expt.



In frame 2, how do charges Q_a and Q_b interact?

note: $\vec{v} \neq 0 \Rightarrow$ B-field $\Rightarrow$ torque(?)

Expt'l Set-Up



$\Longrightarrow$ 'induced' B should try to align plates // to $\vec{V}_\oplus$. Doesn't happen!

//-plate capacitor free to rotate

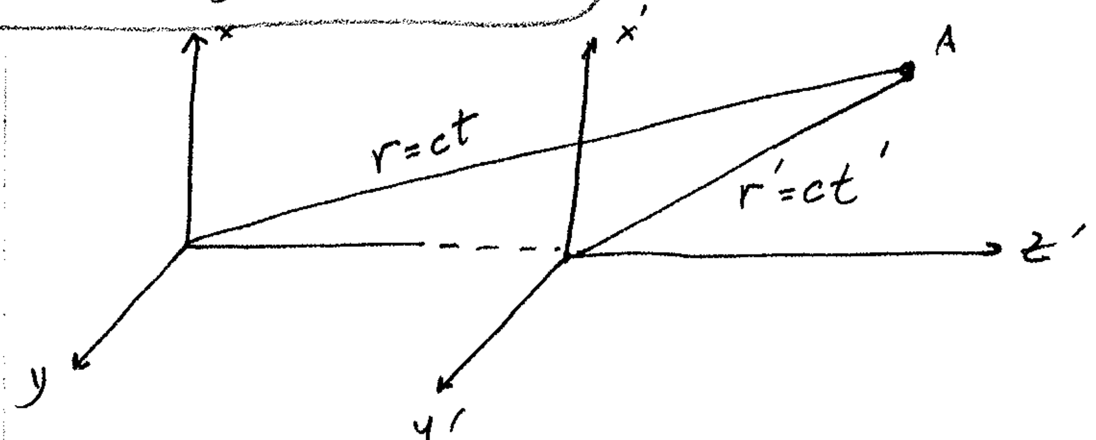
Something deeper than Galilean transformations are needed:
Einstein proposed <u>Special Relativity</u>.

<u>Postulates:</u> ① It is physically impossible to detect the uniform motion of a frame of reference from obs. made entirely within that frame.

② all observers, even when in uniform relative motion, will find the same value c for the speed of light in empty space.

The formal results which follow from these postulates are now derived.

Suppose a flash of light originates at $t=0$ when $z=z'=0$.
What is observed at A?



<u>Observer:</u> $Oxyz$ sees (x,y,z) @ t and $t = r/c$

$Ox'y'z'$ sees (x',y',z') @ t' and $t' = r'/c$

$r \neq r'$ (definitely), but this also implies

$t \neq t'$! time flows differently in among frames.
(because c is constant)

(i) $r^2 = x^2 + y^2 + z^2$ or $0 = x^2 + y^2 + z^2 - (ct)^2$

(ii) $r'^2 = x'^2 + y'^2 + z'^2$ or $0 = x'^2 + y'^2 + z'^2 - (ct')^2$

[The equality set to 0 must hold whenever these pos^ns refer to the arrival of light originating at $\vec{r} = \vec{r}' = 0$ and $t = t' = 0$! They thus must vary together and so,]

$$x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2$$

So, now, the question becomes one of finding the correct transformation of $(\vec{r}, t) \to (\vec{r}', t')$ such that the above holds!

Let: $-(ct)^2 = +(ict)^2$ and $-(ct')^2 = +(ict')^2$, so that

$$r^2 + (ict)^2 = r'^2 + (ict')^2$$

[These correspond to spheres in a 4-dimensional space, (known as Minkowski space*), and are invariant to rotations in 4-dimensional space.]

* note that Euclidean space would lack the "i" → no minus sign in the interval.

Lorentz Transformation I.

(i) $r^2 - c^2t^2 = r'^2 - c^2t'^2$

To effect this transformation, consider the following linear* transformation:

$$\begin{cases} z = A_t t' + A_z z' \\ t = B_t t' + B_z z' \end{cases} \quad \text{and} \quad \begin{cases} x = x' \\ y = y' \end{cases}$$

for motion // to z-axis

As a start, consider

C. Ray

N_2O ⟵ $z = z' = 0,\ t = t' = 0$

C. Ray hits O, N molecules → π-mesons

π-meson

⟸ $z' = 0,\ t' = \tau_\pi$

disintegrates

$\tau_\pi \simeq 2.6 \times 10^{-8}$ s

π-meson decays after $\tau_\pi \simeq 2.6 \times 10^{-8}\,\mathrm{s} = t'$

what are z and t?

* A non-linear transformation would lead to forces in one frame which don't exist in the other. Uniform motion in one frame would be accelerated in another!

frame moves w/ the π-meson, $z' = 0$

So; (i) In lab frame, $(z/t) = \beta c = v$

or

$$z = (\beta c t) = vt$$

or

$$z^2 = (\beta^2 c^2 t^2) = v^2 t^2$$

(ii) $r^2 - c^2 t^2 = r'^2 - c^2 t'^2$; same events

$$z^2 - c^2 t^2 = \cancel{z'^2}^{0} - c^2 t'^2$$

$$v^2 t^2 - c^2 t^2 = -c^2 t'^2 \Rightarrow c^2 t^2 = c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1} t'^2$$

$$\begin{cases} t = (1-\beta^2)^{-1/2} t' \\ \text{and} \\ z = c\beta (1-\beta^2)^{-1/2} t' \end{cases}$$

Thus, we must have

$$\begin{cases} z = \beta c (1-\beta^2)^{-1/2} t' + A_z z' \\ t = (1-\beta^2)^{-1/2} t' + B_{z/c} z' \end{cases}$$

<u>In general</u>,

$$r^2 - c^2 t^2 = r'^2 - c^2 t'^2$$

or

$$\left[\beta c(1-\beta^2)^{-1/2}t' + A_z z'\right]^2 - \left[c(1-\beta^2)^{-1/2}t' + B_z \frac{cz'}{c}\right]^2 = z'^2 - c^2t'^2$$

or

$$t'^2\left[\frac{\beta^2c^2}{1-\beta^2} - \frac{c^2}{1-\beta^2} + c^2\right] + t'\left[2\beta c\frac{A_z z'}{\sqrt{1-\beta^2}} - 2\frac{B_z z' c}{\sqrt{1-\beta^2}}\right]$$

$$+\left[A_z^2 z'^2 - B_z^2 z'^2 - z'^2\right] = 0$$

$$\frac{2ct'z'}{\sqrt{1-\beta^2}}\left[\beta A_z - B_z\right] + z'^2\left[A_z^2 - B_z^2 - 1\right] = 0$$

to be generally valid:

$$\left.\begin{array}{l} A_z = \frac{1}{\beta}B_z \\ A_z^2 = B_z^2 + 1 \end{array}\right\} \longrightarrow \begin{array}{l} A_z^2 = (1-\beta^2)^{-1} \\ B_z^2 = \beta^2(1-\beta^2)^{-1} \end{array}$$

and

$$z = \beta c(1-\beta^2)^{-1/2}t' + (1-\beta^2)^{-1/2}z'$$

$$t = (1-\beta^2)^{-1/2}t' + \frac{\beta}{c}(1-\beta^2)^{-1/2}z'$$

Letting: $\gamma = (1-\beta^2)^{-1/2}$, and $\beta = v/c$

$$z = \gamma\left[z' + vt'\right]$$

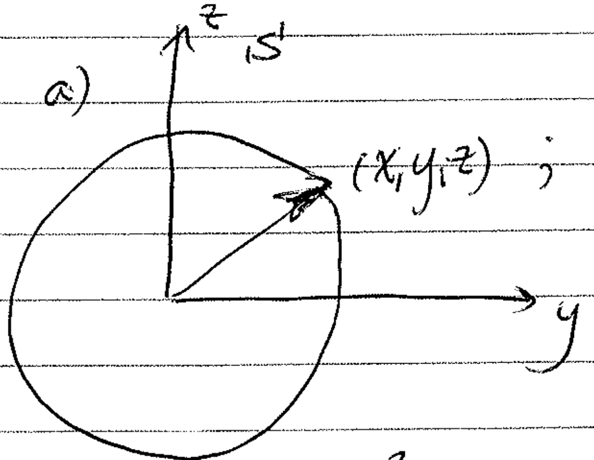
$$t = \gamma\left[\frac{v}{c^2}z' + t'\right]$$

Lorentz Transformation II. (follows)

Lorentz Transformations as Rotations in Minkowski Space (pseudo-Euclidean space)

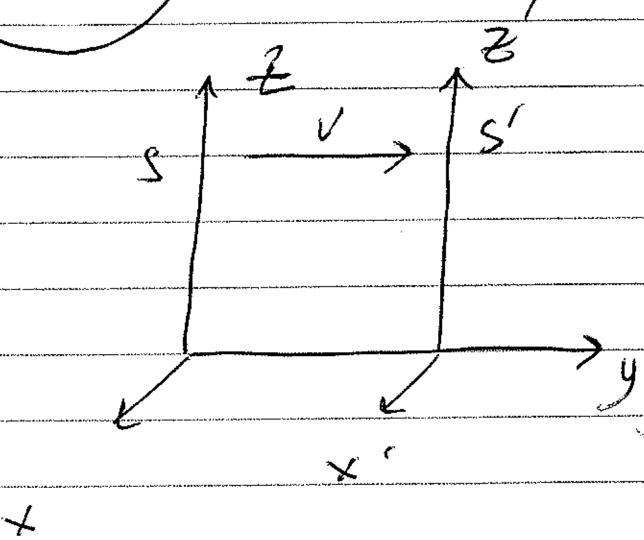
Speed of light is constant in all inertial frames, c.

a)



$$; \quad c^2t^2 = x^2 + y^2 + z^2$$

b)



at $t = t' = 0$, the light wave moves out

$$\Rightarrow c^2t'^2 = x'^2 + y'^2 + z'^2$$

c, speed of wave, is the same in S & S'

$$\boxed{c^2t^2 - x^2 - y^2 + z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2} \longleftarrow$$ Minkowski space, ("pseudo-Euclidean space")

Q: What is the appropriate transformation that takes $\vec{r} \rightarrow \vec{r}\,'$ to satisfy above condition?

