INTRODUCTION TO COMPLEX ANALYSIS

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Abstract. These notes were prepared to accompany Math 5351 (Introduction to Complex Variables) at Missouri S&T. They are based on an earlier version that accompanied Math 185 (Introduction to Complex Analysis) at UC Berkeley in Spring 2015.

Contents

1. Introduction

The primary references used while preparing these notes were the textbook of Stein and Shakarchi (specifically Chapters 1–3, 5, and 8 therein), as well as the textbook of Gamelin (specifically the proof of the prime number theorem in Chapter XIV therein).

1.1. Primer on Analysis and Metric Space Topology.

1.1. Primer on Analysis and Metric Space Topology. We begin by reviewing some fundamental concepts from analysis and topology that will be needed throughout the course.

For set complements, we will write

$$X\backslash Y = \{x \in X : x \notin Y\}.$$

We recall the fundamental notions of norms and metrics.

Definition 1.1 (Norm)**.** Let X be a vector space over $\mathbb{R}$. A **norm** on X is a function $\rho : X \to [0, \infty)$ such that

- for all $x \in X$, $c \in \mathbb{R}$, $\quad \rho(cx) = |c|\rho(x)$
- for all $x \in X$, $\quad \rho(x) = 0 \implies x = 0$
- for all $x, y \in X$, $\quad \rho(x + y) \leq \rho(x) + \rho(y)$ $\quad$ (triangle inequality)

Definition 1.2 (Metric)**.** Let X be a non-empty set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ such that

- for all $x, y \in X$ $\quad d(x, y) = d(y, x)$
- for all $x, y \in X$ $\quad d(x, y) = 0 \implies x = y$
- for all $x, y, z \in X$ $\quad d(x, z) \leq d(x, y) + d(y, z)$ $\quad$ (triangle inequality).

If X is a vector space over $\mathbb{R}$ with a norm ρ, then we may define a metric d on X by

$$d(x, y) = \rho(x - y).$$

However, not every metric arises in this way. For example, if we take the discrete metric on $\mathbb{R}$ (i.e. $d(x, y) = 1$ for any $x \neq y$), this cannot arise from any norm on $\mathbb{R}$, since the homogeneity condition would require $d(\frac{1}{2}x, \frac{1}{2}y) = \frac{1}{2}d(x, y)$, which fails for any $x \neq y$.

Suppose X is a non-empty set with metric d. For $x \in X$ and $r > 0$ we define the **ball of radius** r **around** x by

$$B_r(x) := \{y \in X : d(x, y) < r\}. \tag{1.1}$$

We call a set $S \subset X$ **open** if

$$\text{for all} \quad x \in S \quad \text{there exists} \quad r > 0 \quad \text{such that} \quad B_r(x) \subset S.$$

Given $S \subset X$ and $T \subset S$, we call T **open in** S if $T = S \cap R$ for some open $R \subset X$.

One can readily check that the following properties hold:

- $\emptyset$ is open, $\quad X$ is open,
- *any* union of open sets is open,
- *finite* intersections of open sets are open.

These conditions show that this definition of open sets produces what is called a 'topology'. We therefore call the definition of open sets appearing in (1.1) the **metric space topology**.

Suppose $S \subset X$ and $x \in S$. We call x an **interior point** if there exists $r > 0$ such that $B_r(x) \subset S$. The set of interior points of S is denoted S°. A set S is open if and only if $S = S^\circ$ (exercise).

We call a set $S \subset X$ **closed** if $X \backslash S$ is open. Note that 'closed' does **not** mean 'not open'. For example, the sets $\emptyset$ and X are always both open and closed.

Given a set $S \subset X$ we define the **closure** of S by

$$\overline{S} = \bigcap \{T \subset X : S \subset T \text{ and } T \text{ is closed}\}.$$

A set S is closed if and only if $S = \overline{S}$ (exercise).

A point $x \in X$ is called a **limit point** of $S \subset X$ if

$$\text{for all} \quad r > 0 \quad [B_r(x) \backslash \{x\}] \cap S \neq \emptyset.$$

A set is closed if and only if it contains all of its limit points (exercise).

The **boundary** of S is defined by $\overline{S} \backslash S^\circ$. It is denoted ∂S.

Let $S \subset X$. An **open cover** of S is a collection of open sets $\{U_\alpha\}$ (indexed by some set A) such that

$$S \subset \bigcup_{\alpha \in A} U_\alpha.$$

We call S **compact** if every open cover has a finite subcover. That is, for any open cover $\{U_\alpha\}_{\alpha \in A}$ of S, there exists a **finite** set $B \subset A$ such that

$$S \subset \bigcup_{\alpha \in B} U_\alpha.$$

We record here a few useful facts concerning compact sets in metric spaces.

Lemma 1.3. *In a metric space, every compact set is closed.*

Proof. Suppose that $S \subset X$ is compact and that $y \in X \backslash S$. For every $x \in S$, we have $d(x,y) > 0$, and so with $r_x := \frac{1}{10} d(x,y)$ we have $B_{r_x}(x) \cap B_{r_x}(y) = \emptyset$. Now, the collection $\{B_{r_x}(x) : x \in S\}$ gives an open cover of S, and hence by compactness there exists a finite subcover. That is, there exists $x_1, \dots, x_N$ so that $S \subset \cup_{j=1}^N B_{r_j}(x_j)$, where $r_j = r_{x_j}$. Now let $r = \min\{r_j : j = 1, \dots N\} > 0$. Then we claim $B_r(y) \subset X \backslash S$. To see this, suppose instead that there exists $x \in B_r(y) \cap S$. Then $x \in B_{r_j}(x_j)$ for some x_j. But then

$$d(x_j, y) \leq d(x_j, x) + d(x, y) < r_j + r \leq 2r_j = \tfrac{1}{5} d(x_j, y),$$

yielding a contradiction. In conclusion, we have shown that for any $y \in X \backslash S$, there exists $r_0 > 0$ so that $B_{r_0}(y) \subset X \backslash S$. This shows that $X \backslash S$ is open, so that S is closed. $\square$

Lemma 1.4. *A closed subset of a compact metric space is compact.*

Proof. Suppose (X, d) is compact and $S \subset X$ is closed. If $\{U_\alpha\}$ is an open cover of S, then $\{U_\alpha\}$ together with $[X \backslash S]$ is an open cover of X. There exists a finite subcover of X, which we may assume consists of $U_1, \dots, U_N, X \backslash S$. In this case, $U_1, \dots, U_N$ necessarily covers S. $\square$

Finally, we have the following important theorem about nested compact sets in metric spaces:

Theorem 1.5 (Cantor's intersection theorem)**.** *Let (X, d) be a metric space. Suppose $\{S_k\}_{k=1}^{\infty}$ is a collection of non-empty compact subsets of X such that $S_{k+1} \subset S_k$ for each k. Then*

$$\bigcap_{k=1}^{\infty} S_k \neq \emptyset.$$

Proof. Exercise! □

A set $S \subset X$ is **connected** if it **cannot** be written in the form

$$S = A \cup B,$$

where A, B are disjoint, non-empty, and open in S. This is a bit of a strange definition, in the sense that connectedness is defined in terms of what it is not. Nonetheless, connected means essentially what you think it means (especially if we stick to reasonably intuitive topologies like the metric space topology). Later, we will introduce the notion of path-connectedness, which is even more intuitive and is equivalent to connectedness in many scenarios.

1.2. **Sequences and Convergence.** A **sequence** in a metric space (X, d) is a function $x : \mathbb{N} \to X$. We typically write $x_n = x(n)$ and denote the sequence by $\{x_n\}_{n\in\mathbb{N}}$, $\{x_n\}_{n=1}^{\infty}$, or even by $\{x_n\}$.

Suppose $x : \mathbb{N} \to X$ is a sequence and N is an infinite (ordered) subset of $\mathbb{N}$. The restriction $x : N \to X$, denoted $\{x_n\}_{n\in N}$ is called a **subsequence** of $\{x_n\}_{n\in\mathbb{N}}$.

We often denote subsequences by $\{x_{n_k}\}_{k=1}^{\infty}$ (with the understanding that $N = \{n_k : k \in \mathbb{N}\}$).

Definition 1.6 (Cauchy sequence)**.** A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is **Cauchy** if

$$\text{for all} \quad \varepsilon > 0 \quad \text{there exists} \quad N \in \mathbb{N} \quad \text{such that} \quad n, m \geq N \implies d(x_n, x_m) < \varepsilon.$$

Definition 1.7 (Convergent sequence)**.** A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) **converges** to $\ell \in X$ if

$$\text{for all} \quad \varepsilon > 0 \quad \text{there exists} \quad N \in \mathbb{N} \quad \text{such that} \quad n \geq N \implies d(x_n, \ell) < \varepsilon.$$

We write $\lim_{n\to\infty} x_n = \ell$, or $x_n \to \ell$ as $n \to \infty$. We call ℓ the **limit** of the sequence.

We say the space (X, d) is **complete** if every Cauchy sequence converges. This is a property of the metric, as opposed to the property of the topology generated by the metric (see Exercise 1.3).

We have the following important characterization of compact sets in metric spaces.

Theorem 1.8. *Let (X, d) be a metric space. A set $S \subset X$ is compact if and only if every sequence in S has a subsequence that converges to a point in S.*

Proof. $\implies$: Suppose S is compact and $\{y_k\}$ is a sequence in S. Consider the open cover of S by the balls $\{B_1(x) : x \in S\}$. By compactness, there exists a finite subcover $\{B_1(x_j)\}_{j=1}^{N}$. At least one of these balls, say $B_1(x^1)$ contains infinitely many terms of the sequence y_k, giving us a subsequence we may denote by y_k^1. Now $\overline{B_1(x^1)}$ is a closed subset of S and hence is compact. Therefore, arguing as above, we may find finitely many balls of radius $\frac{1}{2}$ that cover $\overline{B_1(x^1)}$, and at least one of these balls (say $B_{\frac{1}{2}}(x^2)$) must contain infinitely many terms of the sequence y_k^1, giving us a second subsequence y_k^2. We now repeat this process, finding balls of the form $B_{1/j}(x^j)$ that contain subsequences y_k^j, with y_k^{j+1} a subsequence of y_k^j.

Now, observe that the sets $\overline{B_{1/j}(x^j)}$ are (by construction) a nested sequence of non-empty compact sets, and hence (by Theorem 1.5) there exists a point z in their intersection. Note also that $z \in S$; for this one can argue as follows: z is a limit point of S by construction (unless perhaps $z = x^j$ for some j, in which case $z \in S$ anyway).

We now consider the diagonal sequence $z_k := y_k^k$ and show that $z_k \to z$. To this end, let $\varepsilon > 0$ and choose $K > \frac{2}{\varepsilon}$. Then for $k \geq K$ we have $z_k \in B_{1/K}(x^K)$. As $z \in B_{1/K}(x^K)$, we deduce

$$d(z_k, z) \leq d(z_k, x^K) + d(x^K, z) < \tfrac{2}{K} < \varepsilon \quad \text{for all} \quad k \geq K.$$

This shows $z_k \to z$, as desired.

$\Longleftarrow$: Now suppose that every sequence in S has a convergent subsequence, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of S.

We claim that there exists $r > 0$ so that

$$\text{for all} \quad x \in S, \quad \text{there exists} \quad \alpha \in A \quad \text{such that} \quad B_r(x) \subset U_\alpha. \tag{1.2}$$

Suppose not. Then we may find balls B_n of radius $1/n$ that are not contained in any U_α. We choose $x_n \in B_n$ for each n, and pass to a subsequence so that $x_{n_k} \to z \in S$. Now, choose β so that $z \in U_\beta$. As U_β is open, we may find $\varepsilon > 0$ so that $B_{3\varepsilon}(z) \subset U_\beta$. Then for k large enough, we have $x_{n_k} \in B_\varepsilon(z)$. Choosing k possibly even larger (so that $1/n_k < \varepsilon$), we deduce $B_{n_k} \subset B_{3\varepsilon}(z) \subset U_\beta$, contradicting the fact that the balls B_n do not belong to any U_α by construction.

Now choose $r > 0$ as in (1.2). We claim that there exist finitely balls $\{B_k\}_{k=1}^N$ of radius r that cover S. If not, we may construct a sequence of points y_n so that

$$y_n \notin \cup_{k=1}^{n-1} B_r(y_k).$$

However, this sequence cannot have a convergent subsequence, since any ball of radius $r/2$ can contain at most one y_k. Now these balls B_k have radius r and hence each is contained in some set U_{α_k} by (1.2). In particular, we deduce that $S \subset \cup_{k=1}^N U_{\alpha_k}$, yielding the desired finite subcover. $\square$

1.3. **Limits and Continuity.** Suppose (X, d) and $(Y, \tilde{d})$ are metric spaces and $f : X \to Y$.

Definition 1.9 (Limit)**.** Suppose $x_0 \in X$ and $\ell \in Y$. We write

$$\lim_{x \to x_0} f(x) = \ell, \quad \text{or} \quad f(x) \to \ell \quad \text{as} \quad x \to x_0$$

if

$$\text{for all} \quad \{x_n\}_{n=1}^\infty \subset X, \quad \lim_{n \to \infty} x_n = x_0 \implies \lim_{n \to \infty} f(x_n) = \ell.$$

Equivalently, $\lim_{x \to x_0} f(x) = \ell$ if

$$\begin{aligned} &\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ &\text{for all } x \in X \quad d(x, x_0) < \delta \implies \tilde{d}(f(x), \ell) < \varepsilon. \end{aligned}$$

Definition 1.10 (Continuity)**.** The function f is **continuous** at $x_0 \in X$ if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

If f continuous at each $x \in X$ we say f is continuous on X.

Definition 1.11 (Uniform continuity)**.** The function is **uniformly continuous** on X if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that}$$
$$\text{for all } x, \tilde{x} \in X \quad d(x, \tilde{x}) < \delta \implies \tilde{d}(f(x), f(\tilde{x})) < \varepsilon.$$

We will also make use of the following 'little-oh' notation.

Definition 1.12 (Little-oh notation)**.** Suppose (X, d) is a metric space and Y is a vector space over $\mathbb{R}$ with norm ρ. Let $f, g : X \to Y$ and $x_0 \in X$. We write

$$f(x) = o(g(x)) \quad \text{as} \quad x \to x_0$$

if

$$\text{for all } \varepsilon > 0 \quad \text{there exists } \delta > 0 \quad \text{such that}$$
$$d(x, x_0) < \delta \implies \rho(f(x)) < \varepsilon \rho(g(x)).$$

1.4. **Real Analysis.** Finally we recall a few definitions from real analysis.

Let $S \subset \mathbb{R}$.

- $M \in \mathbb{R}$ is an **upper bound** for S if
$$\text{for all} \quad x \in S, \quad x \leq M.$$
- $m \in \mathbb{R}$ is a **lower bound** for S if
$$\text{for all} \quad x \in S, \quad x \geq m.$$
- $M^* \in \mathbb{R}$ is the **supremum** of S if
  - M^* is an upper bound for S, and
  - for all $M \in \mathbb{R}$, if M is an upper bound for S then $M^* \leq M$
- $m_* \in \mathbb{R}$ is the **infimum** of S if
  - m_* is a lower bound for S, and
  - for all $m \in \mathbb{R}$, if m is a lower bound for S then $m_* \geq m$
- If S has no upper bound, we define $\sup S = +\infty$.
- If S has no lower bound, we define $\inf S = -\infty$.

Finally, if $\{x_n\}$ is a real sequence, then

$$\limsup_{n \to \infty} x_n := \lim_{n \to \infty} \left(\sup_{m \geq n} x_m \right), \qquad \liminf_{n \to \infty} x_n := \lim_{n \to \infty} \left(\inf_{m \geq n} x_m \right).$$

1.5. **Exercises.**

Exercise 1.1. Recall the definition of S° and $\overline{S}$ from Section 1.1.

- Show that S is open if and only if $S = S^\circ$
- Show that S is closed if and only if $S = \overline{S}$.
- Show that S is closed if and only if S contains all of its limit points.

Exercise 1.2. Prove Theorem 1.5. (*Hint:* Argue by contradiction. If the intersection is empty, then the collection of the complements of the S_k yields an open cover of X and hence of S_1. Now use compactness...)

Exercise 1.3. Let $\mathbb{N}$ denote the natural numbers $\{1, 2, 3, \dots\}$. Let $d_1(n, m) = |n - m|$ be the standard metric, and define $d_2 = (n, m) = \frac{|n-m|}{nm}$. First, show that that d_2 is a metric. Next, show that for both d_1 and d_2, the 'metric space topology' coincides with the 'discrete topology', that is, the topology in which single points (and hence all sets) are open. Finally, show that $(\mathbb{N}, d_1)$ is a complete metric space, while $(\mathbb{N}, d_2)$ is not.

Exercise 1.4. Prove that the two definitions of limit appearing in Definition 1.9 are equivalent.

2. The Complex Plane

2.1. **Definitions.** The **complex plane**, denoted $\mathbb{C}$, is the set of expressions of the form

$$z = x + iy,$$

where x and y are real numbers and i is an (imaginary) number that satisfies

$$i^2 = -1.$$

We call x the **real part** of z and write $x = \operatorname{Re} z$. We call y the **imaginary part** of z and write $y = \operatorname{Im} z$. If $x = 0$ or $y = 0$, we omit it. That is, we write $x + i0 = x$ and $0 + iy = iy$.

Notice that $\mathbb{C}$ is in one-to-one correspondence with $\mathbb{R}^2$ under the map $x + iy \mapsto (x, y)$. Under this correspondence we call the x-axis the **real axis** and the y-axis the **imaginary axis**.

Addition in $\mathbb{C}$ corresponds to addition in $\mathbb{R}^2$:

$$(x + iy) + (\tilde{x} + i\tilde{y}) = (x + \tilde{x}) + i(y + \tilde{y}).$$

We **define** multiplication in $\mathbb{C}$ as follows:

$$(x + iy)(\tilde{x} + i\tilde{y}) = (x\tilde{x} - y\tilde{y}) + i(x\tilde{y} + \tilde{x}y).$$

Addition and multiplication satisfy the associative, distributive, and commutative properties, as one can readily check. Furthermore we have an additive identity, namely 0, and a multiplicative identity, namely 1. We also have additive and multiplicative inverses. Thus, $\mathbb{C}$ has the **algebraic** structure of a **field**.

2.2. **Topology.** The complex plane $\mathbb{C}$ inherits a **norm** and hence a **metric space** structure from $\mathbb{R}^2$: if $z = x + iy$ then we define the **norm** (or **length**) of z by

$$|z| = \sqrt{x^2 + y^2},$$

and for $z, w \in \mathbb{C}$ we define the **distance** between z and w by $|z - w|$.

We equip $\mathbb{C}$ with the metric space topology. Thus we have notions of open/closed sets, compact sets, connected sets, convergent sequences, continuous functions, and so on.

Definition 2.1 (Bounded set, diameter)**.** A set $\Omega \subset \mathbb{C}$ is **bounded** if

$$\text{there exists} \quad R > 0 \quad \text{such that} \quad \Omega \subset B_R(0).$$

If Ω is a bounded set, its **diameter** is defined by

$$\operatorname{diam}(\Omega) = \sup_{w,z \in \Omega} |z - w|.$$

The **Heine–Borel theorem** in $\mathbb{R}^2$ gives the following characterization of compact sets in $\mathbb{C}$.

Theorem 2.2. *A set $\Omega \subset \mathbb{C}$ is compact if and only if it is closed and bounded.*

Proof. Let us present the general case for $S \subset \mathbb{R}^n$.

$\Longrightarrow$: Suppose S is compact. We know from Lemma 1.3 that S is closed. To see that S is bounded, consider the open cover of S by balls of radius one around each point, and then take a finite subcover. We then have that S is contained in a finite union of bounded sets, and hence is bounded.

$\Longleftarrow$: Now suppose S is closed and bounded, and we will show that every sequence in S has a convergent sequence. This implies that S is compact by Theorem 1.8.

First consider the case $n = 1$, and let $\{x_n\}$ be a sequence in S. Note that $\{x_n\}$ is necessarily a bounded sequence. We will construct a monotonic subsequence. To this end, let

$$N = \{n : x_n \geq x_m \quad \text{for all} \quad m > n\}.$$

If N is infinite, then we can take the subsequence $\{x_n\}_{n\in N}$, which is monotone decreasing. If instead N is finite, then take $n_1 = \max N + 1$. By definition, there exists $n_2 > n_1$ so that $x_{n_2} > x_{n_1}$, and then $n_3 > n_2$ so that $x_{n_3} > x_{n_2}$, and so on. That is, we can constrict a monotone increasing subsequence. In particular, we now have a monotone subsequence of $\{x_n\}$. Noting that bounded monotone subsequences converge (to the infimum or supremum), we have our convergent subsequence. As S is closed, the limit necessarily belongs to S as well. This handles the $n = 1$ case.

For the case of $n > 1$, we again take an arbitrary sequence $\{x_n\}$ in S, which is necessarily bounded. It follows that each component if bounded. Thus we can apply the argument above to find a subsequence along which the first component converges. We can then take a subsequence of this subsequence along which the second component converges, and continuing in this way we can find a subsequence along which every component converges. It follows that the sequence x_n converges along this subsequence, and again the limit must belong to S since S is closed. This completes the proof. □

We note that the completeness of $\mathbb{R}^2$ implies completeness of $\mathbb{C}$ (that is, Cauchy sequences converge).

2.3. **Geometry.** Polar coordinates in $\mathbb{R}^2$ lead to the notion of the **polar form** of complex numbers. In particular, any nonzero $(x, y) \in \mathbb{R}^2$ may be written

$$(x, y) = (r\cos\theta, r\sin\theta)$$

where $r = \sqrt{x^2 + y^2} > 0$ and $\theta \in \mathbb{R}$ is only uniquely defined up to a multiple of 2π.

Thus we can write any nonzero $z \in \mathbb{C}$ as

$$z = r[\cos\theta + i\sin\theta]$$

for some $\theta \in \mathbb{R}$. We call θ the **argument** of z and write $\theta = \arg(z)$.

By considering Taylor series and using $i^2 = -1$, we can write

$$\cos\theta + i\sin\theta = e^{i\theta}$$

(exercise). Thus for any $z \in \mathbb{C}\backslash\{0\}$ we can write z in **polar form**:

$$z = re^{i\theta}, \qquad r = |z|, \quad \theta = \arg(z).$$

The polar form clarifies the geometric meaning of multiplication in $\mathbb{C}$. In particular if $w = \rho e^{i\phi}$ and $z = re^{i\theta}$, then

$$wz = r\rho\, e^{i(\phi+\theta)}.$$

Thus multiplication by z consists of dilation by $|z|$ and rotation by $\arg(z)$.

For $z = x + iy \in \mathbb{C}$ we define the **complex conjugate** of z by

$$\bar{z} = x - iy.$$

That is, $\bar{z}$ is the reflection of z across the real axis. Note that if $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$.

We also note that

$$\operatorname{Re} z = \tfrac{1}{2}(z + \bar{z}) \quad \text{and} \quad \operatorname{Im} z = -\tfrac{i}{2}(z - \bar{z}).$$

Furthermore $|z|^2 = z\bar{z}$ (exercise).

2.4. **The Extended Complex Plane.** Let $\mathbb{S} \subset \mathbb{R}^3$ be the sphere of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$. The function

$$\Phi : \mathbb{S}\backslash\{(0,0,1)\} \to \mathbb{C}$$

defined by

$$\Phi((x, y, z)) = \frac{x}{1-z} + i\frac{y}{1-z}$$

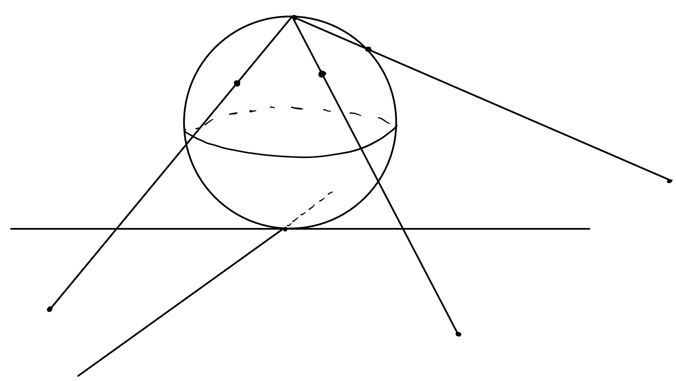
is called the **stereographic projection** map. This function is a bijection, with the inverse

$$\Phi^{-1} : \mathbb{C} \to \mathbb{S}\backslash\{(0,0,1)\}$$

given by

$$\Phi^{-1}(x + iy) = \left(\frac{x}{1+x^2+y^2}, \frac{y}{1+x^2+y^2}, \frac{x^2+y^2}{1+x^2+y^2}\right).$$

We depict this map in the following figure:



Note that $|x + iy| \to \infty$ if and only if $\Phi^{-1}(x + iy) \to (0, 0, 1)$. Thus we can identify $(0, 0, 1)$ with "the point at infinity", denoted ∞.

We call $\mathbb{S}$ the **Riemann sphere**. We call $\mathbb{C}$ together with ∞ the **extended complex plane**, denoted $\mathbb{C} \cup \{\infty\}$. We identify $\mathbb{C} \cup \{\infty\}$ with $\mathbb{S}$ via stereographic projection.

2.5. **Exercises.**

Exercise 2.1. Show that $|z|^2 = z\bar{z}$.

Exercise 2.2. Use Taylor series expansions to show that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Exercise 2.3. For all $z \in \mathbb{C}\backslash\{0\}$ there exists a unique $w \in \mathbb{C}\backslash\{0\}$ such that $zw = 1$, which we denote by $\frac{1}{z}$ or z^{-1}. Given $z = x + iy \in \mathbb{C}\backslash\{0\}$, compute the real and imaginary parts of z^{-1}.

Exercise 2.4. Describe the following sets in $\mathbb{C}$ geometrically and draw a picture of each.

- $\{z \in \mathbb{C} : |z - a| = |z - b|\}$, where $a, b \in \mathbb{C}$,
- $\{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$,
- $\{z \in \mathbb{C} : \mathrm{Re}\,(az + b) > 0\}$, where $a, b \in \mathbb{C}$,
- $\{z \in \mathbb{C} : |z| = \mathrm{Re}\,(z) + 1\}$.

3. Holomorphic Functions

3.1. **Definitions.** The definition of the complex derivative mirrors the definition for the real-valued case.

Definition 3.1 (Holomorphic)**.** Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$. The function f is **holomorphic** at $z_0 \in \Omega$ if there exists $\ell \in \mathbb{C}$ such that

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \ell. \tag{3.1}$$

We write $\ell = f'(z_0)$ and call $f'(z_0)$ the **derivative** of f at z_0.

A synonym for holomorphic is **(complex) differentiable**. If f is holomorphic at each point of Ω, we say f is holomorphic on Ω. If f is holomorphic on all of $\mathbb{C}$, we say that f is **entire**.

Remark. In (3.1) we consider **complex**-valued h. This will have surprisingly drastic consequences for the notion of complex differentiability.

Theorem 3.2. *The usual algebraic rules for derivatives hold:*

- $(f+g)'(z) = f'(z) + g'(z)$
- $(\alpha f)'(z) = \alpha f'(z)$ *for* $\alpha \in \mathbb{C}$
- $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$
- $\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2}$ *provided* $g(z) \neq 0$

Moreover the usual "chain rule" holds: $(f \circ g)'(z) = f'(g(z))g'(z)$.

Proof. As in the real-valued case, these all follow from the definition of the derivative and limit laws. $\square$

Thus complex derivatives share the algebraic properties of real-valued differentiation. However, due to the structure of complex multiplication, complex differentiation turns out to be very different.

3.2. **The Cauchy–Riemann Equations.** Suppose $f : \mathbb{C} \to \mathbb{C}$. For $(x, y) \in \mathbb{R}^2$, define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) := \mathrm{Re}\,[f(x+iy)] \quad \text{and} \quad v(x,y) := \mathrm{Im}\,[f(x+iy)].$$

Note that as mappings we may identify $f : \mathbb{C} \to \mathbb{C}$ with $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = (u(x,y), v(x,y)).$$

The question of differentiability is more subtle.

Proposition 3.3 (Cauchy–Riemann equations)**.** *The function f is holomorphic at $z_0 = x_0 + iy_0$ with derivative $f'(z_0)$ if and only if u, v are differentiable at (x_0, y_0) and satisfy*

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0,y_0) &= \frac{\partial v}{\partial y}(x_0,y_0) = Re\,[f'(z_0)], \\ \frac{\partial v}{\partial x}(x_0,y_0) &= -\frac{\partial u}{\partial y}(x_0,y_0) = Im\,[f'(z_0)]. \end{aligned}$$

Proof. We first note f is differentiable at z_0 with derivative $f'(z_0)$ if and only if

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + o(|z-z_0|) \quad \text{as} \quad z \to z_0.$$

Recalling the definition of multiplication in $\mathbb{C}$ and breaking into real and imaginary parts, this is equivalent to

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} \operatorname{Re}[f'(z_0)] & -\operatorname{Im}[f'(z_0)] \\ \operatorname{Im}[f'(z_0)] & \operatorname{Re}[f'(z_0)] \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$+\, o\big(\sqrt{|x-x_0|^2+|y-y_0|^2}\big) \quad \text{as} \quad (x,y)\to(x_0,y_0).$$

On the other hand, u and v are differentiable at (x_0,y_0) if and only if

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} \frac{\partial u}{\partial x}(x_0,y_0) & \frac{\partial u}{\partial y}(x_0,y_0) \\ \frac{\partial v}{\partial x}(x_0,y_0) & \frac{\partial v}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$+\, o\big(\sqrt{|x-x_0|^2+|y-y_0|^2}\big) \quad \text{as} \quad (x,y)\to(x_0,y_0).$$

The result follows. (See the exercises for another derivation of the Cauchy–Riemann equations, as well.) □

Example 3.1 (Polynomials)**.** If $f:\mathbb{C}\to\mathbb{C}$ is a polynomial, i.e.

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

for some $a_i\in\mathbb{C}$, then f is holomorphic (indeed, entire) with derivative

$$f'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

Example 3.2. Let $f:\mathbb{C}\backslash\{0\}\to\mathbb{C}$ be defined by $f(z)=\frac{1}{z}$. Then f is holomorphic, with

$$f':\mathbb{C}\backslash\{0\}\to\mathbb{C} \quad \text{given by} \quad f'(z) = -\tfrac{1}{z^2}.$$

Example 3.3 (Conjugation)**.** Consider the function $f:\mathbb{C}\to\mathbb{C}$ defined by $f(z)=\bar{z}$, which corresponds to $F:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$F(x,y) = (x,-y).$$

That is, $u(x,y)=x$ and $v(x,y)=-y$. Note that F is infinitely differentiable as a function on $\mathbb{R}^2$. Indeed,

$$\nabla F \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

However, f does not satisfy the Cauchy–Riemann equations, since

$$\tfrac{\partial u}{\partial x} = 1, \quad \text{but} \quad \tfrac{\partial v}{\partial y} = -1.$$

Thus f is **not** holomorphic.

In the exercises you will show $f(z)=\bar{z}$ is not holomorphic by another method.

3.3. **Power Series.** Given $\{a_n\}_{n=0}^\infty\subset\mathbb{C}$, we can define a new sequence $\{S_n\}_{n=0}^\infty$ of **partial sums** by

$$S_N := \sum_{n=0}^N a_n.$$

If the sequence S_N converges, we denote the limit by $\sum_{n=0}^\infty a_n$ and say the **series** $\sum_n a_n$ **converges.** Otherwise we say **the series** $\sum_n a_n$ **diverges**.

If the (real) series $\sum_n |a_n|$ converges, we say $\sum_n a_n$ **converges absolutely**.

Lemma 3.4. *The series $\sum_n a_n$ converges if and only if*

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that}$$
$$n > m \geq N \implies \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

Proof. See Exercise 3.5. □

Corollary 3.5.
(i) If $\sum_n a_n$ converges absolutely, then $\sum_n a_n$ converges.
(ii) If $\sum_n a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof. See Exercise 3.6. □

Given a sequence $\{a_n\}_{n=0}^{\infty} \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$, a **power series** is a function of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Theorem 3.6. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and define the **radius of convergence** $R \in [0, \infty]$ via*

$$R = [\limsup |a_n|^{1/n}]^{-1},$$

with the convention that $0^{-1} = \infty$ and $\infty^{-1} = 0$. Then
- *$f(z)$ converges absolutely for $z \in B_R(z_0)$,*
- *$f(z)$ diverges for $z \in \mathbb{C} \backslash \overline{B_R(z_0)}$.*

Proof. Suppose $R \notin \{0, \infty\}$ (you should *check* these cases separately). Further suppose that $z_0 = 0$. (You should *check* the case $z_0 \neq 0$.)