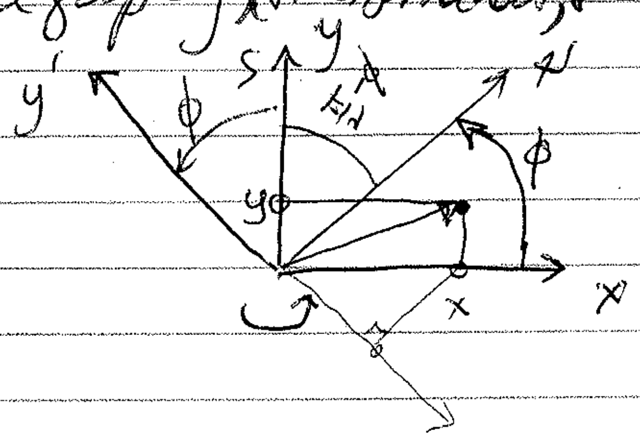
A: Let's consider a 3-D analogy in Euclidean space, a simple rotation

Euclidean Space

for a vector in 3D, consider the vector $\vec{r}$.

We form its norm $\boxed{\vec{r}\cdot\vec{r} = x^2+y^2+z^2}$

To be a vector, this length must be invariant to a group of coordinate transformations that include rotations.



Consider the rotation about the z-axis

$$\Rightarrow \begin{cases} x' = x\cos\phi + y\cos\left(\frac{\pi}{2}-\phi\right) \\ y' = y\cos\phi - x\cos\left(\frac{\pi}{2}-\phi\right) \\ z' = z \end{cases}$$

$$\left.\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi \\ z' &= z \end{aligned}\right\} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Rotation Matrix about z-axis}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

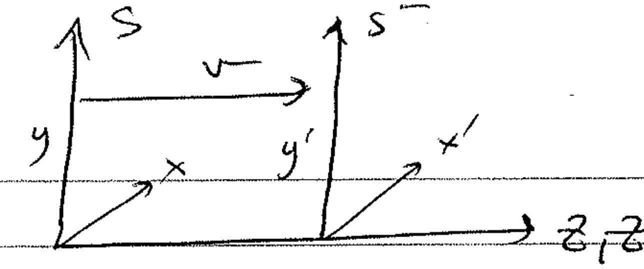
note:

$$|\vec{r}'|^2 = x^2\cos^2\phi + y^2\sin^2\phi + \cancel{2xy\cos\phi\sin\phi} + x^2\sin^2\phi + y^2\cos^2\phi - \cancel{2xy\sin\phi\cos\phi} + z'^2$$

$$\boxed{|\vec{r}'|^2 = x^2+y^2+z^2} \checkmark$$

note: ① $R(\phi)R(\theta) \neq R(\theta)R(\phi)$, unless about same axis

② $R(\phi)$ leaves the internal invariant, and depends only the "event."



rotation along z-axis for simplicity

$$\begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & +\sin\phi & \cos\phi \end{pmatrix}}_{\text{rotation matrix about axis} \perp \text{to } (z,t) \text{ in } -\phi \text{ direction}} \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

pseudo-rotation (about axis $\perp$ z,t) in pseudo-Euclidean space (Minkowski space)

$$\begin{cases} x' = x \\ y' = y \\ z' = z\cos\phi - ict\sin\phi \\ ict' = +z\sin\phi + ict\cos\phi \end{cases}$$

(z, t, z', t') are real $\Rightarrow$ $\sin\phi$ must be imaginary and $\cos\phi$ must be real

<u>let</u>: η ($\equiv$ rapidity) $= i\phi$

$$\Rightarrow \sin\phi = \frac{e^{i(-i\eta)} - e^{-i(-i\eta)}}{2i} = -i\sinh\eta$$

$$\cos\phi = \frac{e^{i(-i\eta)} + e^{-i(-i\eta)}}{2} = \cosh\eta$$

Start here

$$\Rightarrow \begin{cases} x' = x, \quad y' = y \\ z' = z\cosh\eta \text{ [redacted] } ct\sinh\eta \\ ict' = \text{[redacted] } iz\sinh\eta + ict\cosh\eta \end{cases}$$

Q: Does this leave the length invariant?

$$x'^2+y'^2+z'^2 \text{ [redacted] } c^2t'^2 = x^2+y^2+z^2\cosh^2\eta + c^2t^2\sinh^2\eta \text{ [redacted] } \cancel{2ctz\cosh\eta\sinh\eta}$$

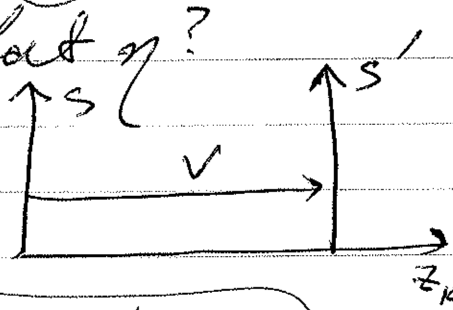
$$- z^2\sinh^2\eta - c^2t^2\cosh^2\eta + \cancel{2zct\sinh\eta\cosh\eta}$$

$$= x^2+y^2+z^2\underbrace{(\cosh^2\eta - \sinh^2\eta)}_{1} + c^2t^2\underbrace{(\sinh^2\eta - \cosh^2\eta)}_{-1}$$

$$= x^2+y^2+z^2-c^2t^2 \quad \checkmark$$

A: yes

Q: what η?



the frame S' moves w/speed v in z direction in frame S

Suppose S' moves at speed v with respect to S.

we have

$$\boxed{\begin{aligned} z' &= z\cosh\eta \text{ [redacted] } ct\sinh\eta \\ \underbrace{dz'}_{=0} &= dz\cosh\eta \text{ [redacted] } c\,dt\sinh\eta \end{aligned}}$$

$$\frac{dz}{dt} = +c\,\frac{\sinh\eta}{\cosh\eta} = +c\tanh\eta = v$$

frame S' moves at speed v w/respect to frame S

and $\boxed{\tanh\eta = +\frac{v}{c} = \beta}$

t/c,
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\eta & +i\sinh\eta \\ 0 & 0 & -i\sinh\eta & \cosh\eta \end{pmatrix}$$
← goal. We need $\sinh\eta$ & $\cosh\eta$

a) $\frac{\sinh\eta}{\cosh\eta} = +\frac{v}{c} \rightarrow \frac{\sinh^2\eta}{\cosh^2\eta} = \beta^2 = \frac{\cosh^2\eta - 1}{\cosh^2\eta} = \beta^2$

$\rightarrow 1-\beta^2 = +\frac{1}{\cosh^2\eta} \rightarrow \boxed{\cosh\eta = \sqrt{\frac{1}{1-\beta^2}}}$

b) $1 = \cosh^2\eta - \sinh^2\eta = \frac{1}{1-\beta^2} - \sinh^2\eta \rightarrow \sinh^2\eta = \frac{\beta^2}{1-\beta^2}$

$\boxed{\sinh\eta = \beta\sqrt{\frac{1}{1-\beta^2}}}$

$$\Longrightarrow \boxed{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & +i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}}$$
{ Lorentz transformation for boost along z-axis

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

and inverse transform

$$\begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix}$$

compare to (12.24) in text. Note $x^0 = ct$ while we use ict. The "i" is equivalent to the metric $g_{\mu\nu}$; we talk about this later

$\boxed{\text{Transformations (Lorentz) contain all the kinematic information of } \underline{\text{special relativity}}}$

<u>note</u>: ① $G(\vec{v}_1)G(\vec{v}_2) = G(\underbrace{\vec{v}_1 + \vec{v}_2}_{\vec{v}_3}) = G(\vec{v}_2)G(\vec{v}_1)$

for a Galilean transformation, velocities add

<u>Galilean transformations commute</u>

② $L(\vec{v}_1)\,L(\vec{v}_2) = L(\vec{v}_3)$

that for Lorentz transformations, the group property holds as for Galilean transformations, but

$$L(\vec{v}_1)\,L(\vec{v}_2) = L(\vec{v}_3) \neq L(\vec{v}_2)\,L(\vec{v}_1)$$

unless the velocities (boosts) are parallel

$$\Rightarrow \vec{v}_1 + \vec{v}_2 \neq \vec{v}_3$$

<u>Lorentz transformations do not commute, in general</u>

alternatively, can consider rot$^{\underline{ns}}$ in 4-dim space

Consider a rot$^{\underline{n}}$ in (z, ict) plane ; $x \to x'$, $y \to y'$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

$x \to x'$; $y \to y'$;

and

$$\Rightarrow \begin{cases} z = z'\cos\theta - ict'\sin\theta \\ \text{and} \\ ict = z'\sin\theta + ict'\cos\theta \end{cases}$$

$\Rightarrow$ $\begin{cases} \cos\theta \text{ must be real and} \\ \sin\theta \text{ must be imaginary!} \end{cases}$

$\Rightarrow$ θ is imaginary. Let $\theta = i\eta$ then

$$\cos\theta = \frac{e^{-\eta} + e^{+\eta}}{2} \qquad \sin\theta = \frac{e^{-\eta} - e^{+\eta}}{2i}$$

$$= \cosh\eta \qquad\qquad = i\sinh\eta$$

and $\quad x \to x'$, $y \to y'$

$$z = z'\cosh\eta - ict'(i\sinh\eta)$$

$$ict = z'i\sinh\eta + ict'\cosh\eta$$

Consider a stationary point in $Ox'y'z'$. Take the differentials of z and t;

$$\begin{cases} dz = d(\cancel{z'\cosh\eta}) + (c\sinh\eta)\,dt' \\ c\,dt = d(\cancel{z'\sinh\eta}) + (c\cosh\eta)\,dt' \end{cases}$$

$$\Longrightarrow \frac{dz}{dt} = +c\tanh\eta \equiv \text{Velocity in } Oxyz \text{ frame!} \equiv v$$