If $|z| < R$ then we may choose $\varepsilon > 0$ small enough (depending on z) that

$$(R^{-1} + \varepsilon)|z| < 1.$$

By definition of lim sup,

$$\text{there exists } N \in \mathbb{N} \quad \text{such that} \quad n \geq N \implies |a_n|^{1/n} \leq R^{-1} + \varepsilon.$$

Thus for $n \geq N$ we have

$$|a_n|\,|z|^n \leq [\ (R^{-1} + \varepsilon)|z|\ ]^n.$$

Using the "comparison test" with the (real) geometric series

$$\sum [\ (R^{-1} + \varepsilon)|z|\ ]^n$$

we deduce that $\sum a_n z^n$ converges absolutely.

If $|z| > R$ then we may choose $\varepsilon > 0$ small enough (depending on z) that

$$(R^{-1} - \varepsilon)|z| > 1.$$

By definition of lim sup, there exists a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k}|^{1/n_k} \geq R^{-1} - \varepsilon.$$

Thus along this subsequence

$$|a_{n_k}|\,|z|^{n_k} \geq [\,(R^{-1} - \varepsilon)|z|\,]^{n_k} > 1.$$

Thus $\lim_{n\to\infty} a_n z^n \not\to 0$, which implies that $\sum a_n z^n$ diverges. □

Remark 3.7. We call $B_R(0)$ the **disc of convergence**. The behavior of f (i.e. convergence vs. divergence) on $\partial B_R(0)$ is a more subtle question.

Definition 3.8. Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$. We call f **analytic** if there exists $z_0 \in \mathbb{C}$ and $\{a_n\}_{n=0}^\infty \subset \mathbb{C}$ such that the power series

$$\sum_{n=0}^\infty a_n (z - z_0)^n$$

has a positive radius of convergence and there exists $\delta > 0$ such that

$$f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n \quad \text{for all} \quad z \in B_\delta(z_0).$$

Example 3.4 (Some familiar functions)**.**

- We define the exponential function by
$$e^z = \sum_{n=0}^\infty \frac{z^n}{n!}.$$
- We define the cosine function by
$$\cos z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!}.$$
- We define the sine function by
$$\sin z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Analytic functions are holomorphic:

Theorem 3.9. *Suppose* $f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$ *has disc of convergence* $B_R(z_0)$ *for some* $R > 0$.

Then f *is holomorphic on* $B_R(z_0)$*, and its derivative* f' *is given by the power series*

$$f'(z) = \sum_{n=0}^\infty n a_n (z - z_0)^{n-1},$$

which has the same disc of convergence.

(By induction f *is infinitely differentiable, and all derivatives are obtained by termwise differentiation.)*

Proof. Let us suppose

$$z_0 = 0.$$

(You should check the case $z_0 \neq 0$.)

We define

$$g(z) = \sum_{n=0}^\infty n a_n z^{n-1}.$$

First notice that since $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\limsup_{n\to\infty} |n a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n},$$

and hence g also has radius of convergence equal to R.

We now let $w \in B_R(0)$ and wish to show that $g(w) = f'(w)$, that is,

$$\lim_{h \to 0} \frac{f(w+h) - f(w)}{h} = g(w).$$

To this end, we first note that for any $N \in \mathbb{N}$ we may write

$$f(z) = \underbrace{\sum_{n=0}^{N} a_n z^n}_{:= S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{:= E_N(z)}.$$

We now choose $r > 0$ such that $|w| < r < R$ and choose $h \in \mathbb{C}\backslash\{0\}$ such that $|w + h| < r$.

We write

$$\begin{aligned} \frac{f(w+h) - f(w)}{h} - g(w) &= \frac{S_N(w+h) - S_N(w)}{h} - S_N'(w) && (1) \\ &\quad + S_N'(w) - g(w) && (2) \\ &\quad + \frac{E_N(w+h) - E_N(w)}{h}. && (3) \end{aligned}$$

Now let $\varepsilon > 0$.

For (3) we use the fact that

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

and $|w + h|, |w| < r$ to estimate

$$\left| \frac{E_N(w+h) - E_N(w)}{h} \right| \le \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(w+h)^n - w^n}{h} \right| \le \sum_{n=N+1}^{\infty} |a_n| n\, r^{n-1}.$$

As g converges absolutely on $B_R(0)$ we may choose $N_1 \in \mathbb{N}$ such that

$$N \ge N_1 \implies \sum_{n=N+1}^{\infty} |a_n| n\, r^{n-1} < \tfrac{\varepsilon}{3}.$$

For (2) we use that $\lim_{N\to\infty} S_N'(w) = g(w)$ to find $N_2 \in \mathbb{N}$ such that

$$N \ge N_2 \implies |S_N'(w) - g(w)| < \tfrac{\varepsilon}{3}.$$

Now we fix $N > \max\{N_1, N_2\}$. For (1) we now take $\delta > 0$ so that

$$|h| < \delta \implies \left| \frac{S_N(w+h) - S_N(w)}{h} - S_N'(w) \right| < \tfrac{\varepsilon}{3} \quad \text{and} \quad |w + h| < r.$$

Collecting our estimates we find

$$|h| < \delta \implies \left| \frac{f(w+h) - f(w)}{h} - g(w) \right| < \varepsilon,$$

as needed. □

Remark 3.10. We just showed that analytic functions are holomorphic. Later we will prove that that the converse is true as well! (In particular, holomorphic functions are automatically infinitely differentiable!)

3.4. **Curves in the Plane.**

Definition 3.11 (Curves)**.**

- A **parametrized curve** is a continuous function $z : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$.
- Two parametrizations
$$z : [a, b] \to \mathbb{C} \quad \text{and} \quad \tilde{z} : [c, d] \to \mathbb{C}$$
are **equivalent** if there exists a continuously differentiable bijection $t : [c, d] \to [a, b]$ such that $t'(s) > 0$ and $\tilde{z}(s) = z(t(s))$.
- A parametrized curve $z : [a, b] \to \mathbb{C}$ is **smooth** if
$$z'(t) := \lim_{h \to 0} \frac{z(t + h) - z(t)}{h}$$
exists and is continuous for $t \in [a, b]$. (For $t \in \{a, b\}$ we take one-sided limits.)
- The family of parametrizations equivalent to a smooth parametrized curve $z : [a, b] \to \mathbb{C}$ determines a **(smooth) curve** $\gamma \subset \mathbb{C}$, namely
$$\gamma = \{z(t) : t \in [a, b]\},$$
with an orientation determined by $z(\cdot)$.
- Given a curve γ we define γ^- to be the same curve with the opposite orientation. If $z : [a, b] \to \mathbb{C}$ is a parametrization of γ, we may parametrize γ^- by
$$z^-(t) = z(b + a - t), \quad t \in [a, b].$$
- A parametrized curve $z : [a, b] \to \mathbb{C}$ is **piecewise-smooth** if there exist points
$$a = a_0 < a_1 < \cdots < a_n = b$$
such that $z(\cdot)$ is smooth on each $[a_k, a_{k+1}]$. (We call the restrictions of z to $[a_k, a_{k+1}]$ the **smooth components** of the curve.)
- The family of parametrizations equivalent to a piecewise-smooth parametrized curve determines a **(piecewise-smooth) curve**, just like above.
- Suppose $\gamma \subset \mathbb{C}$ is a curve and $z : [a, b] \to \mathbb{C}$ is a parametrization of γ. We call $\{z(a), z(b)\}$ the **endpoints** of γ. We call γ **closed** if $z(a) = z(b)$. We call γ **simple** if $z : (a, b) \to \mathbb{C}$ is injective.

Example 3.5. Let $z_0 \in \mathbb{C}$ and $r > 0$. Consider the curve
$$\gamma = \partial B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$
The **positive orientation** is given by
$$z(t) = z_0 + re^{it}, \quad t \in [0, 2\pi],$$
while the **negative orientation** is given by
$$z(t) = z_0 + re^{-it}, \quad [0, 2\pi].$$
By default we will consider positively oriented circles.

Definition 3.12 (Path-connected)**.** A set $\Omega \subset \mathbb{C}$ is **path-connected** if for all $z, w \in \Omega$ there exists a piecewise-smooth curve in Ω with endpoints $\{z, w\}$.

Definition 3.13 (Component)**.** Let $\Omega \subset \mathbb{C}$ be open and $z \in \Omega$. The **connected component** of z is the set of $w \in \Omega$ such that there exists a curve in Ω joining z to w.

Proposition 3.14. *Let $\Omega \subset \mathbb{C}$ be open. Then Ω is connected if and only if Ω is path connected.*

Proof. See Exercise 3.10. □

Definition 3.15 (Homotopy)**.** Let $\Omega \subset \mathbb{C}$ be an open set. Suppose γ_0 and γ_1 are curves in Ω with common endpoints α and β.

We call γ_0 and γ_1 **homotopic in** Ω if there exists a continuous function $\gamma : [0,1] \times [a,b] \to \Omega$ such that

- $\gamma(0,t)$ is a parametrization of γ_0 such that $\gamma(0,a) = \alpha$ and $\gamma(0,b) = \beta$
- $\gamma(1,t)$ is a parametrization of γ_1 such that $\gamma(1,a) = \alpha$ and $\gamma(1,b) = \beta$
- $\gamma(s,t)$ is a parametrization of a curve $\gamma_s \subset \Omega$ for each $s \in (0,1)$ such that $\gamma(s,a) = \alpha$ and $\gamma(s,b) = \beta$.

Definition 3.16 (Simply connected)**.** An open connected set $\Omega \subset \mathbb{C}$ is called **simply connected** if any two curves in Ω with common endpoints are homotopic.

Definition 3.17 (Integral along a curve)**.** Let $\gamma \subset \mathbb{C}$ be a smooth curve parametrized by $z : [a,b] \to \mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function.

We define the **integral of f along** γ by

$$\int_\gamma f(z)\,dz := \underbrace{\int_a^b f(z(t))z'(t)\,dt}_{\text{Riemann integral}}.$$

(To be precise we can define this in terms of real and imaginary parts.)

Remark 3.18. For this to qualify as a definition, we need to check that the definition is independent of parametrization:

Suppose $\tilde{z} : [c,d] \to \mathbb{C}$ is another parametrization of γ, i.e. $\tilde{z}(s) = z(t(s))$ for $t : [c,d] \to [a,b]$.

Changing variables yields

$$\int_c^d f(\tilde{z}(s))\tilde{z}'(s)\,ds = \int_c^d f(z(t(s)))z'(t(s))t'(s)\,ds = \int_a^b f(z(t))z'(t)\,dt.$$

If γ is piecewise smooth we define the integral by summing over the smooth components of γ:

$$\int_\gamma f(z)\,dz := \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,dt.$$

Example 3.6. Let γ be the unit circle, parametrized by $z(t) = e^{it}$ for $t \in [0, 2\pi]$. Then

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \tfrac{1}{e^{it}} ie^{it}\,dt = 2\pi i.$$

We define the **length** of a smooth curve γ by

$$\text{length}(\gamma) = \int_a^b |z'(t)|\,dt.$$

This definition is also independent of parametrization (*check!*).

The length of a piecewise-smooth curve is the sum of the lengths of its smooth components.

Theorem 3.19 (Properties of integration)**.**

$$\int_\gamma [\alpha f(z) + \beta g(z)]\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz.$$
$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$
$$\left| \int_\gamma f(z)\,dz \right| \leq \sup_{z\in\gamma} |f(z)| \cdot \mathit{length}(\gamma).$$

Proof. The first equality follows from the definition and the linearity of the usual Riemann integral.

For the second equality, we use the change of variables formula and the fact that if $z(t)$ parametrizes γ then $z^-(t) := z(b+a-t)$ parametrizes γ^-.

For the inequality we have that

$$\left| \int_\gamma f(z)\,ds \right| \leq \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\,dt \leq \sup_{z\in\gamma} |f(z)|\,\mathrm{length}(\gamma)$$

for a smooth curve γ. □

We turn to the notion of the winding number of a curve around a point.

Definition 3.20 (Winding number)**.** Let $\gamma \subset \mathbb{C}$ be a closed, piecewise smooth curve. For $z_0 \in \mathbb{C}\backslash\gamma$ we define the **winding number of** γ **around** z_0 by

$$W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Theorem 3.21 (Properties of winding number)**.**

(i) $W_\gamma(z_0) \in \mathbb{Z}$ *(for* $z_0 \in \mathbb{C}\backslash\gamma$*)*

(ii) *if* z_0 *and* z_1 *are in the same connected component of* $\mathbb{C}\backslash\gamma$*, then* $W_\gamma(z_0) = W_\gamma(z_1)$

(iii) *If* z_0 *is in the unbounded connected component of* $\mathbb{C}\backslash\gamma$ *then* $W_\gamma(z_0) = 0$.

Proof. Suppose $z : [0,1] \to \mathbb{C}$ is a parametrization of γ, and define $G : [0,1] \to \mathbb{C}$ by

$$G(t) = \int_0^t \frac{z'(s)}{z(s) - z_0}\,ds.$$

Then G is continuous and (except at possibly finitely many points) differentiable, with

$$G'(t) = \frac{z'(t)}{z(t) - z_0}.$$

We now define $H : [0,1] \to \mathbb{C}$ by

$$H(t) = [z(t) - z_0]e^{-G(t)}.$$

Note that H is continuous and (except at possibly finitely many points) differentiable, with

$$H'(t) = z'(t)e^{-G(t)} - [z(t) - z_0]\frac{z'(t)}{z(t) - z_0}e^{-G(t)} = 0$$

Thus H is constant. In particular since $z_0 \notin \gamma$ and γ is closed,

$$[z(1) - z_0]e^{-G(1)} = [z(0) - z_0]e^{-G(0)} \implies e^{-G(1)} = e^{-G(0)} = 1$$

This implies $2\pi i W_\gamma(z_0) = G(1) = 2\pi i k$ for some $k \in \mathbb{Z}$, which gives (i).

Now (ii) follows since W_γ is continuous and $\mathbb{Z}$-valued.

Finally (iii) follows since $\lim_{|z_0|\to\infty} W_\gamma(z_0) = 0$. □

Example 3.7. Example 3.6 shows that if γ is the unit circle then $W_\gamma(0) = 1$.

Theorem 3.22 (Jordan curve theorem)**.** *Let $\gamma \subset \mathbb{C}$ be a simple, closed, piecewise-smooth curve. Then $\mathbb{C}\backslash\gamma$ is open, with boundary equal to γ.*

Moreover $\mathbb{C}\backslash\gamma$ consists of two disjoint connected sets, say A and B.

Precisely one of these sets (say A) is bounded and simply connected. This is the ***interior*** *of γ.*

The other set (B) is unbounded. This is the ***exterior*** *of γ.*

Finally, there exists a "positive orientation" for γ such that

$$W_\gamma(z) = \begin{cases} 1 & z \in A \\ 0 & z \in B. \end{cases}$$

To prove this theorem would take us too far afield, but for the proof one can look in Appendix B in Stein–Shakarchi.

Definition 3.23. We will call a curve γ satisfying the hypotheses of Theorem 3.22 a **Jordan curve**.

3.5. **Goursat's Theorem and Cauchy's Theorem.** Our next topic concerns the existence of 'primitives' (or antiderivatives) of holomorphic functions.

Definition 3.24 (Primitive)**.** Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. A **primitive** for f on Ω is a function $F : \Omega \to \mathbb{C}$ such that

- F is holomorphic on Ω,
- for all $z \in \Omega$, $F'(z) = f(z)$.

Our first result gives a necessary condition for the existence of primitives.

Theorem 3.25. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be continuous.*

Suppose F is a primitive for f. If γ is a curve in Ω joining α to β, then

$$\int_\gamma f(z)\,dz = F(\beta) - F(\alpha).$$

In particular, if γ is closed and f has a primitive then

$$\int_\gamma f(z)\,dz = 0.$$

Proof. Let $z : [a, b] \to \mathbb{C}$ be a parametrization of γ. If γ is smooth, then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt \\ &= \int_a^b \tfrac{d}{dt}[F \circ z](t)\,dt = F(z(b)) - F(z(a)) = F(\beta) - F(\alpha).\end{aligned}$$

If γ is piecewise-smooth, then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{k=0}^{n-1} F(z(a_{k+1})) - F(z(a_k)) \\ &= F(z(a_n)) - F(z(a_0)) = F(\beta) - F(\alpha).\end{aligned}$$

□

Example 3.8. The function $f(z) = \frac{1}{z}$ does **not** have a primitive in $\mathbb{C}\backslash\{0\}$, since

$$\int_\gamma \tfrac{dz}{z} = 2\pi i \quad \text{for} \quad \gamma = \{z : |z| = 1\}.$$

Corollary 3.26. *Let $\Omega \subset \mathbb{C}$ be open and connected. If $f : \Omega \to \mathbb{C}$ is holomorphic and $f' \equiv 0$, then f is constant.*

Proof. See Exercise 3.11. □

Theorem 3.27 (Goursat's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and $T \subset \Omega$ be a (closed) triangle contained in Ω. If $f : \Omega \to \mathbb{C}$ is holomorphic, then*

$$\int_{\partial T} f(z)\,dz = 0.$$

Lemma 3.28 (Warmup)**.** *Let Ω, T, f as above. If in addition f' is continuous, then*

$$\int_{\partial T} f(z)\,dz = 0.$$

Proof. This uses Green's theorem and the Cauchy–Riemann equations. See Exercise 3.15. □

Proof of Goursat's theorem. First write $T = T^0$.

We subdivide T^0 into four similar subtriangles $T_1^1, \dots, T_4^1$ and note

$$\int_{\partial T^0} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial T_j^1} f(z)\,dz.$$

This implies that

$$\left| \int_{\partial T_j^1} f(z)\,dz \right| \geq \tfrac{1}{4} \left| \int_{\partial T^0} f(z)\,dz \right| \quad \text{for at least one } j.$$

Choose such a T_j^1 and rename it T^1.

Repeating this process yields a nested sequence of triangles

$$T^0 \supset T^1 \supset \dots \supset T^n \supset \dots$$

such that

$$\left| \int_{\partial T^0} f(z)\,dz \right| \leq 4^n \left| \int_{\partial T^n} f(z)\,dz \right|,$$

and

$$\operatorname{diam}(T^n) = (\tfrac{1}{2})^n \operatorname{diam}(T^0), \quad \operatorname{length}(\partial T^n) = (\tfrac{1}{2})^n \operatorname{length}(\partial T^0).$$

Using Cantor's intersection theorem we may find $z_0 \in \cap_{n=0}^{\infty} T^n$. (In fact z_0 is unique.)

As f is holomorphic at z_0, we may write

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{:=g(z)} + h(z)(z - z_0),$$

where $\lim_{z \to z_0} h(z) = 0$.

Since g is *continuously* complex differentiable, the lemma implies

$$\int_{\partial T^n} f(z)\, dz = \underbrace{\int_{\partial T^n} g(z)\, dz}_{=0} + \int_{\partial T^n} h(z)(z - z_0)\, dz.$$

Thus we can estimate

$$\begin{aligned} \left| \int_{\partial T^n} f(z)\, dz \right| &\leq \sup_{z \in \partial T^n} |h(z)| \cdot \mathrm{diam}(T^n) \cdot \mathrm{length}(\partial T^n) \\ &= 4^{-n} \sup_{z \in \partial T^n} |h(z)| \cdot \mathrm{diam}(T^0) \cdot \mathrm{length}(\partial T^0). \end{aligned}$$

Thus

$$\left| \int_{\partial T^0} f(z)\, dz \right| \leq \sup_{z \in \partial T^n} |h(z)| \cdot \mathrm{diam}(T^0) \cdot \mathrm{length}(\partial T^0).$$

We now send $n \to \infty$ and use $\lim_{z \to z_0} h(z) = 0$ to conclude that

$$\int_{\partial T^0} f(z)\, dz = 0.$$

□

Corollary 3.29. *Goursat's theorem holds for polygons.*

Proof. Check! □

Theorem 3.30. *If $z_0 \in \mathbb{C}$, $R > 0$, and $f : B_R(z_0) \to \mathbb{C}$ is holomorphic, then f has a primitive in $B_R(z_0)$.*

Proof. Without loss of generality, we may take $z_0 = 0$.

For $z \in B_R(0)$, let γ_z be the piecewise-smooth curve that joins 0 to z comprised of the horizontal line segment joining 0 to $\mathrm{Re}\,(z)$ and the vertical line segment joining $\mathrm{Re}\,(z)$ to z.

We define

$$F(z) = \int_{\gamma_z} f(w)\, dw.$$

We will show

(i) F is holomorphic on $B_R(0)$ and

(ii) $F'(z) = f(z)$ for $z \in B_R(0)$.

To this end, we consider $z \in B_R(0)$ and $h \in \mathbb{C}$ such that $z + h \in B_R(0)$.

Using Goursat's theorem we deduce

$$F(z + h) - F(z) = \int_{\ell} f(w)\, dw,$$

where ℓ is the line segment joining z to $z + h$.

We now write

$$\begin{aligned}\int_\ell f(w)\,dw &= f(z)\int_\ell dw + \int_\ell [f(w)-f(z)]\,dw \\ &= f(z)h + \int_\ell [f(w)-f(z)]\,dw.\end{aligned}$$

Thus

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &= \left|\tfrac{1}{h}\int_\ell [f(w)-f(z)]\,dw\right| \\ &\leq \tfrac{|h|}{|h|}\cdot\sup_{w\in\ell}|f(w)-f(z)| \\ &\to 0 \quad \text{as} \quad h\to 0.\end{aligned}$$

□

Theorem 3.31. *Let $\Omega\subset\mathbb{C}$ be an open set and $f:\Omega\to\mathbb{C}$ be holomorphic. Suppose γ_0 and γ_1 are homotopic in Ω. Then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Proof. By definition of homotopy we get a (uniformly) continuous function $\gamma:[0,1]\times[a,b]\to\Omega$, where each $\gamma(s,\cdot)$ parametrizes the curve γ_s.

As γ is continuous, the image of $[0,1]\times[a,b]$ under γ (denoted by K) is compact.

Step 1. There exists $\varepsilon>0$ such that for all $z\in K$, $B_{3\varepsilon}(z)\subset\Omega$.

If not, then for all n we may find $z_n\in K$ and $w_n\in B_{1/n}(z)\cap[\mathbb{C}\backslash\Omega]$. As K is compact, there exists a convergent subsequence $z_{n_k}\to z\in K\subset\Omega$. However, by construction $w_{n_k}\to z$. As $\mathbb{C}\backslash\Omega$ is closed, we find $z\in\mathbb{C}\backslash\Omega$, a contradiction.

Choose an $\varepsilon>0$ as in Step 1. By uniform continuity,

$$\begin{aligned}&\text{there exists} \quad \delta>0 \quad \text{such that} \\ &|s_1-s_2|<\delta \implies \sup_{t\in[a,b]}|\gamma(s_1,t)-\gamma(s_2,t)|<\varepsilon.\end{aligned}$$

Step 2. We will show that for any s_1,s_2 with $|s_1-s_2|<\delta$ we have

$$\int_{\gamma_{s_1}} f(z)\,dz = \int_{\gamma_{s_2}} f(z)\,dz. \tag{3.2}$$

For this step we construct points $\{z_j\}_{j=0}^n\subset\gamma_{s_1}$, $\{w_j\}_{j=0}^n\subset\gamma_{s_2}$, and balls $\{D_j\}_{j=0}^n$ in Ω of radius 2ε such that:

- $w_0=z_0$ and $w_n=z_n$ are the common endponts of γ_{s_1} and γ_{s_2}
- for $j=0,\dots,n-1$ we have $z_j,z_{j+1},w_j,w_{j+1}\in D_j$
- $\gamma_{s_1}\cup\gamma_{s_2}\subset\cup_{j=0}^n D_j$.

On each ball D_j Theorem 3.30 implies that f has a primitive. say F_j.

On $D_j\cap D_{j+1}$ the functions F_j and F_{j+1} are both primitives for f, and hence they differ by a constant (see Exercise 3.12).

In particular

$$F_{j+1}(z_{j+1})-F_j(z_{j+1}) = F_{j+1}(w_{j+1})-F_j(w_{j+1}),$$

or, rearranging:

$$F_{j+1}(z_{j+1})-F_{j+1}(w_{j+1}) = F_j(z_{j+1})-F_j(w_{j+1}).$$

Hence

$$\begin{aligned}
\int_{\gamma_{s_1}} f(z)\,dz - \int_{\gamma_{s_2}} f(z)\,dz &= \sum_{j=0}^{n-1}[F_j(z_{j+1}) - F_j(z_j)] - \sum_{j=0}^{n-1}[F_j(w_{j+1}) - F_j(w_j)] \\
&= \sum_{j=0}^{n-1}[F_j(z_{j+1}) - F_j(w_{j+1}) - (F_j(z_j) - F_j(w_j))] \\
&= \sum_{j=0}^{n-1}[F_{j+1}(z_{j+1}) - F_{j+1}(w_{j+1}) - (F_j(z_j) - F_j(w_j))] \\
&= F_n(z_n) - F_n(w_n) - (F_0(z_0) - F_0(w_0)) \\
&= 0.
\end{aligned}$$

Step 3. We now divide $[0,1]$ into finitely many intervals $[s_j, s_{j+1}]$ of length less than δ and apply Step 2 on each interval to deduce that

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

□

We can now give a sufficient condition for the existence of a primitive.

Theorem 3.32 (Cauchy's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be simply connected and $f : \Omega \to \mathbb{C}$ be holomorphic. Then f has a primitive in Ω.*

In particular,

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve $\gamma \subset \Omega$.

Proof. Fix $z_0 \in \Omega$.

For any $z \in \Omega$ let γ_z be a curve in Ω joining z_0 to z and define

$$F(z) = \int_{\gamma_z} f(w)\,dw.$$

(Note that this is well-defined by Theorem 3.31)

For $h \in \mathbb{C}$ sufficiently small, we can write

$$F(z+h) - F(z) = \int_\ell f(w)\,dw,$$

where ℓ is the line segment joining z and $z+h$.

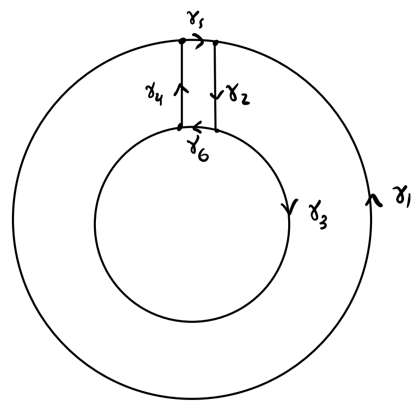
Thus arguing as in the proof of Theorem 3.30 we find that $F'(z) = f(z)$. □

3.6. **The Cauchy Integral Formula and Applications.** We next prove an important 'representation formula' for holomorphic functions and explore some its consequences.

Lemma 3.33. *Suppose $w \in \mathbb{C}$, $R > 0$, and $g : B_R(w)\backslash\{w\} \to \mathbb{C}$ is holomorphic. Then*

$$\int_{\partial B_r(w)} g(z)\,dz = \int_{\partial B_s(w)} g(z)\,dz \quad \textit{for} \quad 0 < r < s < R.$$

Proof. Let $\delta > 0$ be a small parameter. Join $\partial B_r(w)$ to $\partial B_s(w)$ with two vertical line segments a distance δ apart, and denote the curves $\gamma_1, \dots, \gamma_6$ as in the following figure:



By Cauchy's theorem (applied twice), we have

$$\int_{\gamma_1} g(z)\,dz + \int_{\gamma_3} g(z)\,dz = -\left(\int_{\gamma_2} g(z)\,dz + \int_{\gamma_4} g(z)\,dz\right) = \int_{\gamma_5} g(z)\,dz + \int_{\gamma_6} g(z)\,dz.$$

Rearranging gives

$$\int_{\gamma_1} g(z)\,dz - \int_{\gamma_5} g(z)\,dz = \int_{\gamma_6} g(z)\,dz - \int_{\gamma_3} g(z)\,dz,$$

which gives the result.

□

Theorem 3.34 (Cauchy integral formula)**.** *Let Ω be an open set and $f : \Omega \to \mathbb{C}$ holomorphic. Suppose $w \in \Omega$ and B is a ball containing w such that $\overline{B} \subset \Omega$. Then*

$$f(w) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(z)}{z-w}\,dz.$$

Proof. Arguing as in the proof of Lemma 3.33, one can show that it suffices to take $B = B_r(w)$ for some $r > 0$. (*Check!*)

Step 1. We show

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{\partial B_\varepsilon(w)} \frac{f(z)}{z-w}\,dz = f(w). \tag{3.3}$$

To see this we write

$$\frac{f(z)}{z-w} = \frac{f(z)-f(w)}{z-w} + f(w)\frac{1}{z-w}.$$

Since

$$\lim_{z\to w} \frac{f(z)-f(w)}{z-w} = f'(w)$$

we find that

$$\text{there exist}\quad \varepsilon_0 > 0,\ C > 0 \quad\text{such that}\quad |z-w| < \varepsilon_0 \implies \left|\frac{f(z)-f(w)}{z-w}\right| < C.$$

Thus for $\varepsilon < \varepsilon_0$ we have

$$\left| \frac{1}{2\pi i} \int_{\partial B_\varepsilon(w)} \frac{f(z) - f(w)}{z - w} \, dz \right| \leq \frac{2C\pi\varepsilon}{2\pi} = C\varepsilon.$$

In particular

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial B_\varepsilon(w)} \frac{f(z) - f(w)}{z - w} \, dz = 0. \quad (*)$$

On the other hand for any $\varepsilon > 0$

$$\frac{1}{2\pi i} \int_{\partial B_\varepsilon(w)} \frac{dz}{z - w} = W_{\partial B_\varepsilon(w)}(w) = 1. \quad (**)$$

Putting together $(*)$ and $(**)$ we complete Step 1.

Step 2. Using the lemma and the fact that

$$\frac{f(z)}{z - w}$$

is holomorphic in $\Omega \backslash \{w\}$, we find that

$$\int_{\partial B_r(w)} \frac{f(z)}{z - w} \, dz = \int_{\partial B_\varepsilon(w)} \frac{f(z)}{z - w} \, dz \quad \text{for all} \quad 0 < \varepsilon < r.$$

Thus by Step 1,

$$\frac{1}{2\pi i} \int_{\partial B_r(w)} \frac{f(z)}{z - w} \, dz = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial B_\varepsilon(w)} \frac{f(z)}{z - w} \, dz = f(w).$$

□

3.7. **Corollaries of the Cauchy Integral Formula.** We now record some important consequences of the Cauchy integral formula.

Corollary 3.35. *Holomorphic functions are analytic (and hence infinitely differentiable).*

More precisely: let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. Then for all $z_0 \in \Omega$ we can expand f in a power series centered at z_0 with radius of convergence at least $\inf_{z \in \mathbb{C} \backslash \Omega} |z - z_0|$.

Proof. Let $z_0 \in \Omega$ and choose

$$0 < r < \inf_{z \in \mathbb{C} \backslash \Omega} |z - z_0|.$$

By the Cauchy integral formula we have

$$f(w) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{z - w} \, dz \quad \text{for all} \quad w \in B_r(z_0).$$

Now for $z \in \partial B_r(z_0)$ and $w \in B_r(z_0)$ we have $|w - z_0| < |z - z_0|$, so that

$$\frac{1}{z - w} = \frac{1}{(z - z_0) - (w - z_0)} = \frac{1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^n.$$

Here we have used the geometric series expansion, and we note that the series converges **uniformly** for $z \in \partial B_r(z_0)$.

In particular, for $w \in B_r(z_0)$ we have

$$f(w) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(z)}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n dz$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right)(w-z_0)^n$$

This shows that f has a power series expansion at w.

Moreover since

$$\frac{f(z)}{(z-z_0)^{n+1}}$$

is holomorphic in $\Omega\backslash\{z_0\}$ we can use Lemma 3.33 above to see that the integrals

$$\frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

are independent of r.

Thus f has a power series expansion for all $w \in B_r(z_0)$, with the same coefficients for each w. □

Remark 3.36. From the proof of Corollary 3.35 and termwise differentiation we deduce the **Cauchy integral formulas:**

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad \text{for} \quad 0 < r < \inf_{z\in\mathbb{C}\backslash\Omega}|z-z_0|.$$

From these identities we can read off the **Cauchy inequalities:**

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n}\sup_{z\in\partial B_r(z_0)}|f(z)| \qquad \text{for} \quad 0 < r < \inf_{z\in\mathbb{C}\backslash\Omega}|z-z_0|.$$

Next we have Liouville's theorem.

Corollary 3.37 (Liouville's theorem)**.** *Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded. Then f is constant.*

Proof. The Cauchy inequalities imply

$$|f'(z)| \leq \frac{1}{r}\sup_{w\in\mathbb{C}}|f(w)|$$

for any $r > 0$. As f is bounded, this implies $f'(z) \equiv 0$, which implies that f is constant. □

Corollary 3.38 (Fundamental theorem of algebra)**.** *Let $f(z) = a_nz^n+\cdots+a_1z+a_0$ with $a_n \neq 0$. Then there exist $\{w_j\}_{j=1}^n$ such that*

$$f(z) = a_n(z-w_1)(z-w_2)\cdots(z-w_n).$$

Proof. Without loss of generality assume $a_n = 1$.