So, $\tanh\eta = \dfrac{\sinh\eta}{\cosh\eta}$ and

$$\cosh^2\eta - \sinh^2\eta = \frac{1}{4}\left[e^{-2\eta}+e^{2\eta}+2\right] - \frac{1}{4}\left[e^{-2\eta}+e^{2\eta}-2\right]$$

$$= 1$$

and $\tanh\eta = \dfrac{\sqrt{\cosh^2\eta - 1}}{\cosh\eta} \longrightarrow \cosh\eta = \dfrac{1}{\sqrt{1-\tanh^2\eta}}$

$$= \sqrt{\frac{1}{1-(v/c)^2}} = \gamma$$

$$\sinh\eta = \cosh\eta\tanh\eta$$

$$= \frac{v/c}{\sqrt{1-(v/c)^2}} = \frac{v}{c}\gamma$$

$$\Longrightarrow \begin{cases} x = x' \\ y = y' \\ z = \gamma(z' + vt') \\ t = \gamma(\frac{v}{c^2}z' + t') \end{cases}$$

and inverse transformations of

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma(z - vt) \\ t' = \gamma\left(-\frac{v}{c^2} z + t\right) \end{cases}$$

Simply let $\vec{x} \to \vec{x}'$ and $v \to -v$

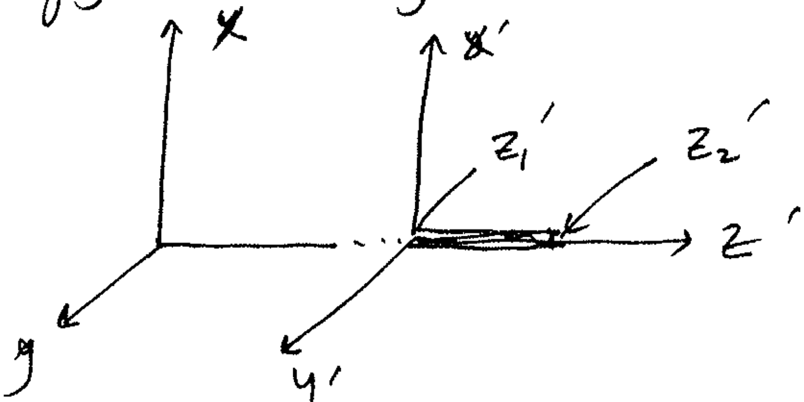
all kinematic results of Sp. Relativity are contained in [illegible]

Properties of Lorentz transformations

(A) **Length**: Consider a rigid stick of length

$$\boxed{L_0 = z_2' - z_1'}$$

lying at rest along the z' axis



(a) The observer $Oxyz$ ascribes to the stick a length
$(z_2 - z_1)$
at time
$t = t_1 = t_2$

In from S', L_0 given in terms of (z,t) quantities

$$\Rightarrow (z_2' - z_1') = \gamma(z_2 - vt_2) - \gamma(z_1 - vt_1) = L_0$$

$$= \gamma(z_2 - z_1) - \gamma v \underbrace{(t_2 - t_1)}_{t_1 = t_2 \Rightarrow (t_2 - t_1)}$$

and so

$$L = (z_2 - z_1) = \left(\frac{z_2'}{\gamma} + vt_2\right) - \left(\frac{z_1'}{\gamma} + vt_1\right)$$

$$= \frac{1}{\gamma}(z_2' - z_1') = L_0/\gamma < L_0 \, !$$

that is, the stationary observer sees a "contracted" stick!

(b) Suppose Oxyz holds the stick of length L_0; so,

$$L_0 = z_2 - z_1$$

what does $O'x'y'z'$ measure? at time $t' = t_1' = t_2'$, measure z_2', z_1' and so

$$L' = z_2' - z_1' = \gamma\left[z_2 - vt_2 - z_1 + vt_1\right]$$

$$= \gamma(z_2 - z_1) - \gamma v \underbrace{(t_2 - t_1)}_{\left(\frac{v}{c^2} z_2 + \frac{1}{\gamma} t_2' - \frac{v}{c^2} z_1 - \frac{1}{\gamma} t_1'\right)}$$

evaluate at $t' = t_1', t_2'$

$$= \gamma\left[(z_2 - z_1) - \left(\frac{v}{c}\right)^2 (z_2 - z_1)\right]$$

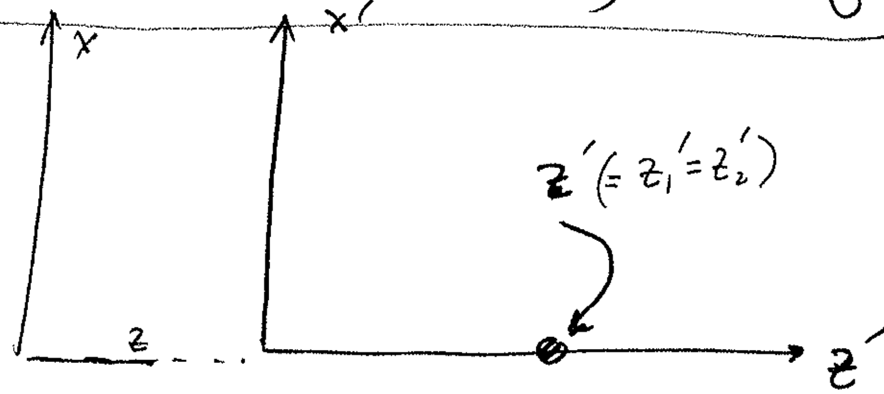
$$L' = \frac{1}{\gamma}(z_2 - z_1) \quad \checkmark$$

So, $O'x'y'z'$ also sees a "contracted" stick

(c) the sol^{n} to this paradox is that, one observer sets $t = t_1 = t_2$ while the other sets $t' = t_1' = t_2'$ $\Rightarrow$ different observations are made!

Ⓑ Time Dilation

Suppose observer $O'x'y'z'$ sees an event happen at $t' = t_1', t_2'$ @ $z_1' = z_2' = z'$ ($\Delta z' = 0$)
that is, the event is stationary in his frame



and $\Delta t' = t_2' - t_1'$

Observer $Oxyz$ sees an event happen at two differents z's; z_1, z_2 at two different times; t_1, t_2

ⓐ $z_1 = \gamma(z_1' + vt_1')$; $z_2 = \gamma(z_1' + vt_2')$

$\boxed{\Delta z = v\gamma\Delta t'} \left\{ z_2 - z_1 = \gamma(z_1' + vt_2') - \gamma(z_1' + vt_1') \right.$

ⓑ $\Delta t = t_2 - t_1 = \gamma\left[t_2' + \frac{v}{c^2}z_1' - t_1' - \frac{v}{c^2}z_1'\right]$

$$= \gamma\left[t_2' - t_1'\right]$$

$$\boxed{\Delta t = \gamma\Delta t' > \Delta t'}$$

Thus, the stationary observer measures a $\Delta t > \Delta t'$ of the moving frame clock!

$\equiv$

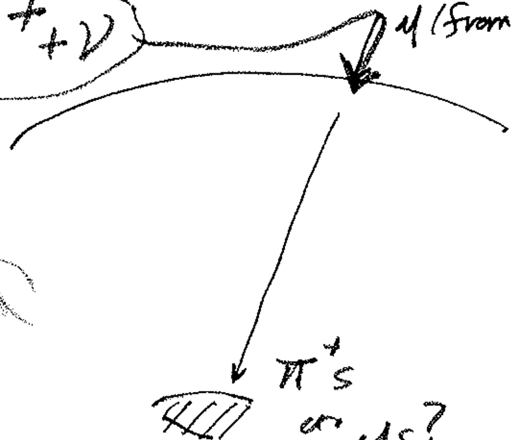
and vice versa $\Rightarrow$ any moving clock appears to run slower by a factor of γ^{-1}!

This time dilation is experimentally verified by fast moving fundamental particles. Return to π-mesons (actually μ's)

Suppose you see muons (μ) "raining" down on the atm.

$\pi^+ \rightarrow \mu^+ + \nu$

μ (from CRs) (actually generated by Cosmic Rays striking the atm.)



at rest

$\tau_\mu \simeq 2.2\ \mu s$

π^+s or μs?

$$\Rightarrow d = \tau_\mu V = 7\times 10^4\,\text{cm}\left(\frac{V}{c}\right) < 1\,\text{km}$$

β

But μ's reach the surface of the earth! (atm >> 1 k

# of muons (μ)

ⓐ $\dfrac{dN}{dt'} = -\dfrac{N}{\tau_\mu}$ in the frame of the μ ← decay of μs in their rest frame

ⓑ In the frame of a stationary observer, $dt = \gamma dt'$

and, since the # of decays is invariant,

$$\frac{dN}{dt} = -\frac{N}{\gamma\tau_\mu}$$ and the "half-life" becomes $\gamma\tau_\mu$

So, the decay per unit length becomes

$$N = N_0 \exp\left(-t/\gamma\tau_\mu\right)$$

and $vt = x \Rightarrow t = x/v$

$$N(x) = N_0 \exp\left(-x/v\gamma\tau_\mu\right)$$

To have a reasonable # of μ's travel several hundred km to the $\oplus$

$$\Rightarrow +x \simeq v\gamma\tau_\mu \Rightarrow \gamma \simeq + \frac{x}{v\tau_\mu}$$

$$\simeq 150\left(\frac{x}{10^2\,km}\right)\left(\frac{v}{c}\right)^{-1}$$

or $\beta \simeq 1 - \frac{1}{\gamma^2} = 0.999956$

$\rightarrow v \simeq c$

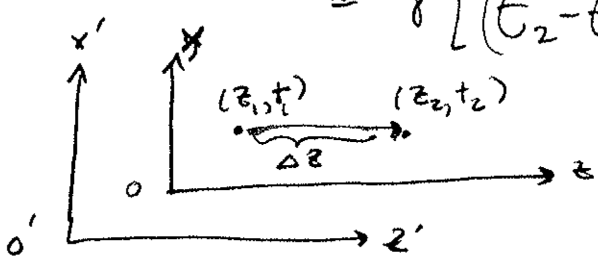
(c) Simultaneity

Observer $Oxyz$ sees an event at (z_1, t_1), (z_2, t_2)
Observer $O'x'y'z'$ sees the same event with a time interval of

$$\Delta t' = t_2' - t_1' = \gamma\left[t_2 - \frac{v}{c^2}z_2 - t_1 + \frac{v}{c^2}z_1\right]$$

$$= \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(z_2 - z_1)\right]$$

figure



Suppose the event was simultaneous to Oxyz
$\Rightarrow \Delta t = 0$!

and $\Delta t' = \gamma\left[-\frac{v}{c^2}(z_2 - z_1)\right] \neq 0$ unless $\Delta z = 0$

and so, in general, the events do not have to appear simultaneously to both observers.