Suppose first that

$$f(z) \neq 0 \quad \text{for all} \quad z \in \mathbb{C}.$$

Then the function $\frac{1}{f}$ is entire. Moreover, we claim it is **bounded**.

To see this write

$$f(z) = z^n + z^n\left(\tfrac{a_{n-1}}{z} + \cdots + \tfrac{a_0}{z^n}\right) \quad \text{for} \quad z \neq 0.$$

As $\lim_{|z|\to\infty} \frac{1}{z^k} = 0$ for all $k \geq 1$

$$\text{there exists} \quad R > 0 \quad \text{such that} \quad |z| > R \implies \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| < \frac{1}{2}.$$

Thus

$$|z| > R \implies |f(z)| \geq \tfrac{1}{2}|z|^n \geq \tfrac{1}{2}R^n \implies \left|\frac{1}{f(z)}\right| \leq 2R^{-n}.$$

On the other hand, since f is continuous and non-zero on the compact set $\overline{B_R(0)}$, there exists $\varepsilon > 0$ such that

$$|z| \leq R \implies |f(z)| \geq \varepsilon \implies \left|\frac{1}{f(z)}\right| \leq \varepsilon^{-1}.$$

Thus

$$\text{for all} \quad z \in \mathbb{C} \quad \left|\frac{1}{f(z)}\right| \leq 2R^{-n} + \varepsilon^{-1},$$

that is, $\frac{1}{f}$ is bounded.

Thus Liouville's theorem implies that $\frac{1}{f}$ (and hence f) is constant, which is a contradiction.

We conclude that

$$\text{there exists} \quad w_1 \in \mathbb{C} \quad \text{such that} \quad f(w_1) = 0.$$

We now write $z = (z - w_1) + w_1$ and use the binomial formula to write

$$f(z) = (z - w_1)^n + b_{n-1}(z - w_1)^{n-1} + \cdots + b_1(z - w_1) + b_0$$

for some $b_k \in \mathbb{C}$.

Noting that $b_0 = f(w_1) = 0$, we find

$$f(z) = (z - w_1)[(z - w_1)^{n-1} + \cdots + b_2(z - w_1) + b_1] =: (z - w_1)g(z).$$

We now apply the arguments above to the degree $n - 1$ polynomial $g(z)$ to find $w_2 \in \mathbb{C}$ such that $g(w_2) = 0$.

Proceeding inductively we find that $P(z)$ has n roots $\{w_j\}_{j=1}^n$ and factors as

$$f(z) = (z - w_1)(z - w_2)\cdots(z - w_n),$$

as was needed to show. $\square$

We next have a converse of Goursat's theorem.

Corollary 3.39 (Morera's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be continuous. If*

$$\int_{\partial T} f(z)\,dz = 0$$

for all closed triangles $T \subset \Omega$, then f is holomorphic in Ω.

Proof. Recall that to prove Theorem 3.30 (the existence of primitives for holomorphic functions in a disk) we needed (i) continuity and (ii) the conclusion of Goursat's theorem.

For this theorem we are given both (i) and (ii) and hence we may conclude that f has a primitive in any disk contained in Ω.

Thus for any $w \in \Omega$ there exists $r > 0$ and a holomorphic function $F : B_r(w) \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in B_r(w) \subset \Omega$.

Using Corollary 3.35 we conclude that $F' = f$ is holomorphic at w, as needed. $\square$

We also have the following useful corollary.

Corollary 3.40. *Let $\Omega \subset \mathbb{C}$ be open. Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$. Suppose f_n converges to $f : \Omega \to \mathbb{C}$ "locally uniformly", that is, for any compact $K \subset \Omega$ we have $f_n \to f$ uniformly on K. Then f is holomorphic on Ω.*

Proof. Let $T \subset \Omega$ be a closed triangle.

Note that as $f_n \to f$ uniformly we have f is continuous on T.

By Goursat's theorem we have

$$\int_{\partial T} f_n(z)\,dz = 0 \quad \text{for all} \quad n.$$

Thus since $f_n \to f$ uniformly on T we have

$$\int_{\partial T} f(z)\,dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)\,dz = 0.$$

As T was arbitrary, Morera's theorem implies that f is holomorphic on Ω. □

Remark 3.41. Contrast this to the real-valued case: every continuous function on $[0,1]$ can be uniformly approximated by polynomials (this is Weierstrass's theorem), but not every continuous function is differentiable.

Remark 3.42. Under the hypotheses of Corollary 3.40 we also get that f_n' converge to f' locally uniformly. In fact, this is true for higher derivatives as well. (See Exercise 3.20.)

Remark 3.43. In practice one uses Corollary 3.40 to construct holomorphic functions (perhaps with a prescribed property) as a series of the form

$$F(z) = \sum_{n=1}^\infty f_n(z).$$

A related idea is to construct holomorphic functions of the form

$$f(z) = \int_a^b F(s,z)\,ds.$$

See Exercise 3.21.

We next turn to a remarkable "uniqueness theorem" for holomorphic functions.

Theorem 3.44 (Uniqueness theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected and let $z_0 \in \Omega$. Suppose $\{z_k\}_{k=1}^\infty \subset \Omega\backslash\{z_0\}$ satisfies $\lim_{k\to\infty} z_k = z_0$.*

Suppose $f, g : \Omega \to \mathbb{C}$ are holomorphic and $f(z_k) = g(z_k)$ for each k. Then $f \equiv g$ in Ω.

Proof. First we note that it suffices to consider the case $g = 0$. (*Check!*)

As f is holomorphic at z_0, we may find $r > 0$ such that

$$f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n \quad \text{for} \quad z \in B_r(z_0).$$

Step 1. We show $f(z) = 0$ for $z \in B_r(z_0)$.

By continuity we have $f(z_0) = 0$.

Let $z \in B_r(z_0)\backslash\{z_0\}$. If $f(z) \neq 0$, then we choose m to be the smallest integer such that $a_m \neq 0$.

We can then write
$$f(z) = a_m(z-z_0)^m(1+g(z))$$
where
$$g(z) := \sum_{n=m+1}^{\infty} \tfrac{a_n}{a_m}(z-z_0)^{n-m} \to 0 \quad \text{as} \quad z \to z_0.$$
Thus there exists $\delta > 0$ such that
$$|z-z_0| < \delta \implies |g(z)| < \tfrac{1}{2} \implies 1+g(z) \neq 0.$$
Choosing k large enough that $|z_k - z_0| < \delta$ and recalling $z_k \neq z_0$ we find
$$0 = f(z_k) = a_m(z_k-z_0)^m(1+g(z_k)) \neq 0,$$
a contradiction.

Step 2. We use a "clopen" argument.

Define the set
$$S = \text{interior}(\{z \in \Omega : f(z) = 0\}).$$
This set is open by definition. Moreover by Step 1, $z_0 \in S$. Thus $S \neq \emptyset$.

Finally we claim that S is closed in Ω.

To see this we suppose $\{w_n\}_{n=1}^{\infty} \subset S$ converges to some $w_0 \in \Omega$. We need to show $w_0 \in S$.

To see this we first note that by continuity $f(w_0) = 0$.

Next, arguing as in Step 1, we find $\delta > 0$ such that $f(z) = 0$ for all $z \in B_\delta(w_0)$. This shows $w_0 \in S$.

As Ω is connected and S is nonempty, open in Ω, and closed in Ω, we conclude that $S = \Omega$, as was needed to show. □

Definition 3.45. Suppose Ω and Ω' are open connected subsets of $\mathbb{C}$ with $\Omega \subsetneq \Omega'$. If $f : \Omega \to \mathbb{C}$ and $F : \Omega' \to \mathbb{C}$ are holomorphic and $f(z) = F(z)$ for $z \in \Omega$, we call F **the analytic continuation** of f into Ω'.

Remark 3.46. By the uniqueness theorem, a holomorphic function can have at most one analytic continuation.

3.8. **Exercises.**

Exercise 3.1. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \bar{z}$. Use the **definition of the derivative** to show that f is not holomorphic at any point.

Exercise 3.2. Fix $w \in \mathbb{D}$ and define the **Blaschke factor**
$$F(z) = \frac{w-z}{1-\bar{w}z} \quad \text{for} \quad z \in \mathbb{D}.$$
Show the following:
- $F : \mathbb{D} \to \mathbb{D}$, and $F : \partial\mathbb{D} \to \partial\mathbb{D}$,
- F is a bijection on $\mathbb{D}$,
- F is holomorphic on $\mathbb{D}$.

Exercise 3.3. Let $f : \mathbb{C} \to \mathbb{C}$ and define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by
$$u(x,y) = \text{Re}\,[f(x+iy)], \quad v(x,y) = \text{Im}\,[f(x+iy)].$$
Suppose f is holomorphic at some $z_0 = x_0 + iy_0 \in \mathbb{C}$.
- Use the **definition of the derivative** to show that

$$f'(z_0) = \tfrac{\partial u}{\partial x}(x_0,y_0) + i\tfrac{\partial v}{\partial x}(x_0,y_0) \quad \text{and} \quad f'(z_0) = -i\tfrac{\partial u}{\partial y}(x_0,y_0) + \tfrac{\partial v}{\partial y}(x_0,y_0). \quad (*)$$

- Use $(*)$ to derive the Cauchy–Riemann equations.

Exercise 3.4. Suppose $f:\mathbb{C}\to\mathbb{C}$ is holomorphic. Show the following:

- if $\mathrm{Re}\,(f)$ is constant, then f is constant,
- if $\mathrm{Im}\,(f)$ is constant, then f is constant,
- if $|f|$ is constant, then f is constant.

Exercise 3.5. Prove Lemma 3.4.

Exercise 3.6. Prove Corollary 3.5.

Exercise 3.7. Let $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ be finite sequences in $\mathbb{C}$, and define $B_k=\sum_{n=1}^k b_n$, with the convention $B_0=0$. Prove the **summation by parts formula**:

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1}(a_{n+1}-a_n)B_n.$$

Exercise 3.8. Show the following:

- the power series $\sum_n nz^n$ does not converge for any $z\in\partial\mathbb{D}$,
- the power series $\sum_n \frac{1}{n^2}z^n$ converges for all $z\in\partial\mathbb{D}$,
- the power series $\sum_n \frac{1}{n}z^n$ converges for all $z\in\partial\mathbb{D}$ except for $z=1$.

Exercise 3.9. Show that if f is holomorphic at $z\in\mathbb{C}$ then f is continuous at z.

Exercise 3.10. Let $\Omega\subset\mathbb{C}$ be open. Show that Ω is connected if and only if it is path connected.

Exercise 3.11. Let $\Omega\subset\mathbb{C}$ be open and connected and $f:\Omega\to\mathbb{C}$ be holomorphic. Show that if $f'(z)=0$ for all $z\in\Omega$ then f is constant.

Exercise 3.12. Suppose $\Omega\subset\mathbb{C}$ is open and connected and $f:\Omega\to\mathbb{C}$ is continuous. Show that if F and $\tilde F$ are both primitives for f in Ω then the function $F-\tilde F$ is constant.

Exercise 3.13.

- Show that $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ is simply connected.
- Find an open connected subset of $\mathbb{C}$ that is **not** simply connected. (Explain why your example meets all of the stated requirements.)

Exercise 3.14. Let γ be a circle with positive orientation.

- Suppose γ is centered at the origin. Evaluate the integrals

$$\int_\gamma z^n\,dz \quad \text{for} \quad n\in\mathbb{Z}. \quad (*)$$

- Suppose γ does not contain the origin. Evaluate the integrals $(*)$.

Exercise 3.15. Let $\Omega\subset\mathbb{C}$ be open and $f:\Omega\to\mathbb{C}$ be holomorphic. Further assume that f' is continuous. Use **Green's theorem** to show that

$$\int_{\partial T} f(z)\,dz = 0$$

for any triangle $T\subset\Omega$.

Exercise 3.16. Show that the relation "is homotopic to" is an equivalence relation. That is,

(i) any curve is homotopic to itself,
(ii) if γ_0 is homotopic to γ_1, then γ_1 is homotopic to γ_0,
(ii) if γ_0 is homotopic to γ_1 and γ_1 is homotopic to γ_2, then γ_0 is homotopic to γ_2.

Exercise 3.17. Let $r, R > 0$ and $z_0, z_1 \in \mathbb{C}$. Construct a **continuous** $F : [0,1] \times [0,1] \to \mathbb{C}$ such that

- the function $t \mapsto F(0,t)$ is a parametrization of $\partial B_r(z_0)$,
- the function $t \mapsto F(1,t)$ is a parametrization of $\partial B_R(z_1)$,
- for each $s \in (0,1)$ the function $t \mapsto F(s,t)$ parametrizes a closed curve in $\mathbb{C}$.

Exercise 3.18. Prove this stronger version of the Cauchy integral formula: let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. If $z_0 \in \Omega$ and B is any ball containing z_0 such that $\overline{B} \subset \Omega$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - z_0}\, dz.$$

Hint. Use the version of the Cauchy integral formula we proved in class, and argue as in the proof of "Lemma 4.33" from class.

Exercise 3.19. Let $K \subset \mathbb{C}$ be compact and $f : K \to \mathbb{C}$ be continuous. Suppose that $f(z) \neq 0$ for all $z \in K$. Show that

$$\text{there exists} \quad \delta > 0 \quad \text{such that} \quad \text{for all} \quad z \in K \quad |f(z)| \geq \delta.$$

Exercise 3.20. Let $\Omega \subset \mathbb{C}$ be open and suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge uniformly to $f : \Omega \to \mathbb{C}$. Show that for $\delta > 0$ we have that $f_n' \to f'$ uniformly on the set

$$K_\delta := \{z \in \Omega : \overline{B_\delta(z)} \subset \Omega\}.$$

Hint. Use the Cauchy inequalities. You may take for granted that f is holomorphic (since we proved this in class).

Exercise 3.21. Let $\Omega \subset \mathbb{C}$ be open. Suppose $F : [0,1] \times \Omega \to \mathbb{C}$ is continuous and satisfies

$$\text{for all} \quad s \in [0,1] \quad \text{the function} \quad z \mapsto F(s,z) \quad \text{is holomorphic on } \Omega.$$

Show that the function $f : \Omega \to \mathbb{C}$ defined by $f(z) = \int_0^1 F(s,z)\, ds$ is holomorphic on Ω. *Hint.* Use the definition of the Riemann integral to show that f is the (locally uniform) limit of holomorphic functions.

Exercise 3.22. Can every continuous function on the set $\{z \in \mathbb{C} : |z| \leq 1\}$ be approximated uniformly by polynomials? If so, prove it. If not, give a counterexample.

Exercise 3.23. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and satisfies $|f(z)| \leq C(1 + |z|)^n$ for some $C > 0$ and some integer n (for all $z \in \mathbb{C}$). Show that f is a polynomial of degree at most n.
Hint. We proved the case $n = 0$ in class (Liouville's theorem). For the general case use the Cauchy inequalities to show that $f^{(n+k)}(0) = 0$ for all integers $k > 0$. Why does this solve the problem?

Exercise 3.24. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire.

(i) Show that if $f(z) = 0$ for uncountably many $z \in \mathbb{C}$ then $f \equiv 0$.

(ii) Suppose that for each $z_0 \in \mathbb{C}$ at least one coefficient in the power series expansion at z_0 is zero. Prove that f is a polynomial.

4. Meromorphic Functions

4.1. Isolated Singularities.

Definition 4.1 (Isolated singularity)**.** If $z_0 \in \mathbb{C}$ and $f : \Omega \backslash \{z_0\} \to \mathbb{C}$ for some open set Ω, we call z_0 an **isolated singularity** (or **point singularity**) of f.

Example 4.1. The following functions have isolated singularities at $z = 0$.

(i) $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $f(z) = z$
(ii) $g : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $g(z) = \frac{1}{z}$
(iii) $h : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $h(z) = e^{\frac{1}{z}}$.

Theorem 4.2 (Riemann's removable singularity theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and let $z_0 \in \Omega$. Suppose $f : \Omega\backslash\{z_0\} \to \mathbb{C}$ satisfies*

(i) *f is holomorphic on $\Omega\backslash\{z_0\}$*
(ii) *f is bounded on $\Omega\backslash\{z_0\}$.*

Then f may be extended uniquely to a holomorphic function $F : \Omega \to \mathbb{C}$.

Remark 4.3. We call the point z_0 in Theorem 4.2 a **removable singularity** of f.