Simultaneity depends on the observer!

note: If $\Delta z = 0$, but $\Delta x, \Delta y \neq 0$, can still be simultaneous $\Rightarrow$ above does not say the events are at the same point in space.

Causality:

$$\Delta t' = \gamma\left[\underbrace{(t_2 - t_1)}_{\Delta t} - \frac{v}{c^2}(z_2 - z_1)\right]$$

the - sign $\Rightarrow \Delta t'$ and Δt can have different signs!

In Oxyz frame, $\Delta t > 0$

$\Rightarrow$ if $\Delta t - \frac{v}{c^2}(z_2 - z_1) < 0$, $\Delta t' < 0$

or, if $\frac{(z_2 - z_1)}{\Delta t} > \frac{c^2}{v} > c$ (for $v < c$)

for the reversal to occur!

that is, the disturbance which caused the events at
z_1 and z_2 had to propagate $\ni$

$$\frac{\Delta z}{\Delta t} > c \quad \left(\text{for } v<c \right)$$

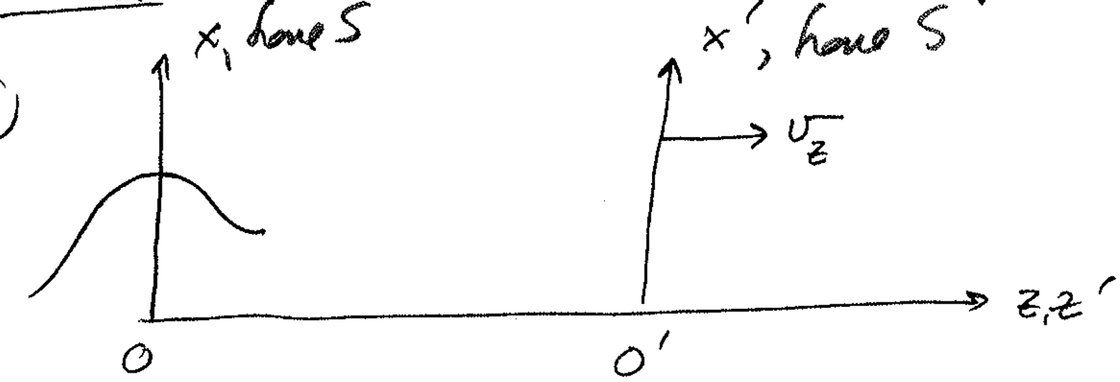
We require a maximum signal propagation speed of c to violate causality.

ⓓ Doppler Shift

The time dilation effect also influences the period T or T' and hence frequency, $\omega = 2\pi/T$ and $\omega' = 2\pi/T'$ ascribed to harmonic waves in relative motion.

Consider:

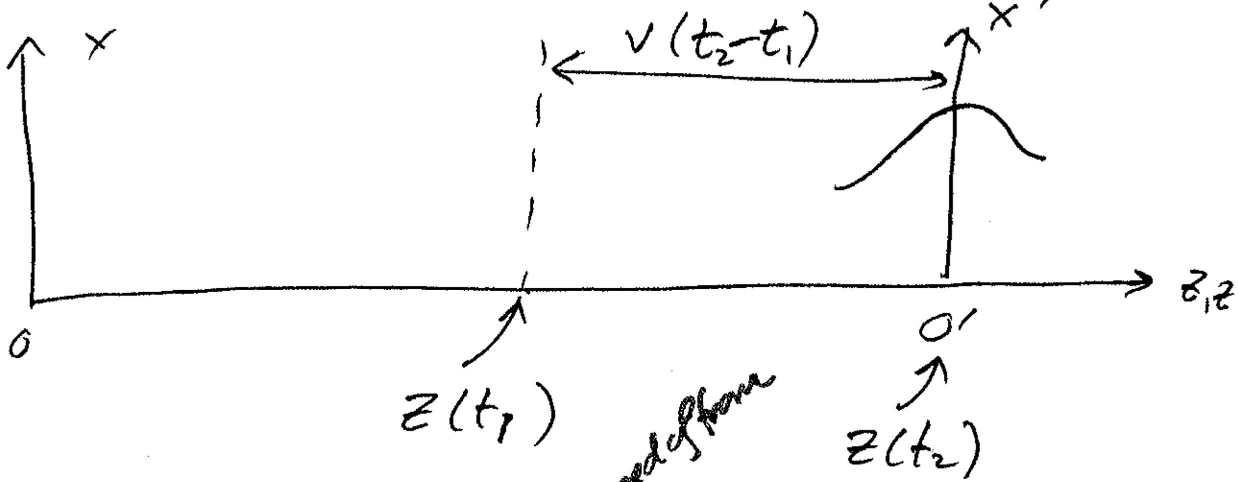
1



at time $t=t_1$ a wave crest passes by Observer O. at this time, O' is at

$$z = z(t_1)$$

② let t_2 be the instant (by O's clock) when the same crest passes O′ (at $z_z(t_2)$))



③ the phase velocity of the wave is speed of frame c and so,

$$z(t_2)\left[\equiv \underbrace{z(t_1)+v(t_2-t_1)}_{\text{pos}^{\text{n}}\text{ of observer}}\right] = \underbrace{c(t_2-t_1)}_{\text{distance wave traveled}}$$

So that

$$(c-v)(t_2-t_1) = z(t_1)$$

④ If the wave has period $\Delta t_1 = T = 2\pi/\omega$ by O's clock, then the next crest leaves O at his time of (t_1+T), when O′ is at $z(t_1+T) = z(t_1)+vT$ and will arrive at O′ at moment $t_2+\Delta t_2$ as given by

"catchy" up formula $\Longrightarrow$

$$\underbrace{(c-v)}_{\text{relative speed}}(\underbrace{t_2+\Delta t_2}_{\text{arrival}}\underbrace{-t_1-\Delta t_1}_{\text{departure}\;-(t_1+\underbrace{\Delta t_1}_{T})}) = \underbrace{z(t_1)+vT}_{z(t_1+T)}$$

$\Delta t_1 = T$

(a) The difference in these 2 arrival times is

$$(c-v)(\Delta t_2 - T) = vT$$

or

$$\boxed{\Delta t_2 = \frac{T}{1-\frac{v}{c}}}$$

this is the time as measured on O's clock for the new period of the wave w/respect to O'. If O' were moving nonrelativistically, this is the period he would also measure and hence

$$\boxed{\omega_{NR} \simeq \left(1-\frac{v}{c}\right)\omega} \quad ; \text{since } \Delta t_2' \simeq \Delta t_2$$

(b) If the frame O'x'y'z' moves relativistically, then $\Delta t_2' \neq \Delta t_2$ and time dilation must be taken into account! This is a time interval between events at rest in O' (passage of crests past O' origin) and so,

$$\Delta t_2 = \gamma \, \Delta t_2'$$

is dilated w/respect to $\Delta t_2' = T' = 2\pi/\omega'$!

7 So $\Delta t_2 = \frac{T}{1-\beta} = \Delta t_2 = \gamma \Delta t_2'$

$$\Longrightarrow \Delta t_2' = \frac{1}{\gamma}\frac{T}{1-\beta}$$

and

$$\omega' = \gamma(1 \mp \beta)\omega$$

"$-$" $\rightarrow$ $\omega \downarrow$ as $\beta \uparrow$ $\Longrightarrow$ redshift
"$+$" $\rightarrow$ $\omega \uparrow$ as $\beta \uparrow$ $\Longrightarrow$ blueshift

for $\beta (\geq 0)$ and $-$ Redshift

(i) $\omega' = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega$

$\beta \to 1 \Rightarrow \omega' \to 0$, or $\lambda \to \infty$

(ii) $\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda \rightarrow \frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \equiv (Z+1)$

and

$$\frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \underset{\beta\to 0}{\simeq} \beta \simeq Z$$

Relativistic Addition of Velocities

$$\left.\begin{aligned} x &= x' \\ y &= y' \\ z &= \gamma(z'+vt') \\ t &= \gamma(t'+v/c^2 z') \end{aligned}\right\} \quad (A)$$

take differentials of (A) and form

$$\frac{dx}{dt} = \frac{dx'}{\gamma(dt'+\frac{v}{c^2}dz')} = \frac{dx'/dt'}{\gamma(1+\frac{v}{c^2}dz'/dt')}$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt'+\frac{v}{c^2}dz')} = \frac{dy'/dt'}{\gamma(1+\frac{v}{c^2}dz'/dt)}$$

$$\frac{dz}{dt} = \frac{\gamma(dz'+vdt')}{\gamma(dt'+\frac{v}{c^2}dz')} = \frac{dz'/dt'+v}{1+\frac{v}{c^2}dz'/dt'}$$

let: $\vec{u} = \frac{d\vec{r}}{dt}$, $\vec{u}' = \frac{d\vec{r}'}{dt'}$ and $\vec{u}_T = u_x\hat{x} + u_y\hat{y}$

then

$$u_z = \frac{u_z'+v}{1+\frac{v}{c^2}u_z'}, \quad \vec{u}_T = \frac{\vec{u}_T'}{\gamma(1+\frac{v}{c^2}u_z')}$$

for the velocities u_z and u_z' observed in the O and O' frames!

again, the inverse transformation sets $u \to u'$ and $v \to -v$

for motion // z

$$u \to u_z \to \frac{u_z' + v}{1 + \frac{v u_z'}{c^2}} \leq c$$

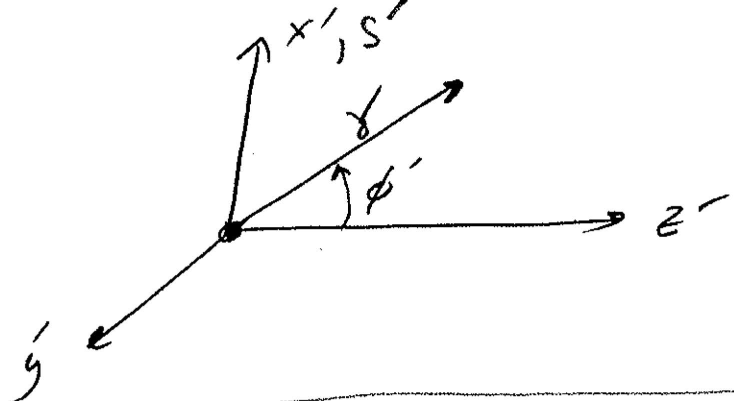
e.g., (i) $\underbrace{u_z' = c}_{\substack{\text{photon} \\ \text{(or any massless} \\ \text{particle)}}} \implies \dfrac{c + v}{1 + v/c} = c\left(\dfrac{1 + v/c}{1 + v/c}\right) = c$!

(ii) $v = c \implies \dfrac{u_z' + c}{1 + \frac{u_z'}{c}} = c$!

(iii) if $v = u_z' = c \implies \dfrac{c + c}{1 + 1} = c$!

"Headlight" Effect

Consider a moving source (frame $O'x'y'z'$) w/ velocity $\vec{v}$ which emits a photon at an angle ϕ' with respect to its z' axis (in x'-z' plane)



Q: What angle ϕ, does this photon make with the rest frame?, S

1) In $O'x'y'z'$: $v_x' = c\sin\phi'$, $v_z' = c\cos\phi$, $v_y' = 0$

Using the velocity transformation we get:

2)
$$v_z = \frac{c\cos\phi' + v}{1 + \frac{v}{c^2}c\cos\phi'}, \quad v_x = \frac{c\sin\phi'}{\gamma(1 + \frac{v}{c^2}c\cos\phi')}$$

$$v_y = 0$$

3) and the observed $\measuredangle$ in $Oxyz$ is

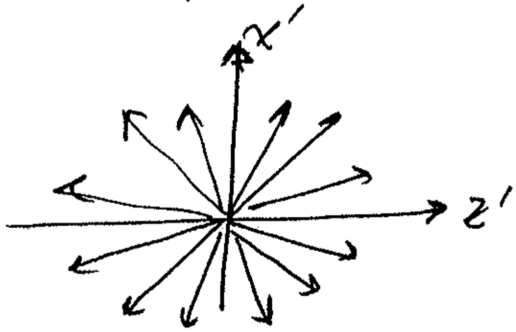
$$\tan\phi = \frac{v_x}{v_z} = \frac{\sin\phi'}{\gamma(\cos\phi' + \frac{v}{c})}$$

<u>Let</u>: $\beta = \frac{v}{c} \to 1 \Rightarrow \gamma = (1-\beta^2)^{-1/2} \longrightarrow \infty$

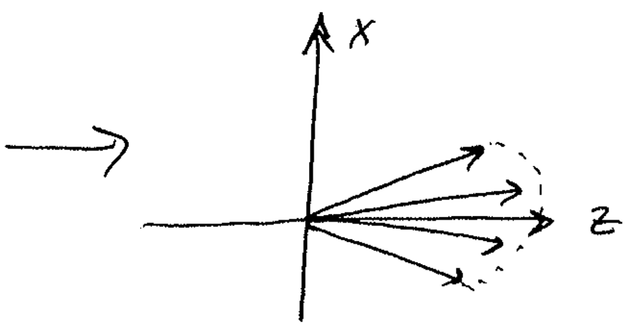
$$\tan\phi \longrightarrow \frac{1}{\infty}\left[\frac{\sin\phi'}{1+\cos\phi'}\right] \to 0$$

unless $1+\cos\phi' \to 0$
(i.e., $\phi' = \pi$)

$\therefore \phi \to 0$, unless $\phi' = \pi$



If photons emitted isotropically in moving frame



photons are beamed into the forward direction (dir$^{\underline{n}}$ of motion)!

$\boxed{\text{Recall: we've gone to c.g.s.}}$ ← No, Jackson goes to c.g.s.

Frame Independent Representation

Consistency w/ relativistic principles requires that expressions for physical expectations are equally applicable in every equiv. reference frame, e.g.,

$$\vec{F} = m\ddot{\vec{u}}$$

is invariant under Galilean transformations. Maxwell's eqns. are not invariant under Galilean transformations, but are expected to be invariant under Lorentz transformations.

The notation here is meant to make sure (or find the representation) where these expectations are true.

We find that physical laws that are expressed in 4-tensor form automatically satisfy Einstein's postulates of relativity. They are manifestly covariant.

[A theorem of tensor calculus states that if "two tensors are equal in any coordinate system, then they are equal in any other coordinate system." Thus if we express the laws of physics in the form of 4-vectors, then they will be valid in any coordinate system.]

Structure of Space-time

We have been using,

(x, y, z, ict) and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix}$$

for our representations (discussions) so far. Let's change up a little bit.

(a) We switch to

$$x^0 = ct \quad \text{and} \quad \beta = \frac{v}{c}$$

$\Rightarrow$ we change the unit of time from seconds to meters; 1 meter of x^0 corresponds to $t = \frac{1\,m}{c}$

$\Rightarrow$ $\boxed{x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z}$ ← **Change order**

$$\Rightarrow \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

for a boost along x^1

or $x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3)$ ← superscript → contravariant

and the transformation is given by

~~$(x'^\alpha) = a\ x$ Lorentz transformation~~

$\mu \to$ row

$$X'^{\mu} = \sum_{\beta=0}^{3} \Lambda^{\mu}{}_{\nu} X^{\nu}$$

Lorentz transformation

Formula for transformation of coordinates from frame S to frame S' for a boost along x^1 axis.

<u>Q: What is the form for $\Lambda^{\mu}{}_{\nu}$?</u>

$$\begin{cases} X'^0 = \gamma X^0 - \gamma\beta X^1 + 0 + 0 \\ X'^1 = -\gamma\beta X^0 + \gamma X^1 + 0 + 0 \\ X^2 = 0 + 0 + X^2 + 0 \\ X^3 = 0 + 0 + 0 + 1 \end{cases}$$

note: look at x'^0 relation; $X'^0 \to \alpha = 0$, $\nu = 0 \to 3$

(i) $\frac{\partial X'^0}{\partial X^0} = \gamma = \Lambda^0{}_0$ (iii) $\frac{\partial X'^0}{\partial X^2} = 0 = \Lambda^0{}_2$

(ii) $\frac{\partial X'^0}{\partial X^1} = -\gamma\beta = \Lambda^0{}_1$ (iv) $\frac{\partial X'^0}{\partial X^3} = 0 = \Lambda^0{}_3$

look at x'^1 relation; $x'^1 \to \alpha = 1$, $\nu = 0 \to 3$

(i) $\frac{\partial X'^1}{\partial X^0} = -\gamma\beta = \Lambda^1{}_0$ (iii) $\frac{\partial X'^1}{\partial X^2} = 0 = \Lambda^1{}_2$

(ii) $\frac{\partial X'^1}{\partial X^1} = \gamma = \Lambda^1{}_1$ (iv) $\frac{\partial X'^1}{\partial X^3} = 0 = \Lambda^1{}_3$

repeated covariant + contravariant index ⇒ summation, Einstein summation convention

and we can write

$$x'^{\alpha} = \sum_{\nu=0}^{3} \left(\frac{\partial x'^{\alpha}}{\partial x^{\nu}}\right) x^{\nu} = \Lambda^{\alpha}{}_{\nu}\, x^{\nu}$$

form of transfer vector

raised index signifies a <u>contravariant vector</u>

(b) the position 4-vector,

$$x^{\mu} = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

is a contravariant variant and transforms as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

(c)

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse transform, $\Lambda^{-1\,\mu}{}_{\nu}$, is found from $\beta \to -\beta$

$$\Lambda^{-1\,\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\Lambda^{\mu}{}_{\nu} \Lambda^{-1\mu}{}_{\nu} = \delta_{\alpha}{}^{\beta}$

<u>note:</u> $\Lambda^{\mu}{}_{\nu} \neq \Lambda_{\mu}{}^{\nu}$ for general tensors. This is true for the Lorentz transformation because it is symmetric.

(d) We define a covariant transformation as one where the object transforms as the coordinate

$$x'_{\mu} = \left(\frac{\partial x^{\nu}}{\partial x'^{\mu}}\right) x_{\nu}$$

$$= \Lambda^{-1\nu}{}_{\mu} x_{\nu}$$

<u>recall:</u>

$$x'^{\mu} = \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) x^{\nu}$$

(e) Any object that transforms a coordinate, r^{μ}, is a 4-vector, that is, any object that transforms as

$$\bar{A}^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu}$$

is a 4-vector.

Also, more generally, we have that if an object transforms as

$$T^{\alpha\beta} = \Lambda^{\alpha}{}_{\delta} \Lambda^{\beta}{}_{\gamma} T^{\delta\gamma}$$

then it is a 4-tensor. And so on ... to n-tensors

Tensors (scalars, 4-vectors, 4-tensors, ...) are covariant (and compact formulations).

Q: Is it easy to construct 4-tensors?
A: Yes, because if we construct 4-tensors out of 4-vectors, we're golden!

A useful guide is the "Quotient Rule." The quotient of 2 4-tensors is a 4-tensor.

the "quotient" ~~tensor~~ of tensors is difficult, but an alternative formulation states that,

"if the contraction of an entity $T_{\alpha\ldots}^{\beta\ldots}$ with an arbitrary 4-tensor, $B_{\mu\ldots}^{\nu\ldots}$ produces a 4-tensor $A_{\kappa\ldots}^{\lambda\ldots}$, then $T_{\alpha\ldots}^{\beta\ldots}$ is a 4-tensor."