Proof of Theorem 4.2. As Ω is open we may find $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$.
For $z \in B_r(z_0)$ let us define

$$F(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{w - z}\, dw.$$

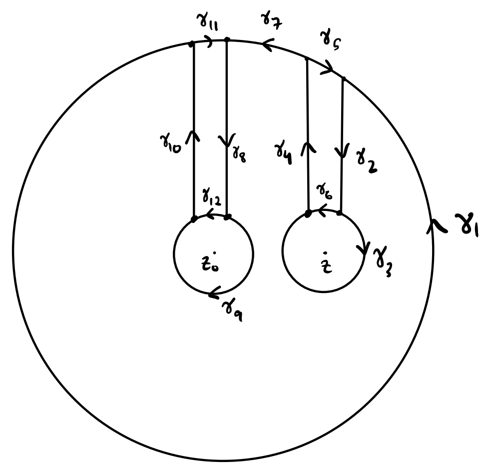
We first note that $F : B_r(z_0) \to \mathbb{C}$ is holomorphic (cf. the exercises).
We will show that

$$f(z) = F(z) \quad \text{for} \quad z \in B_r(z_0)\backslash\{z_0\},$$

which implies (by the "uniqueness theorem") that F extends to a holomorphic function on the connected component A of Ω containing z_0, and $f(z) = F(z)$ for $z \in A\backslash\{z_0\}$.
Let $z \in B_r(z_0)\backslash\{z_0\}$ and let $\varepsilon > 0$ be small enough that

$$\overline{B_\varepsilon(z_0)} \cup \overline{B_\varepsilon(z)} \subset B_r(z_0).$$

(Without loss of generality assume $\mathrm{Re}\,(z) > \mathrm{Re}\,(z_0)$. This only helps the picture.)
Let $\delta > 0$ be a small parameter. Join $\partial B_\varepsilon(z)$ and $\partial B_\varepsilon(z_0)$ up to $\partial B_r(z_0)$ with two pairs of lines, each a distance δ apart. We define the curves $\gamma_1, \ldots, \gamma_{12}$ as in the following figure:



Let us define

$$A_j = \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(w)}{w-z}\,dw \quad \text{for} \quad j = 1, \dots, 12.$$

Using Cauchy's theorem we deduce

- $A_1 + A_2 + A_3 + A_4 + A_7 + A_8 + A_9 + A_{10} = 0$,
- $A_{11} + A_8 + A_{12} + A_{10} = 0$,
- $A_5 + A_2 + A_6 + A_4 = 0$.

Combining these equalities yields

$$A_1 - A_5 + A_7 - A_{11} = A_6 - A_3 + A_{12} - A_9,$$

or:

$$\underbrace{\frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{w-z}\,dw}_{F(z)} = \underbrace{\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z)} \frac{f(w)}{w-z}\,dw}_{I} + \underbrace{\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z_0)} \frac{f(w)}{w-z}\,dw}_{II}. \quad (*)$$

Note that $I = f(z)$ for all small $\varepsilon > 0$ by the Cauchy integral formula. For II we note that for any $0 < \varepsilon < \frac{1}{2}|z - z_0|$,

$$\begin{aligned} |II| &= \left|\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z_0)} \frac{f(w)}{w-z}\,dw\right| \\ &\leq \frac{2\pi\varepsilon \sup_{w\in\Omega\backslash\{z_0\}} |f(w)|}{2\pi \inf_{w\in\partial B_\varepsilon(z_0)} |w-z|} \\ &\leq \varepsilon\,\frac{\sup_{w\in\Omega\backslash\{z_0\}} |f(w)|}{\frac{1}{2}|z-z_0|}. \end{aligned}$$

Since f is bounded on $\Omega\backslash\{z_0\}$, we find

$$\lim_{\varepsilon\to 0} |II| = 0.$$

Thus sending $\varepsilon \to 0$ in $(*)$ implies $F(z) = f(z)$, as was needed to show. $\square$

Example 4.2. The function $f(z) = z$ on $\mathbb{C}\backslash\{0\}$ has a removable singularity at $z = 0$.

Definition 4.4 (Pole)**.** Let $\Omega \subset \mathbb{C}$ be an open set, $z_0 \in \Omega$, and $f : \Omega\backslash\{z_0\} \to \mathbb{C}$.

If there exists $r > 0$ such that the function $g : B_r(z_0) \to \mathbb{C}$ defined by

$$g(z) := \begin{cases} \frac{1}{f(z)} & z \in B_r(z_0)\backslash\{z_0\} \\ 0 & z = z_0 \end{cases}$$

is holomorphic on $B_r(z_0)$, we say f has a **pole at** z_0.

Example 4.3. The function $f(z) = \frac{1}{z}$ defined on $\mathbb{C}\backslash\{0\}$ has a pole at $z = 0$.

Proposition 4.5. *Suppose $f : \Omega\backslash\{z_0\} \to \mathbb{C}$ is holomorphic with an isolated singularity at z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole then by definition $\frac{1}{f(z)} \to 0$ as $z \to z_0$. In particular $|f(z)| \to \infty$ as $z \to z_0$.

On the other hand, suppose $|f(z)| \to \infty$ as $z \to z_0$. Then $\frac{1}{f(z)} \to 0$ as $z \to z_0$. In particular $\frac{1}{f}$ is bounded as $z \to z_0$.

Thus $\frac{1}{f}$ has a holomorphic extension in some ball around z_0, which (by continuity) must be given by the function "g" defined in Definition 4.4. In particular f has a pole at z_0. □

Definition 4.6 (Essential singularity)**.** Let $\Omega \subset \mathbb{C}$ be an open set, $z_0 \in \Omega$, and $f : \Omega\backslash\{z_0\} \to \mathbb{C}$ be holomorphic. If z_0 is neither a removable singularity nor a pole, we call z_0 an **essential singularity**.

Example 4.4. The function $f(z) = e^{\frac{1}{z}}$ defined on $\mathbb{C}\backslash\{0\}$ has an essential singularity at $z = 0$.

The behavior of a function near an essential singularity is wild:

Theorem 4.7 (Casorati–Weierstrass theorem)**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose $f : B_r(z_0)\backslash\{z_0\}$ is holomorphic with an essential singularity at z_0. Then the image of $B_r(z_0)\backslash\{z_0\}$ under f is dense in $\mathbb{C}$, that is,*

$$\begin{aligned} &\textit{for all} \quad w \in \mathbb{C} \quad \textit{for all} \quad \varepsilon > 0 \\ &\textit{there exists} \quad z \in B_r(z_0)\backslash\{z_0\} \quad \textit{such that} \quad |f(z) - w| < \varepsilon. \end{aligned}$$

Proof. Suppose not. Then there exists $w \in \mathbb{C}$ and $\varepsilon > 0$ such that

$$|f(z) - w| \geq \varepsilon \quad \text{for all} \quad z \in B_r(z_0)\backslash\{z_0\}.$$

We define

$$g : B_r(z_0)\backslash\{z_0\} \qquad \text{by} \qquad g(z) = \frac{1}{f(z) - w}.$$

Note that g is holomorphic on $B_r(z_0)\backslash\{z_0\}$ and bounded by $\frac{1}{\varepsilon}$.

Thus g has a removable singularity at z_0 and hence may be extended to be holomorphic on $B_r(z_0)$.

If $g(z_0) \neq 0$ then the function

$$z \mapsto f(z) - w$$

is holomorphic on $B_r(z_0)$. Thus f is holomorphic at z_0, a contradiction.

If $g(z_0) = 0$ then the function

$$z \mapsto f(z) - w$$

has a pole at z_0. Thus f has a pole at z_0, a contradiction. □

Definition 4.8 (Meromorphic)**.** Let $\Omega \subset \mathbb{C}$ be open and $\{z_n\}$ be a (finite or infinite) sequence of points in Ω with no limit points in Ω. A function $f : \Omega \backslash \{z_1, z_2, \dots\}$ is called **meromorphic on** Ω if

(i) f is holomorphic on $\Omega \backslash \{z_1, z_2, \dots\}$
(ii) f has a pole at each z_n.

Definition 4.9 (Singularities at infinity)**.** Suppose that $f : \mathbb{C} \backslash B_R(0) \to \mathbb{C}$ is holomorphic for some $R > 0$. Define $F : B_{1/R}(0) \backslash \{0\} \to \mathbb{C}$ by $F(z) = f(1/z)$.

We say f has a **pole at infinity** if F has a pole at $z = 0$. Similarly, f can have a **removable singularity at infinity**, an **essential singularity at infinity**.

If f is meromorphic on $\mathbb{C}$ and either has a pole or removable singularity at infinity, we say f is **meromorphic on the extended plane.**

Our next task is to understand the behavior of meromorphic functions near poles.

Theorem 4.10. *Let $\Omega \subset \mathbb{C}$ be open and $z_0 \in \Omega$. Suppose f has a pole at z_0. Then there exists a unique integer $m > 0$ and an open set $U \ni z_0$ such that*

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n \quad \textit{for} \quad z \in U.$$

Remark 4.11. We call m the **multiplicity** (or **order**) of the pole at z_0. If $m = 1$ we call the pole **simple.**

We call the function

$$g(z) = \sum_{n=-m}^{-1} a_n (z - z_0)^n$$

the **principal part** of f at z_0.

The coefficient a_{-1} is called the **residue** of f at z_0, denoted $\text{res}_{z_0} f$, for which we can deduce the following formula:

$$\text{res}_{z_0} f = \lim_{z \to z_0} \tfrac{1}{(m-1)!} (\tfrac{d}{dz})^{m-1} [(z - z_0)^m f(z)].$$

We also introduce the following convention: if f is holomorphic at z_0, we define $\text{res}_{z_0} f = 0$.

Lemma 4.12. *Suppose $\Omega \subset \mathbb{C}$ is open and connected and $z_0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$ be holomorphic and not identically zero.*

If $f(z_0) = 0$ then there exists an open set $U \ni z_0$, a unique integer $m > 0$, and a holomorphic function $g : U \to \mathbb{C}$ such that

- *$f(z) = (z - z_0)^m g(z)$ for $z \in U$,*
- *$g(z) \neq 0$ for $z \in U$.*

Remark 4.13. We call m the **multiplicity** (or **order**) of the zero at z_0.

Proof. We can write f in a power series in some ball around z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

As f is not identically zero, there is some smallest integer $m > 0$ such that $a_m \neq 0$.

Thus

$$f(z) = (z - z_0)^m [a_m + a_{m+1}(z - z_0) + \cdots] =: (z - z_0)^m g(z).$$

Note that g is analytic, and hence holomorphic. Moreover $g(z_0) = a_m \neq 0$, so that g is non-zero in some open set around z_0.

For the uniqueness of m, suppose we may write

$$f(z) = (z - z_0)^m g(z) = (z - z_0)^n h(z)$$

with $h(z_0) \neq 0$ and $n \neq m$. Without loss of generality, suppose $n > m$. Then we find

$$g(z) = (z - z_0)^{n-m} h(z) \to 0 \quad \text{as} \quad z \to z_0,$$

a contradiction. $\square$

Lemma 4.14. *Suppose f has a pole at $z_0 \in \mathbb{C}$. Then there exists an open set $U \ni z_0$, a unique integer $m > 0$, and a holomorphic function $h : U \to \mathbb{C}$ such that*

- *$f(z) = (z - z_0)^{-m} h(z)$ for $z \in U$,*
- *$h(z) \neq 0$ for $z \in U$.*

Proof. We apply the lemma above to the function $\frac{1}{f}$. $\square$

Proof of Theorem 4.10. We apply Lemma 4.14 and write

$$f(z) = (z - z_0)^{-m} h(z)$$

for z in an open set $U \ni z_0$. The series expansion for f now follows from the power series expansion for the holomorphic function h. $\square$

We can now classify the possible meromorphic functions on the extended complex plane.

Theorem 4.15. *If f is meromorphic on the extended complex plane, then f is a rational function. (That is, f is the quotient of polynomials.)*

Proof. We define $F : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $F(z) = f(1/z)$.

By assumption, F has a pole or removable singularity at 0; thus it is holomorphic in $B_r(0)\backslash\{0\}$ for some $r > 0$.

This implies that f has at most one pole in $\mathbb{C}\backslash\overline{B_{1/r}(0)}$ (namely, the possible pole at infinity).

We next note f can have at most finitely many poles in $\overline{B_{1/r}(0)}$, say $\{z_k\}_{k=1}^n$.

For each $k \in \{1, \ldots, n\}$ we may write

$$f(z) = g_k(z) + h_k(z),$$

where g_k is the principal part of f at z_k and h_k is holomorphic in an open set $U_k \ni z_k$. Note that g_k is a polynomial in $1/(z - z_k)$.

Furthermore (if there is a pole at infinity) we can write

$$F(z) = \tilde{g}_\infty(z) + \tilde{h}_\infty(z)$$

where $\tilde{g}_\infty$ is the principal part of F at 0 and $\tilde{h}_\infty$ is holomorphic in an open set containing 0. Note that $\tilde{g}_\infty$ is a polynomial in $1/z$.

We define $g_\infty(z) = \tilde{g}_\infty(1/z)$ and $h_\infty(z) = \tilde{h}_\infty(1/z)$.

Now consider the function

$$H(z) = f(z) - g_\infty(z) - \sum_{k=1}^n g_k(z).$$

Notice that H has removable singularities at each z_k, so that we may extend H to be holomorphic on all of $\mathbb{C}$.

Moreover, $z \mapsto H(1/z)$ is bounded near $z = 0$, which implies H is bounded near infinity.

In particular, we have H is bounded on $\mathbb{C}$ so that (by Liouville's theorem) H must be constant, say $H(z) \equiv C$.

Rearranging we have

$$f(z) = C + g_\infty(z) + \sum_{k=1}^{n} g_k(z),$$

which implies that f is a rational function, as needed. □

4.2. **The Residue Theorem and Evaluation of Some Integrals.**

Theorem 4.16 (Residue theorem). *Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be meromorphic on Ω. Let $\gamma \subset \Omega$ be a simple closed curve such that f has no poles on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{w \in interior\ \gamma} res_w f.$$

Remark 4.17. Note that if f is holomorphic on Ω, this formula reproduces Cauchy's theorem.

Proof. We define $S = \text{interior}(\gamma)$.

To begin, we notice that there can only be finitely many poles in S, say $\{z_j\}_{j=0}^n$. (*Why?*)

We treat the case of one pole z_0; it should be clear how to generalize the proof to more poles.

As f is holomorphic in $S\backslash\{z_0\}$, a familiar argument using Cauchy's theorem shows

$$\int_\gamma f(z)\,dz = \int_{\partial B_\varepsilon(z_0)} f(z)\,dz \qquad \text{for all small } \varepsilon > 0.$$

(cf. the proof of Lemma 3.33).

From Theorem 4.10 we can write

$$f(z) = \underbrace{\sum_{n=-m}^{-1} a_n (z-z_0)^n}_{:=g(z)} + h(z),$$

where h is holomorphic.

As the Cauchy integral formulas imply

$$\frac{(k-1)!}{2\pi i}\int_{\partial B_\varepsilon(z_0)} \frac{dz}{(z-z_0)^k} \;=\; (k-1)^{st} \text{ derivative of 1 at } z_0 \;=\; \begin{cases} 1 & k=1 \\ 0 & k>1, \end{cases}$$

we deduce

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z_0)} g(z)\,dz = a_{-1} = \text{res}_{z_0} f.$$

On the other hand, Cauchy's theorem says

$$\int_{\partial B_\varepsilon(z_0)} h(z)\,dz = 0.$$

We conclude

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \mathrm{res}_{z_0} f,$$

as was needed to show. □

The main use of the residue theorem is the computation of integrals.

Example 4.5 (Shifting the contour)**.** Consider the integral

$$F(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} e^{-\pi x^2}\,dx \quad \text{for} \quad \xi \geq 0.$$

(This integral evaluates the Fourier transform of the function $x \mapsto e^{-\pi x^2}$ at the point ξ.)

We first note that $F(0) = 1$. (*Check!*)

For $\xi > 0$ we complete the square in the integrand to write

$$F(\xi) = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\,dx.$$

"Formally" we would like to make a substitution $y = x + i\xi$, $dy = dx$, to see that

$$F(\xi) = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi y^2}\,dy = e^{-\pi\xi^2}F(0) = e^{-\pi\xi^2}.$$

To make this argument precise we introduce the function $f(z) = e^{-\pi z^2}$, which we note is entire.

For $R > 0$ we let γ_R be the boundary of the rectangle with vertices $\pm R, \pm R + i\xi$, oriented counter clockwise.