Pf Consider $\overset{\text{"object"}}{T^{\alpha\beta}}\, \overset{\text{4-vectors}}{B_\beta = A^\alpha}$. Transform to S'

$$T'^{\alpha\beta} B'_\beta = \underbrace{\Lambda^\alpha{}_\kappa \Lambda^\beta{}_\lambda T^{\kappa\lambda}}_{T'^{\alpha\beta}} \Lambda^{-1\delta}{}_\beta B_\delta = A'^\alpha = \Lambda^\alpha{}_\beta A^\beta$$

$$T'^{\alpha\beta} \Lambda^{-1\kappa}{}_\beta B_\kappa = \Lambda^\alpha{}_\beta A^\beta$$

$$\Lambda^\beta{}_\lambda T^{\kappa\lambda} \Lambda^\alpha{}_\kappa \Lambda^{-1\kappa}{}_\beta B_\kappa = \Lambda^\alpha{}_\beta A^\beta$$

Transform

$$T'^{\alpha\beta} B'_{\beta} = A'^{\alpha}$$

$$T'^{\alpha\beta} \Lambda^{-1\kappa}{}_{\beta} B_{\kappa} = \Lambda^{\alpha}{}_{\lambda} A^{\lambda}$$

$$\Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} T^{\mu\nu} \Lambda^{-1\kappa}{}_{\beta} B_{\kappa} = \Lambda^{\alpha}{}_{\lambda} A^{\lambda}$$

$$\Lambda^{\alpha}{}_{\mu} T^{\mu\nu} \delta_{\nu}{}^{\kappa} B_{\kappa} = \Lambda^{\alpha}{}_{\lambda} A^{\lambda}$$

$$\Lambda^{\alpha}{}_{\mu} T^{\mu\kappa} B_{\kappa} = \Lambda^{\alpha}{}_{\lambda} A^{\lambda}$$

Consider the phase φ of an arbitrary wave,

$$\varphi = \omega t - \vec{k} \cdot \vec{r} = k_\alpha r^\alpha$$

where

$$\varphi = \omega t - \vec{k} \cdot \vec{r} = k_0 ct + k_1 x^1 + k_2 x^2 + k_3 x^3$$

$$\Rightarrow \boxed{\begin{aligned} k_\alpha &= \left(\frac{\omega}{c}, -k_1, -k_2, -k_3 \right) \\ &= \left(\frac{\omega}{c}, -\vec{k} \right) \end{aligned}}$$

(i) the phase φ is a scalar invariant to some transformation b/c for a wave all observers count the same # of wave fronts

(ii) r^α is an arbitrary 4-vector

(i) & (ii) $\Rightarrow$ k_α is a 4-vector by the quotient rule

Doppler Shift

$$k_\mu' = k_\kappa \Lambda^{-1\,\kappa}{}_{\mu}$$

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\omega}{c} \\ +k_1 \\ +k_2 \\ +k_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} k_0' = -\frac{\omega'}{c} = -\gamma\frac{\omega}{c} + \beta\gamma k_1 \\ k_1' = k_1' = -\beta\gamma\frac{\omega}{c} + \gamma k_1 \\ k_2' = k_2 \\ k_3' = k_3 \end{cases}$$

(a) look at k_0'. Assume $k_2 = k_3 = 0$; light ~~moving~~ moving toward the observer

$$-\frac{\omega'}{c} = -\gamma\frac{\omega}{c} + \beta\gamma k$$

$$= -\gamma\frac{\omega}{c} + \beta\gamma\frac{\omega}{c}$$

$$-\frac{\omega'}{c} = \frac{\omega}{c}\gamma(\beta - 1) \Longrightarrow \frac{\omega'}{\omega} = \cancel{\frac{\gamma}{1-\beta}}$$

$$= \gamma(1-\beta)$$

$$\boxed{\frac{\omega'}{\omega} = \sqrt{\frac{1-\beta}{1+\beta}}}$$

note: classical Doppler Effect, $\frac{\omega'}{\omega} \simeq (1-\beta)$

Metrics, Norms, & Lorentz Transformations

We want to look intervals in a second, so let's write down the scalar product (dot product) in our current formulation:

$$-a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3 \text{, (Minkowski Space)}$$

for 2 arbitrary 4-vectors, $\vec{a}, \vec{b}$, which is invariant under Lorentz transformation, $\Lambda^{\mu}{}_{\nu}$. The covariant form of a 4vector $\vec{a}\,(a^{\mu})$ goes to

$$a_{\mu} = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3)$$

$\Rightarrow$ same as a^{μ} except for the a_0 component sign

$\Rightarrow$ Scalar product $= \sum_{\mu=0}^{3} a_{\mu} b^{\mu}$, by Einstein convention

$$= a_{\mu} b^{\mu} = -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3$$

or

$$= -a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$$

Intervals

$$\Delta x^{\mu} = (x^{\mu}_A - x^{\mu}_B) \quad \text{for 2 events, A \& B}$$

$$\text{Interval} = (\Delta x)_{\mu} (\Delta x)^{\mu}$$

for Spec. Rel. (and in Cartesian)

$$= -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

In general, we write ds^2 (the norm or metric) as

(i) $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$

$g_{\alpha\beta}$ = metric tensor

for Special Relativity, $g_{00} = -1$
$g_{11} = g_{22} = g_{33} = +1$

(ii) Contravariant form is $g^{\alpha\beta} \left(= g_{\alpha\beta}\right)$

(iii) $\underbrace{g_{\alpha\gamma} g^{\gamma\beta}}_{\text{contraction of } g_{\alpha\beta} g^{\delta\gamma}} = \underbrace{g_{\alpha 0} g^{0\beta}} + \underbrace{g_{\alpha 1} g^{1\beta}} + \underbrace{g_{\alpha 2} g^{2\beta}} + \underbrace{g_{\alpha 3} g^{3\beta}}$

$= \delta_{\alpha=0}^{\beta=0} + \delta_{\alpha=1}^{\beta=1} + \delta_{\alpha=2}^{\beta=2} + \delta_{\alpha=3}^{\beta=3}$

$= \delta_\alpha^\beta$

(iv) we note: $ds^2 = g\, dx_\mu dx^\mu$ from "dot product"

$= g_{\alpha\beta}\, dx^\alpha dx^\beta$ from (i) above

$\Rightarrow$ can change contravariant dx^α to covariant dx_β

from $dx_\beta = g_{\alpha\beta}\, dx^\alpha$

and $dx^\alpha = g^{\alpha\beta}\, dx_\beta$

$g_{\alpha\beta}$, $g^{\alpha\beta}$ raises & lowers indices the metric ($g_{\alpha\beta}$, $g^{\alpha\beta}$) raises and lowers indices, changes contravariant $\rightleftarrows$ covariant

Return to

$$\boxed{\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2}$$

or

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = dx_\beta\, dx^\beta$$

(1) for two arbitrary events, Δs^2 can take 3 signs:

(A) $\Delta s^2 > 0 \Rightarrow$ spacelike (light cannot travel between 2 events)

$(\Delta x^2 + \Delta y^2 + \Delta z^2) > c^2\Delta t^2$; called spacelike b/c if events are simultaneous ($\Delta t = 0$) and so only spatial separation occurs

(B) $\Delta s^2 < 0 \Rightarrow$ timelike (light can travel between locations quite comfortably)

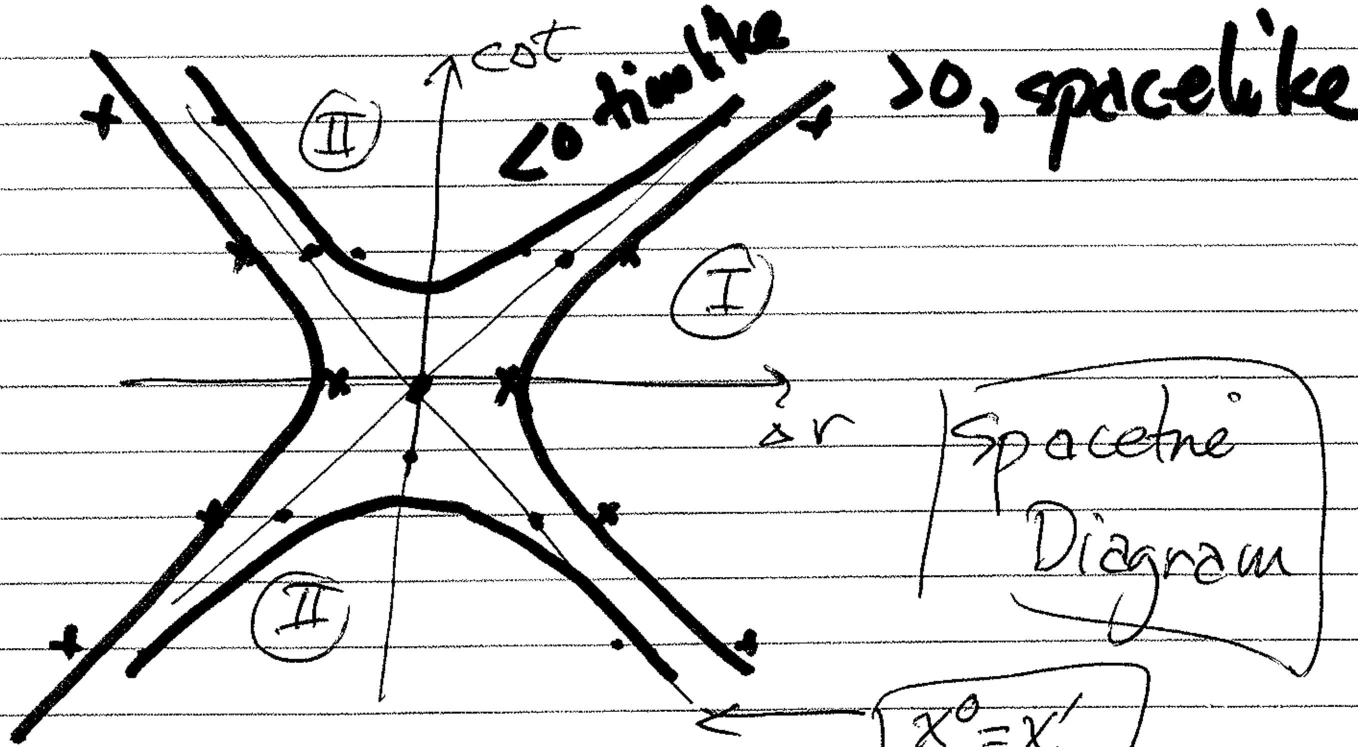
$c^2\Delta t^2 > (\Delta x^2 + \Delta y^2 + \Delta z^2)$; appropriate for case when events are separated temporally; and so can be at the same location

(C) $\Delta s^2 = 0 \Rightarrow$ lightlike

$$\Delta s^2 \Rightarrow c^2 = \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2} \longrightarrow$$ events in question are connected by signal travelling at c.

$\Delta r^2 - c^2\Delta t^2 = A < 0$ (II)

$\Delta r^2 - c^2\Delta t^2 = ? = A > 0$ (I)



Spacetime Diagram

$x^0 = x^1$

needs "slopes" ≥ 1 for $v < c$

Proper Time

Invariant measure of motion of a particle along its trajectory in space-time; see above

timelike $\Rightarrow$ $\boxed{\Delta r^2 - c^2\Delta t^2 = A_{(t)} < 0}$

A defines the trajectory $\qquad \tau \equiv$ proper time

$$\Rightarrow (ds)^2 = (d\vec{r})^2 - (c\,dt)^2 = -c^2 \underbrace{d\tau^2}$$

$$\Rightarrow d\tau^2 = \frac{1}{c^2}(ds)^2 \Rightarrow d\tau = \sqrt{-\frac{1}{c^2} dx_\mu dx_\mu}$$

and $\boxed{\frac{d\tau}{dt} = \sqrt{\frac{-1}{c^2}\left(-c^2\frac{dt^2}{dt^2} + \frac{(d\vec{r})^2}{dt^2}\right)} = \sqrt{1 - \frac{u^2}{c^2}} = \frac{1}{\gamma}}$

$\boxed{\text{Causality}}$

interval; if $\begin{cases} >0 \to \text{spacelike} \\ <0 \to \text{timelike} \end{cases}$

$$-c^2dt'^2+dx'^2+dy'^2+dz'^2 = -c^2dt^2+dx^2+dy^2+dz^2$$

$\boxed{\text{motion along } x\text{-axis, } \beta_x \text{, and look at } dt'}$

$$-c^2dt'^2+dx'^2 = -c^2dt^2+dx^2; \text{ look at } dt'$$

$$-c^{\cancel{2}}\left[\gamma\left\{-\frac{v}{c^2}x_2+t_2\right\}-\gamma^{\cancel{2}}\left\{-\frac{v}{c^2}x_1+t_1\right\}\right] = -c^2dt'^2$$

consider an event in S' that occurs at $x_1'=x_2'$
$\Rightarrow$ stationary (but what about y', z'?)

$$\Rightarrow dt'^{\cancel{2}} = \gamma\left\{(t_2-t_1)-\frac{v}{c^2}(x_2-x_1)\right\}$$

sign difference $\Rightarrow$ dt' & dt can have different signs $\Rightarrow$ causality can be broken

$$\Rightarrow \overbrace{(t_2-t_1)}^{\Delta t>0}-\frac{v}{c^2}(x_2-x_1) < 0$$

$$\Rightarrow \Delta x > \frac{c^2}{v}\Delta t \Rightarrow \frac{\Delta x}{\Delta t} > \frac{c}{v}(c)$$

for $dt'<0$ $\Rightarrow$ reversal, need faster than light signal propagation

In tems of interval,

$$\Delta x > \left(\frac{c^2}{v}\right)\Delta t$$

$$\Rightarrow -c^2dt^2 + dx^2 + dy^2 + dz^2 > 0$$

$\Rightarrow$ spacelike interval

Let's move on to electrodynamics

a) Recall, a 4-vector is a quantity that transforms as

$$V'^{\alpha} = \Lambda^{\alpha}{}_{\beta} V^{\beta}$$

when the coordinates transform as

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}$$

More precisely, V^{β} is a contravariant 4-vector.
A covariant 4-vector transforms as

$$V'_{\alpha} = \Lambda_{\alpha}{}^{\beta} V_{\beta}$$

$$\text{where } \Lambda_{\alpha}{}^{\beta} = \underbrace{g_{\alpha\gamma}}_{\text{lowers } \delta \text{ to } \alpha,}\ \underbrace{g^{\beta\delta}}_{\text{raises } \delta \text{ to } \beta} \Lambda^{\gamma}{}_{\delta}$$

lowers δ to α, ~~lowers~~ raises δ to β

b) "Dot" product

$$U'_{\alpha} V'^{\alpha} = \overbrace{\Lambda_{\alpha}{}^{\gamma} \Lambda^{\alpha}{}_{\beta}}^{\delta_{\alpha}{}^{\alpha}} U_{\gamma} V^{\beta} = U_{\alpha} V^{\alpha}$$

Lorentz Invariant

note: $\Lambda_{\alpha}{}^{\gamma}$ is the inverse of $\Lambda^{\alpha}{}_{\gamma}$

d) Gradient

Consider a scalar function $\phi(x^\alpha)$, define

$$\frac{\partial\phi}{\partial x^\alpha} = \underbrace{\left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}_{\text{derivative with respect to contravariant variable}}\phi$$

$$= \partial_\alpha \phi$$

the differential of ϕ is then

$$d\phi = \partial_\alpha \phi \, dx^\alpha$$

$$= \left(\frac{\partial}{\partial x^\alpha}\phi\right) dx^\alpha$$

scalar invariant

4-vector

by quotient rule ∂_α is a 4-vector

$$\Rightarrow \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}$$

covariant transformation, $\Lambda^\beta{}_\alpha$

$$\Rightarrow \partial_\alpha = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \equiv \text{covariant gradient}$$

and

$$\partial^\alpha = \left(-\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \equiv \text{contravariant gradient}$$

e) <u>4-vector Divergence</u>

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha$$

$$\left(-\frac{\partial}{\partial x^0}A_0 + \frac{\partial}{\partial x^1}A_1 + \frac{\partial}{\partial x^2}A_2 + \frac{\partial}{\partial x^3}A_3\right) = \left(\frac{\partial}{\partial x^0}A^0 + \frac{\partial}{\partial x^1}A^1 + \frac{\partial}{\partial x^2}A^2 + \frac{\partial}{\partial x^3}A^3\right)$$

f) <u>4-vector d'Alembertian</u>

$$\Box^2 = \partial_\alpha \partial^\alpha = -\frac{\partial^2}{\partial x^{0 2}} + \nabla^2$$

invariant, because of divergence

Armed w/ 4-vectors, divergence, and d'Alembertian, gradients, let's do some damage

Tensor Operations (of note)

① Sum of 2 Tensors; if 2 tensors have the same # of covariant and position, and contravariant indices, sum is

$$C_\alpha{}^\beta = A_\alpha{}^\beta + B_\alpha{}^\beta$$

② Outer Product of 2 Tensors: If we have a tensor of rank n and a tensor of rank m, then the outer product is a tensor of rank $(n+m)$

$$C^{\alpha\beta}{}_{\gamma\delta} = A^{\alpha\beta} B_{\gamma\delta}$$

③ Contraction of a tensor; if we sum over 2 indices of a tensor of rank n of which one index is covariant and one is contravariant, the result is a tensor of rank $(n-2)$,

$$S^\alpha{}_\gamma = T^{\alpha\beta}{}_{\beta\gamma}$$

for homework

(3i) In the case we combine the outer product of 2 vectors and a contraction, we get the inner product,

ⓐ $C^\alpha{}_\beta = A^\alpha B_\beta$, outer product

ⓑ $C^\alpha{}_\alpha = D = \text{constant}$, contraction

Comment: $C'^\alpha{}_\alpha = A'^\alpha B'_\alpha = \underbrace{\Lambda^\alpha{}_\mu A^\mu}_{A'^\alpha} \overbrace{\Lambda^{-1}{}^\mu{}_\alpha B_\mu}^{B'_\alpha} = A^\mu \Lambda^\alpha{}_\mu \Lambda^{-1}{}^\mu{}_\alpha$

$$= A^\mu B_\mu = C^\mu{}_\mu$$

Covariance of Electrodynamics

(a) Charge Conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \text{ , where } \vec{J} = \rho \vec{v} \equiv \text{Current Density}$$

find the $\boxed{\text{manifestly covariant}}$ form for the Continuity Equation.

b) Conservation of charge. In some where, we find (experimentally) that q is an invariant

$$\boxed{Q = \rho d^3x = q\,dN \equiv \text{scalar invariant}}$$

$$\Rightarrow \underline{\rho' d^3x' = \rho d^3x = Q}$$

for example, $dx' = \gamma dx$

$\rho' = \frac{\rho}{\gamma}$ contraction

contraction of volume

$\Rightarrow \rho' \uparrow$

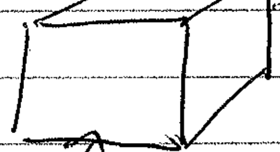
* Note that in a moving frame, S',

$d^3x' = dx'dy'dz'$,

and

$x' = -\beta\gamma ct + \gamma x \Rightarrow \frac{dx'}{dx} = \gamma, \; dt = 0$

$\Rightarrow \boxed{d^3x' = \gamma\, dx\, dy\, dz \Rightarrow \text{3-d volume is } \underline{\text{not}} \text{ invariant}}$



$d^3\vec{r}$

(i) In frame S, consider the volume $d^3\vec{r}$.