By the residue theorem (in this case Cauchy's theorem) we have

$$\int_{\gamma_R} f(z)\,dz = 0 \quad \text{for all} \quad R > 0.$$

We write γ_R as the union of four curves $\gamma_1, \ldots, \gamma_4$, which we parametrize as follows

- $z_1(x) = x, \qquad x \in [-R, R]$
- $z_2(x) = R + ix, \qquad x \in [0, \xi]$
- $z_3(x) = i\xi - x, \qquad x \in [-R, R]$
- $z_4(x) = -R + i\xi - ix, \quad x \in [0, \xi]$.

Thus

$$-\int_{\gamma_3} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_4} f(z)\,dz. \quad (*)$$

Now,

$$\begin{aligned} -\int_{\gamma_3} f(z)\,dz = \int_{-R}^{R} e^{-\pi(i\xi - x)^2}\,dx &= e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2} e^{2\pi i x\xi}\,dx \\ &= e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx \quad (\text{"}u\text{ sub"}) \\ &\to e^{\pi\xi^2}F(\xi) \quad \text{as} \quad R \to \infty. \end{aligned}$$

Similarly

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} e^{-\pi x^2}\,dx \to 1 \quad \text{as} \quad R \to \infty.$$

We now claim that

$$\lim_{R\to\infty}\left(\int_{\gamma_2} f(z)\,dz + \int_{\gamma_4} f(z)\,dz\right) = 0,$$

so that sending $R\to\infty$ in $(*)$ gives

$$e^{\pi\xi^2}F(\xi) = 1, \quad \text{i.e.} \quad F(\xi) = e^{-\pi\xi^2},$$

as we hoped to show.

We deal with γ_2 and leave γ_4 (which is similar) as an exercise. We compute

$$\int_{\gamma_2} f(z)\,dz = i\int_0^{\xi} e^{-\pi(R+ix)^2}\,dx = ie^{-\pi R^2}\int_0^{\xi} e^{\pi x^2}e^{-2\pi iRx}\,dx,$$

so that

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \xi e^{\pi\xi^2}e^{-\pi R^2} \to 0 \quad \text{as} \quad R\to\infty.$$

Example 4.6 (Calculus of residues)**.** We can use the residue theorem to evaluate the integral

$$I = \int_0^{\infty} \frac{dx}{1+x^4}.$$

We define the function

$$f(z) = \frac{1}{1+z^4}.$$

We note that f is meromorphic on $\mathbb{C}$, with poles at the points z such that $z^4 = -1$.

Question. For which $z\in\mathbb{C}$ do we have $z^4=-1$?

As z^4+1 is a polynomial of degree four, the fundamental theorem of algebra tells us we must have four roots (counting multiplicity).

For any such root we must have $|z|^4 = 1$, so that $|z| = 1$ and we may write $z = e^{i\theta}$.

Writing $1 = e^{i\pi}$, we have reduced the question to finding $\theta\in[0,2\pi)$ such that $e^{4i\theta} = e^{i\pi}$. That is,

$$e^{i(4\theta-\pi)} = 1, \quad \text{i.e.} \quad 4\theta - \pi = 2k\pi \quad \text{for some integer } k.$$

We find

$$\theta = \frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}.$$

Thus f has simple poles at

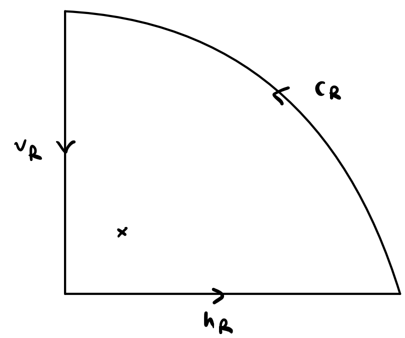
$$z_1 = e^{i\pi/4}, \ \ldots, \ z_4 = e^{7i\pi/4}$$

and we can write

$$f(z) = \prod_{j=1}^{4} \frac{1}{z-z_j}.$$

Now consider the curve γ_R that consists of the three following pieces:

- $h_R = \{x : x\in[0,R]\}$, oriented 'to the right'
- $c_R = \{Re^{i\theta} : 0\leq\theta\leq\pi/2\}$, oriented counter-clockwise,
- $v_R = \{ix : x\in[0,R]\}$, oriented 'downward'.



By the residue theorem we have that

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty} 2\pi i \sum_{w\in\text{interior}(\gamma_R)} \operatorname{res}_w f = 2\pi i\ \operatorname{res}_{z_1} f. \quad (*)$$

Now, we notice that for large R we have

$$\left|\int_{c_R} f(z)\,dz\right| \le \frac{2\cdot\frac{1}{2}\pi R}{R^4} \to 0 \quad \text{as} \quad R\to\infty.$$

On the other hand, we note

$$\lim_{R\to\infty}\int_{h_R} f(z)\,dz = \int_0^\infty \frac{dx}{1+x^4} = I.$$

We can also compute

$$\int_{v_R} f(z)\,dz = -\int_0^R \frac{i\,dx}{1+(ix)^4} = -i\int_0^R \frac{dx}{1+x^4} \to -i\,I \quad \text{as} \quad R\to\infty.$$

Thus sending $R\to\infty$, $(*)$ becomes

$$(1-i)I = 2\pi i\operatorname{res}_{z_1} f, \quad \text{i.e.} \quad I = \frac{2\pi i}{1-i}\ \operatorname{res}_{z_1} f.$$

It remains to compute the residue:

$$\begin{aligned}
\operatorname{res}_{z_1} f = \lim_{z\to z_1}[(z-z_1)f(z)] &= \prod_{j=2}^4 \frac{1}{z_1-z_j} = \frac{1}{z_1^3}\prod_{j=2}^4 \frac{1}{1-\frac{z_j}{z_1}} \\
&= \frac{1}{e^{i\frac{3\pi}{4}}(1-e^{i\frac{\pi}{2}})(1-e^{i\pi})(1-e^{i\frac{3\pi}{2}})} \\
&= \frac{1}{\frac{\sqrt{2}}{2}(-1+i)(1-i)(2)(1+i)} \\
&= \frac{1}{2\sqrt{2}(-1+i)}.
\end{aligned}$$

Thus

$$I = \frac{2\pi i}{1-i}\cdot\frac{1}{2\sqrt{2}(-1+i)} = \frac{2\pi i}{2\sqrt{2}(2i)} = \frac{\pi}{2\sqrt{2}}.$$

In the exercises you will compute

$$\int_0^\infty \frac{dx}{1+x^n}$$

for all integers $n \geq 2$.

4.3. **The Argument Principle and Applications.**

Theorem 4.18 (Argument principle for holomorphic functions)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic, with $f \not\equiv 0$. Let $\gamma \subset \Omega$ be a simple closed curve such that f has no zeros on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \#\{\textit{zeros of } f \textit{ in interior}(\gamma),\ \textit{counting multiplicity}\}.$$

Proof. Let $S = \text{interior}(\gamma)$.

Let $\{z_k\}_{k=1}^n$ denote the (finitely many) zeros of f in S.

As the function $z \mapsto \frac{f'(z)}{f(z)}$ is holomorphic on $S\backslash\{z_k\}_{k=1}^n$, a familiar argument using Cauchy's theorem shows that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^n \frac{1}{2\pi i}\int_{\partial B_k} \frac{f'(z)}{f(z)}\,dz,$$

where $B_k \subset S$ is any sufficiently small ball containing z_k.

Thus it suffices to show that if z_k is a zero of order m_k we have

$$\frac{1}{2\pi i}\int_{\partial B_k} \frac{f'(z)}{f(z)}\,dz = m_k.$$

To this end we use Lemma 4.12 to write

$$f(z) = (z-z_k)^{m_k} g_k(z) \quad \text{for} \quad z \in B_k,$$

where g_k is holomorphic and $g_k(z) \neq 0$ for $z \in B_k$.

Thus

$$f'(z) = m_k(z-z_k)^{m_k-1}g_k(z) + (z-z_k)^m g_k'(z) \qquad (z \in B_k)$$

so that

$$\frac{f'(z)}{f(z)} = \frac{m_k}{z-z_k} + \underbrace{\frac{g_k'(z)}{g_k(z)}}_{\text{holomorphic}} \qquad (z \in B_k).$$

Thus by Cauchy's theorem:

$$\frac{1}{2\pi i}\int_{\partial B_k} \frac{f'(z)}{f(z)}\,dz = \frac{m_k}{2\pi i}\int_{\partial B_k} \frac{dz}{z-z_k} + 0 = m_k\, W_{\partial B_k}(z_k) = m_k,$$

as was needed to show. $\square$

Remark 4.19. Let Ω, f, and γ be as above. Let γ be parametrized by $z(t)$ for $t \in [a,b]$. Consider the curve $f \circ \gamma$, parametrized by $f(z(t))$ for $t \in [a,b]$. Then

$$\begin{aligned} W_{f\circ\gamma}(0) &= \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dz}{z} = \frac{1}{2\pi i}\int_a^b \frac{[f\circ z]'(t)}{f(z(t))}\,dt = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))z'(t)}{f(z(t))}\,dt \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz. \end{aligned}$$

Thus

$$W_{f\circ\gamma}(0) = \#\{\text{zeros of } f \text{ in interior}(\gamma)\}.$$

In particular if f has n zeros inside γ then the argument of $f(z)$ increases by $2\pi n$ as z travels around γ.

(This explains the terminology "argument principle".)

There is also an argument principle for meromorphic functions. It works similarly, but poles count as zeros of negative order.

Theorem 4.20 (Argument principle for meromorphic functions)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be meromorphic. Let $\gamma \subset \Omega$ be a simple closed curve such that f has no zeros or poles on γ. Then*

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz &= \#\{\textit{zeros of } f \textit{ in interior}(\gamma),\ \textit{counting multiplicity}\}\\ &\quad - \#\{\textit{poles of } f \textit{ in interior}(\gamma),\ \textit{counting multiplicity}\}\end{aligned}$$

Proof. Arguing as in the proof of Theorem 4.18, we find that it suffices to show the following:

If z_0 is a pole of f of order m, then

$$\frac{1}{2\pi i}\int_{\partial B} \frac{f'(z)}{f(z)}\,dz = -m$$

where B is any sufficiently small ball containing z_0.

To this end we use Lemma 4.14 to write

$$f(z) = (z - z_0)^{-m}h(z) \qquad (z \in B),$$

where h is holomorphic and $h(z) \neq 0$ for $z \in B$.

Thus

$$f'(z) = -m(z - z_0)^{-m-1}h(z) + (z - z_0)^{-m}h'(z)$$

so that

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_0} + \underbrace{\frac{h'(z)}{h(z)}}_{\text{holomorphic}} \qquad (z \in B).$$

Thus by Cauchy's theorem:

$$\frac{1}{2\pi i}\int_{\partial B} \frac{f'(z)}{f(z)}\,dz = -m\frac{1}{2\pi i}\int_{\partial B}\frac{dz}{z - z_0} + 0 = -m\,W_{\partial B}(z_0) = -m,$$

as needed. □

Corollary 4.21 (Rouché's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and $\gamma \subset \Omega$ be a simple closed curve. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic. If*

$$|f(z)| > |g(z)| \quad \textit{for all} \quad z \in \gamma,$$

then f and $f + g$ have the same number of zeros in the interior of γ.

Remark 4.22. One can interpret Rouché's theorem as follows (we learned this interpretation from R. Killip): if you walk your dog around a flagpole such that the leash length is always less than your distance to the flagpole, then your dog circles the flagpole as many times as you do. ($f \rightsquigarrow$ you, $g \rightsquigarrow$ leash, $f + g \rightsquigarrow$ dog, $0 \rightsquigarrow$ flagpole.)

This theorem remains true if we replace $(*)$ with the weaker hypothesis

$$|g(z)| < |f(z)| + |f(z) + g(z)| \quad \text{for all} \quad z \in \gamma, \quad (*)$$

which means that the flagpole never obscures your view of the dog (see Exercise 4.3).

Proof of Rouché's theorem. For $t \in [0,1]$ consider the holomorphic function $z \mapsto f(z) + tg(z)$.

We first note that

$$|f(z)| > |g(z)| \implies |f(t) + tg(z)| > 0 \quad \text{for} \quad z \in \gamma, \quad t \in [0,1].$$

It follows that the function

$$n(t) := \frac{1}{2\pi i} \int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\, dz$$

is continuous for $t \in [0,1]$.

We now notice that by the argument principle,

$$n(t) = \#\{\text{zeros of } f + tg \text{ inside } \gamma\}.$$

In particular, n is integer-valued. As continuous integer-valued functions are constant, we conclude $n(0) = n(1)$, which gives the result. $\square$

Remark 4.23. Rouché's theorem allows for a very simple proof of the fundamental theorem of algebra. See Exercise 4.4.

With Rouché's theorem in place, we can prove an important topological property of holomorphic functions.

Theorem 4.24 (Open mapping theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, and let $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant. Then*

$$f(\Omega) := \{f(z) : z \in \Omega\} = \{w \in \mathbb{C} \mid \exists\, z \in \Omega : f(z) = w\}$$

is open.

Proof. Let $w_0 \in f(\Omega)$. We need to show that

$$\text{there exists } \varepsilon > 0 \quad \text{such that} \quad B_\varepsilon(w_0) \subset f(\Omega). \quad (*)$$

To this end, we first choose $z_0 \in \Omega$ such that $f(z_0) = w_0$.

As Ω is open and f is non-constant, we may find $\delta > 0$ such that

- $B_\delta(z_0) \subset \Omega$,
- $f(z) \neq w_0$ for $z \in \partial B_\delta(z_0)$.

In particular, as $\partial B_\delta(z_0)$ is compact and f is continuous, we find

$$\text{there exists} \quad \varepsilon > 0 \quad \text{such that} \quad |f(z) - w_0| > \varepsilon \quad \text{for } z \in \partial B_\delta(z_0).$$

We will now show that $B_\varepsilon(w_0) \subset f(\Omega)$, so that $(*)$ holds.

Fix $w \in B_\varepsilon(w_0)$ and write

$$f(z) - w = \underbrace{f(z) - w_0}_{:=F(z)} + \underbrace{w_0 - w}_{:=G(z)}.$$

Note that for $z \in \partial B_\delta(z_0)$ we have

$$|F(z)| > \varepsilon = |G(z)|,$$

so that Rouché's theorem implies that F and $F + G$ have the same number of zeros in $B_\delta(z_0)$.

As $F(z_0) = 0$, we conclude that $F + G$ has at least one zero in $B_\delta(z_0)$. That is,

$$\text{there exists} \quad z \in B_\delta(z_0) \quad \text{such that} \quad f(z) = w.$$

That is, $w \in f(\Omega)$. We conclude $B_\varepsilon(w_0) \subset f(\Omega)$, as was needed to show. □

We turn to one final property of holomorphic functions.

Theorem 4.25 (Maximum principle)**.** *Let $\Omega \subset \mathbb{C}$ be open, bounded, and connected and $f : \Omega \to \mathbb{C}$ holomorphic. If there exists $z_0 \in \Omega$ such that*

$$|f(z_0)| = \max_{z \in \overline{\Omega}} |f(z)|, \quad (*)$$

then f is constant.

In particular, if f is non-constant then $|f|$ attains its maximum on $\partial\Omega$.

Proof #1. Suppose $(*)$ holds for some $z_0 \in \Omega$.

Suppose toward a contradiction that f is not constant.

Then $f(\Omega)$ is open, and hence there exists $\varepsilon > 0$ such that $B_\varepsilon(f(z_0)) \subset f(\Omega)$.

However, this implies that

$$\exists z \in \Omega : |f(z)| > |f(z_0)|,$$

contradicting $(*)$. □

Remark 4.26. The hypothesis that Ω is bounded is essential. Indeed, consider $f(z) = e^{-iz^2}$ on

$$\Omega = \{z : \mathrm{Re}\,(z) > 0,\ \mathrm{Im}\,(z) > 0\}.$$

Then $|f(z)| = 1$ for $z \in \partial\Omega$ but $f(z)$ is unbounded in Ω.

4.4. **The Complex Logarithm.** The function $f(z) = \frac{1}{z}$ is holomorphic in $\mathbb{C}\backslash\{0\}$.

By analogy to the real-valued case, we may expect that f has a primitive in $\mathbb{C}\backslash\{0\}$, namely "log$(z)$."

However, f does not have a primitive in $\mathbb{C}\backslash\{0\}$, since

$$\int_\gamma \frac{dz}{z} = 2\pi i W_\gamma(0),$$

which is nonzero for any closed curve γ enclosing 0.

We next show that we can indeed define a primitive for f, but only in certain subsets of $\mathbb{C}$.