(ii) Construct the quantity

$\underline{\rho\, d^3\vec{r}}$ ≡ invariant charge

$\underline{dr^\alpha}$ ≡ invariant displacement of each charge element

$$\underbrace{\rho\, d^3\vec{r}\, dr^\alpha}_{\text{4-vector}}$$

(iii) $$\rho\, d^3\vec{r}\, dr^\alpha = \rho\, d^3\vec{r}\, dt \left(\frac{dr^\alpha}{dt}\right) \; *$$

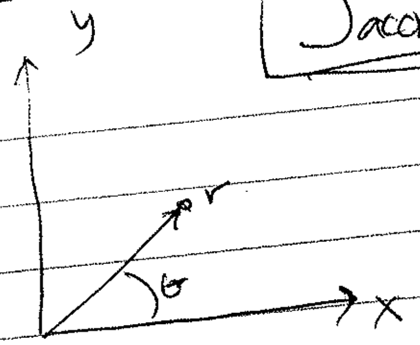
invariant

$$= \rho \underbrace{\boxed{c\, d^3\vec{r}\, dt}}_{\text{scalar invariant (4-D volume)}} \left(\frac{dr^\alpha}{c\,dt}\right)$$

4-volume

left-over $\Rightarrow \rho\left(\frac{dr^\alpha}{dt}\right)$ is a 4-vector

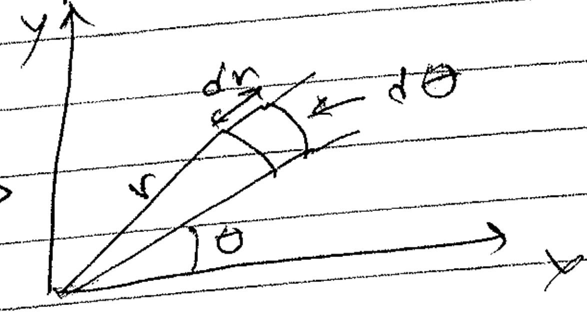
$\boxed{\text{Jacobian}}$



area element

$$dA = dx\,dy$$

Suppose I wanted to write dA in polar coordinates,



$dA = r\,d\theta\,dr$ from a simple geometric argument. However, consider

$$dA = J\,dr\,d\theta \qquad J = \text{Jacobian}$$

However, let me write,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\Rightarrow \boxed{dA = r\,dr\,d\theta}$$

and so, in general,

$$\boxed{d^4x' = \overset{\text{Jacobian}}{J}\,d^4x}$$

for our 4-D volume transformation

*

the 4-D volume is an invariant (while the 3D volume).

$$4D \text{ volume} \equiv dx^0 dx^1 dx^2 dx^3$$

$$d^4x' = J\, d^4x$$

$$\text{Jacobian} = \underbrace{\begin{vmatrix} \frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^0}{\partial x^1} & \frac{\partial x'^0}{\partial x^2} & \frac{\partial x'^0}{\partial x^3} \\ \frac{\partial x'^1}{\partial x^0} & \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^1}{\partial x^3} \\ \frac{\partial x'^2}{\partial x^0} & \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^2}{\partial x^3} \\ \frac{\partial x'^3}{\partial x^0} & \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^3}{\partial x^3} \end{vmatrix}}_{\text{determinant of } \Lambda^{\mu}{}_{\nu}}$$

and

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow J = 1 \implies \boxed{d^4x' = d^4x}$$

for Lorentz transformation

define: (recall: $\vec{J} = \rho\vec{v}$)

(i) $J^\alpha = \rho\left(\frac{dr^\alpha}{dt}\right)$

$$= \rho\left\{\left(\frac{dx^0}{\left[\frac{dx^0}{c}\right]}\right), \frac{dx^1}{\left(\frac{dx^0}{c}\right)}, \frac{dx^2}{\left(\frac{dx^0}{c}\right)}, \frac{dx^3}{\left(\frac{dx^0}{c}\right)}\right\}$$

$$\boxed{J^\alpha = \rho\left(c, \vec{v}\right)}$$

as a 4-vector; Interestingly neither ρ nor $(c, \vec{v})$ is invariant, but their product is 4-vector. How do we reconcile this w/ the Quotient Rule?

recall: $\rho\vec{v} = \vec{J} \Rightarrow \boxed{J^\alpha = (\rho c, \vec{J})}$

Return to Charge Conservation

(i) $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

(ii) $\frac{\partial(\rho c)}{c\,\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \frac{\partial J^0}{\partial x^0} + \vec{\nabla}\cdot\vec{J} = 0$

and

Continuity Equation: $\boxed{\partial_\mu J^\mu = 0}$

h) Potentials; $V, \vec{A}$

In the Lorentz gauge, $\frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$, we have,

$$\left\{\begin{array}{ll} \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - (\vec{\nabla}\cdot\vec{\nabla})\vec{A} = \mu_0 \vec{J} & (1) \\ \text{and } \frac{1}{c^2}\frac{\partial^2}{\partial t^2}V - (\vec{\nabla}\cdot\vec{\nabla})V = \frac{\rho}{\epsilon_0} & (2) \end{array}\right.$$

We earlier noted the symmetry between the wave equations for V & $\vec{A}$. Let's draw the "symmetry" tighter. Divide V equation by $\frac{1}{c}$

$$\Rightarrow \underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}_{\partial (x^0)^2}\left(\frac{V}{c}\right) - \nabla^2\left(\frac{V}{c}\right) = \frac{\rho}{\epsilon_0 c} = \mu_0 \overbrace{\rho c}^{J^0}$$

$\partial_0 \partial^0$

$$\frac{1}{\mu_0\epsilon_0} = c^2 \Rightarrow \frac{1}{\epsilon_0} = \mu_0 c^2$$

Comparison (1) and (2) shows that we can write 1 form to encompass both,

scalar invariant

$$\boxed{\partial_\mu \partial^\mu A^\alpha = -\mu_0 J^\alpha \qquad \left(\frac{V}{c}, \vec{A}\right) \qquad \left(\rho c, \vec{J}\right)}$$

4-vector

4-vector! from Quotient Rule

Lorentz gauge,

$$\frac{1}{c^2}\frac{\partial}{\partial t}V + \vec{\nabla}\cdot\vec{A} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{V}{c}\right) + \vec{\nabla}\cdot\vec{A} = 0$$

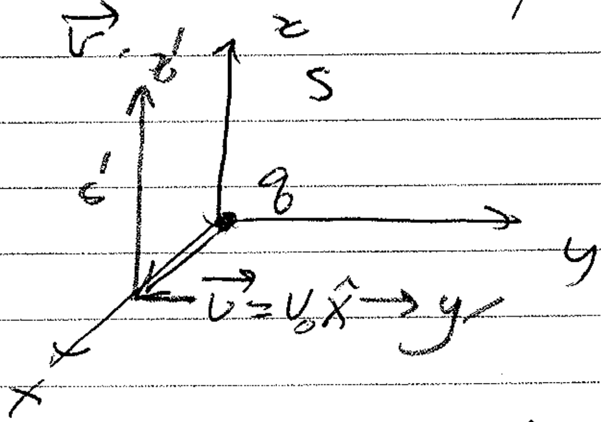
recall $\left(\frac{V}{c}, \vec{A}\right)$ is the potential 4-vector

$$\Rightarrow \boxed{\partial_\mu A^\alpha = 0}$$

Example,

$$A^{\alpha} = \left(\frac{V}{c}, \vec{A} \right)$$ ← take advantage of fact this is a 4-vector

Find the 4-vector potential for a charge moving at constant $\vec{v}$.



Lienard-Wiechert potentials

a) Go to frame S' in which q is at rest (at origin)

$$\Rightarrow V' = \frac{1}{4\pi\epsilon_0} \frac{q}{r'} \text{ in } S' ; \ \vec{A}' = 0 \text{ in } S'$$

$$\begin{pmatrix} V/c \\ A_x \\ A_y \\ A_z \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & +\gamma\beta & 0 & 0 \\ +\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda_{\alpha}^{\ \beta}} \begin{pmatrix} V'/c \\ A_x' \\ A_y' \\ A_z' \end{pmatrix}$$

inverse transform

$$\Rightarrow \frac{V}{c} = \gamma\left(\frac{V'}{c}\right) + \cancel{\gamma\beta A_x'}$$

$$A_x = \gamma\beta\left(\frac{V'}{c}\right) + \cancel{\gamma A_x'}$$

$$A_y = 0$$

$$A_z = 0$$

$$\boxed{\begin{aligned} V &= \gamma V' \\ A_x &= \frac{\beta}{c} V' \\ A_y &= 0 \\ A_z &= 0 \end{aligned}}$$

and $V = \gamma V' = \gamma \dfrac{q}{4\pi\epsilon_0 r'}$

$$= \gamma \frac{q}{4\pi\epsilon_0 \sqrt{(x'^2 + y'^2 + z'^2)}}$$

// coordinate transformation

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\Rightarrow ct = \gamma ct' + \gamma\beta x', \quad x = \gamma\beta ct' + \gamma x', \quad y = y', \quad z = z'$$

$$\Rightarrow \left[\gamma\beta x' = ct - \gamma ct', \quad \gamma x' = x - \gamma\beta ct' \right]$$

$$\Rightarrow \gamma\beta x' = ct - \gamma c \left\{ \frac{x - \gamma x'}{\gamma\beta c} \right\} = ct - \frac{1}{\beta}(x - \gamma x')$$

$$x'\left(\gamma\beta - \frac{\gamma}{\beta}\right) = ct - \frac{x}{\beta}$$

$$x' = \frac{1}{\gamma}\left(\frac{ct - x/\beta}{\beta - 1/\beta}\right) = \frac{1}{\gamma}\left(\frac{\beta ct - x}{\beta^2 - 1}\right)$$

$$= -\left(\frac{\beta ct - x}{\sqrt{1-\beta^2}}\right)$$

$$\Rightarrow \boxed{V = \frac{1}{4\pi\epsilon_0} \frac{q\gamma}{\sqrt{\gamma^2(x - vt)^2 + y^2 + z^2}}}$$

$$\Rightarrow A_x = \gamma\beta\left(\frac{V'}{c}\right) = \beta\left(\frac{V}{c}\right) = \frac{v}{c^2}V$$

$$\boxed{A_x = \left(\frac{1}{4\pi\epsilon_0 c^2}\right)\frac{q\gamma v}{\sqrt{\gamma^2(x-vt)^2+y^2+z^2}}}$$

forms derived in class [in anticipation of above result].