Theorem 4.27 (Existence of logarithm)**.** *Let $\Omega \subset \mathbb{C}$ be simply connected with $1 \in \Omega$ but $0 \notin \Omega$. Then there exists $F : \Omega \to \mathbb{C}$ such that*

(i) *F is holomorphic in Ω,*
(ii) *$e^{F(z)} = z$ for $z \in \Omega$,*
(iii) *$F(r) = \log r$ when $r \in \mathbb{R}$ is sufficiently close to 1.*

We write $F(z) = \log_\Omega z$.

Remark 4.28. By (ii) and the chain rule, we can deduce $F'(z) = \frac{1}{z}$. This will also be clear from the proof of Theorem 4.27.

Proof of Theorem 4.27. For $z \in \Omega$ we let γ be a curve in Ω joining 1 to z and define

$$F(z) = \int_\gamma \frac{dw}{w}.$$

As $0 \notin \Omega$, the function $w \mapsto \frac{1}{w}$ is holomorphic on Ω.

As Ω is simply connected, we note that F is independent of γ.

Arguing as we did long ago (to prove existence of primitives; see Theorems 3.30 and 3.32), we find that F is holomorphic with $F'(z) = \frac{1}{z}$. This proves (i).

For (ii) we compute

$$\tfrac{d}{dz}(ze^{-F(z)}) = e^{-F(z)} - zF'(z)e^{-F(z)} = e^{-F(z)} - z\tfrac{1}{z}e^{-F(z)} = 0.$$

As Ω is connected we deduce that $ze^{-F(z)}$ is constant.

As $e^{-F(1)} = e^0 = 1$, we conclude $ze^{-F(z)} \equiv 1$, which gives (ii).

Finally we note that if $r \in \mathbb{R}$ is sufficiently close to 1 then

$$F(r) = \int_1^r \frac{dx}{x} = \log r,$$

as needed. □

Definition 4.29. If $\Omega = \mathbb{C}\backslash(-\infty, 0]$, we call $\log_\Omega$ the **principal branch** of the logarithm and write $\log_\Omega z = \log z$.

Remark 4.30.

(i) If $z = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi$, so that $z \in \mathbb{C}\backslash(-\infty, 0]$, then we have

$$\log z = \log r + i\theta.$$

Indeed, we can let $\gamma = \gamma_1 \cup \gamma_2$, where $\gamma_1 = [1, r] \subset \mathbb{R}$ and $\gamma_2 = \{re^{it} : t \in [0, \theta]\}$, and compute

$$\log z = \int_1^r \frac{dx}{x} + \int_0^\theta \frac{ire^{it}}{re^{it}}\,dt = \log r + i\theta.$$

(ii) *Beware:* in general, $\log z_1 z_2 \neq \log z_1 + \log z_2$.

Indeed, if $z_1 = z_2 = e^{\frac{2\pi i}{3}}$ then $\log z_1 = \log z_2 = \frac{2\pi i}{3}$, while $\log z_1 z_2 = -\frac{2\pi i}{3}$. (*Check!*)

(iii) One can compute the following series expansion:

$$\log(1 + z) = -\sum_{n=1}^\infty \frac{(-1)^n}{n} z^n \quad \text{for} \quad |z| < 1. \quad (\textit{Check!})$$

(iv) Let $\Omega \subset \mathbb{C}$ be simply connected with $1 \in \Omega$ but $0 \notin \Omega$, and let $\alpha \in \mathbb{C}$. For $z \in \Omega$ we can now define

$$z^\alpha := e^{\alpha \log_\Omega z}.$$

One can *check* that $1^\alpha = 1$, z^n agrees with the "usual" definition, and $(z^{\frac{1}{n}})^n = z$.

We close this section with the following generalization of Theorem 4.27.

Theorem 4.31. *Let $\Omega \subset \mathbb{C}$ be simply connected. Let $f : \Omega \to \mathbb{C}$ be holomorphic and satisfy $f(z) \neq 0$ for any $z \in \Omega$. Then there exists a holomorphic $g : \Omega \to \mathbb{C}$ such that*

$$f(z) = e^{g(z)}.$$

We write $g(z) = \log_\Omega f(z)$.

Proof. Let $z_0 \in \Omega$ and choose $c_0 \in \mathbb{C}$ such that $e^{c_0} = f(z_0)$.

For $z \in \Omega$, we let γ be any curve in Ω joining z_0 to z and define

$$g(z) = c_0 + \int_\gamma \frac{f'(w)}{f(w)}\,dw.$$

As f is holomorphic and non-zero, the function $w \mapsto \frac{f'(w)}{f(w)}$ is holomorphic on Ω.
As Ω is simply connected, we note that g is independent of γ.
We also find that g is holomorphic on Ω, with $g'(z) = \frac{f'(z)}{f(z)}$.
On the other hand, we can compute

$$\frac{d}{dz}\big[f(z)e^{-g(z)}\big] = e^{-g(z)}\big[f'(z) - f(z)g'(z)\big] = e^{-g(z)}\big[f'(z) - f(z)\tfrac{f'(z)}{f(z)}\big] = 0.$$

As Ω is connected we deduce $f(z)e^{-g(z)}$ is constant.
As $e^{g(z_0)} = e^{c_0} = f(z_0)$, we conclude that $f(z) \equiv e^{g(z)}$, as was needed to show. □

4.5. **Exercises.**

Exercise 4.1. Let $z_0 \in \mathbb{C}$ and $R > 0$. Suppose that $f : B_R(z_0)\backslash\{z_0\}$ is holomorphic and that there exist $C > 0$ and $0 < \varepsilon < 1$ such that

$$|f(z)| \leq C|z - z_0|^{-1+\varepsilon} \quad \text{for all} \quad z \in B_R(z_0)\backslash\{z_0\}.$$

Show that the singularity of f at z_0 is removable, that is, there exists a unique holomorphic function $F : B_R(z_0) \to \mathbb{C}$ such that $F(z) = f(z)$ for $z \in B_R(z_0)\backslash\{z_0\}$.

Hint. Follow the proof of Riemann's removable singularity theorem.

Exercise 4.2. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(i) Show that if $a_n \neq 0$ for infinitely many n then f has an essential singularity at infinity.

(ii) Show that if f is injective then $f(z) = a_0 + a_1 z$ with $a_1 \neq 0$.

Hint. For part (ii): if f is a polynomial, you are done (why?). Otherwise by part (i) f has an essential singularity at infinity. In this case, one can use Casorati–Weierstrass and the open mapping theorem to contradict the injectivity of f.

Exercise 4.3. Prove the following stronger version of Rouché's theorem.

Let $\Omega \subset \mathbb{C}$ be open and $\gamma \subset \Omega$ be a simple closed curve. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic functions such that f has no zeros on γ and

$$|g(z)| < |f(z)| + |f(z) + g(z)| \quad \text{for all} \quad z \in \gamma.$$

Then f and $f+g$ have the same number of zeros in the interior of γ. *Hint.* As in the proof of Rouchè's theorem from class, the key step is to prove that $|f(z)+tg(z)| > 0$ for $z \in \gamma$ and $t \in [0, 1]$. To do this, use the triangle inequality two deduce two lower bounds for $|f(z) + tg(z)|$ whose average is $\frac{1}{2}\big(|f(z)| + |f(z) + g(z)| - |g(z)|\big)$.

Exercise 4.4. Use Rouchè's theorem to prove that any degree n polynomial has n zeros.

Exercise 4.5. This exercise demonstrates how to use the Cauchy integral formula and the 'tensor power trick' to prove the maximum principle. Suppose f, Ω are as in Theorem 4.25 and that $|f| \leq M$ on the boundary of Ω.

- Use the Cauchy integral formula to deduce that
$$|f(z)| \leq \tfrac{1}{2\pi}\tfrac{|\partial\Omega|}{\mathrm{dist}(z,\partial\Omega)}M$$
for any $z \in \Omega$.
- Use the same argument applied to f^n to deduce
$$|f(z)|^n \leq \tfrac{1}{2\pi}\tfrac{|\partial\Omega|}{\mathrm{dist}(z,\partial\Omega)}M^n$$
for any $z \in \Omega$.

- Take n^{th} roots and send $n \to \infty$ to deduce $|f(z)| \leq M$ for $z \in \Omega$.

Exercise 4.6. Compute $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$.

Exercise 4.7. Compute $\int_0^\infty \frac{dx}{1+x^n}$ for all integers $n \geq 2$.

Exercise 4.8. Compute $\int_0^{2\pi} \frac{d\theta}{2+\cos^2\theta}$.

Exercise 4.9. Compute $\int_{-\infty}^\infty \frac{x^2}{(1+x^2)(4+x^2)}\,dx$.

Exercise 4.10. Compute $\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}\,dx$ for $a > 0$.

Exercise 4.11. Compute $\int_{-\infty}^\infty \frac{x^3 \sin x}{x^4+16}\,dx$.

Exercise 4.12. Compute $\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}}$ for all integers $n \geq 0$.

Exercise 4.13. Compute $\int_0^\infty \frac{\sin x}{x}\,dx$.

Exercise 4.14. Compute $\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ for $a, b > 0$.

Exercise 4.15. Compute $\int_0^\infty \frac{1-\cos x}{x^2}\,dx$.

Exercise 4.16. Compute $\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx$ for $0 < a < 1$.

Exercise 4.17. Compute $\int_{-\infty}^\infty \frac{x \sin x}{x^2+a^2}\,dx$ for $a > 0$.

Exercise 4.18. Compute $\int_{-\infty}^\infty e^{-2\pi i x\xi} \frac{1}{\cosh(\pi\xi)}\,dx$ for $\xi \in \mathbb{R}$.

Exercise 4.19. Compute $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ for $a, b \in \mathbb{R}$ with $|b| < a$.

5. Entire Functions

We turn to the study of entire functions, in particular the following question: given a sequence $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$, is there an entire function whose zeros are precisely a_k?

By the uniqueness theorem, a **necessary** condition is that $\lim_{k\to\infty} |a_k| \to \infty$. But is this condition also sufficient?

Convention. Throughout this section we always exclude the case $f \equiv 0$.

5.1. **Infinite Products.**

We first turn to the question of infinite products of complex numbers and functions.

Definition 5.1. Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$. We say **the infinite product** $\prod_{n=1}^{\infty}(1+a_n)$ **converges** if the limit $\lim_{N\to\infty} \prod_{n=1}^{N}(1+a_n)$ exists.

The following result gives a useful criterion for convergence.

Theorem 5.2. *Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$. If the series $\sum_n a_n$ converges absolutely, then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges. Moreover the product converges to zero if and only if one of its factors is zero.*

Proof. Recall that

$$\log(1+z) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n \quad \text{for} \quad |z| < 1,$$

with $1+z = e^{\log(1+z)}$. In particular $|\log(1+z)| \leq C|z|$ for $|z| \leq \frac{1}{2}$.

Note that loss of generality, we may assume $|a_n| < \frac{1}{2}$ for all n. (*Why?*)

Thus we can write

$$\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{\sum_{n=1}^{N}\log(1+a_n)}.$$

We now estimate

$$\sum_{n=1}^{N} |\log(1+a_n)| \leq C \sum_{n=1}^{N} |a_n|$$

to see that the series $\sum_n \log(1+a_n)$ converges absolutely.

In particular

$$\text{there exists} \quad \ell \in \mathbb{C} \quad \text{such that} \quad \lim_{N\to\infty} \sum_{n=1}^{N} \log(1+a_n) = \ell.$$

By continuity, we have that $e^{\sum_{n=1}^{N}\log(1+a_n)} \to e^{\ell}$, which shows that $\prod_n(1+a_n)$ converges (to e^{ℓ}).

To conclude the proof we note that if $1+a_n = 0$ for some n then the product is zero, while if $1+a_n \neq 0$ for any n then the product is non-zero since it is of the form e^{ℓ}. □

For products of functions, we have the following.

Theorem 5.3. *Let $\Omega \subset \mathbb{C}$ be open and suppose $F_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions. If there exist $c_n > 0$ such that*

- $|F_n(z) - 1| \leq c_n$ *for all n and all $z \in \Omega$,*
- $\sum_n c_n < \infty$,

then:

(i) *the products* $\prod_{n=1}^{\infty} F_n(z)$ *converge uniformly on* Ω *to a holomorphic function* $F(z)$*, and*
(ii) *if each* F_n *is nonzero on* Ω*, then so is* F*.*

Proof. For $z \in \Omega$ we may write

$$F_n(z) = 1 + a_n(z), \quad \text{with} \quad |a_n(z)| \leq c_n,$$

and argue as in Theorem 5.2 to see that $\prod_n F_n(z)$ converges. Moreover the convergence is uniform in z, since the bounds on $a_n(z)$ are.

Denoting the limit function by $F(z)$, we note that since F is the uniform limit of holomorphic functions, it is holomorphic.

Note that (ii) follows from the second statement in Theorem 5.2. □

5.2. **Weierstrass Infinite Products.** We return to our original question.

Theorem 5.4 (Weierstrass's theorem)**.** *Suppose* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ *satisfies* $\lim_{n\to\infty} |a_n| = \infty$*. Then there exists an entire function* $f : \mathbb{C} \to \mathbb{C}$ *such that* $\{a_n\}$ *are precisely the zeros of* f*.*

Furthermore any other entire function with precisely these zeros is of the form fe^g *for some entire function* g*.*

As a first attempt, one could try

$$f(z) = \prod_{n=1}^{\infty} (1 - \tfrac{z}{a_n}).$$

However, depending on the sequence $\{a_n\}$ this product may not converge.

The solution to this problem (due to Weierstrass in 1894) is to insert factors that guarantee convergence of the product without affecting the zeros.

Definition 5.5. For an integer $k \geq 0$ we define the **canonical factor**s $E_k : \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} &E_0(z) = (1 - z), \\ &E_k(z) = (1 - z)e^{z + z^2/2 + \cdots + z^k/k} \quad (k \geq 1). \end{aligned}$$

We call k the **degree** of E_k.

Note that $E_k(1) = 0$ for all $k \geq 0$. In fact we will prove a rate of convergence to zero as $z \to 1$.

Lemma 5.6 (Bounds for E_k)**.** *For all* k *we have:*
(i) $|z| \leq \frac{1}{2} \implies |1 - E_k(z)| \leq 2e|z|^{k+1}$

Proof. For $|z| \leq \frac{1}{2}$ we can write $\log(1 - z)$ in a power series

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n},$$

with $1 - z = e^{\log(1-z)}$. Thus

$$E_k(z) = e^{\log(1-z) + z + z^2/2 + \cdots + z^k/k} = e^{-\sum_{j=k+1}^{\infty} z^j/j}.$$

We now notice that since $|z| \le \frac{1}{2}$, we have

$$\Big|\sum_{j=k+1}^{\infty} \frac{z^j}{j}\Big| \le |z|^{k+1} \sum_{j=k+1}^{\infty} |z|^{j-k-1} \le |z|^{k+1} \sum_{j=0}^{\infty} (\tfrac{1}{2})^j \le 2|z|^{k+1} \le 1.$$

Thus using the estimate

$$|1 - e^w| \le e|w| \quad \text{for} \quad |w| \le 1, \qquad (\textit{Check!})$$

we find

$$|1 - E_k(z)| = |1 - e^{-\sum_{j=k+1}^{\infty} z^j/j}| \le e\Big|\sum_{j=k+1}^{\infty} \frac{z^j}{j}\Big| \le 2e|z|^{k+1},$$

which gives (i).

□

Proof of Theorem 5.4. We first let

$$m = \#\{n : a_n = 0\} < \infty$$

and then redefine the sequence so that $0 \notin \{a_n\}_{n=1}^{\infty}$.

We define the holomorphic functions $f_N : \mathbb{C} \to \mathbb{C}$ by

$$f_N(z) = z^m \prod_{n=1}^{N} E_n(\tfrac{z}{a_n}).$$

We let $R > 0$. We will use Theorem 5.3 to show that f_N converges (uniformly) in $B_R(0)$.

We define the sets

$$S_1 = \{n : |a_n| \le 2R\}, \quad S_2 = \{n : |a_n| > 2R\}.$$

As $|a_n| \to \infty$, we have $\#S_1 < \infty$. Thus we may write

$$f_N(z) = z^m g_N(z) h_N(z),$$

where g_N, h_N are the holomorphic functions given by

$$g_N(z) = \prod_{n\in S_1,\ n\le N} E_n(\tfrac{z}{a_n}) \quad \text{and} \quad h_N(z) = \prod_{n\in S_2,\ n\le N} E_n(\tfrac{z}{a_n}).$$

Note that $\#S_1 < \infty$ implies that

$$\text{for all} \quad N \ge N_0 := \#S_1, \quad g_N = g_{N_0}.$$

Now for $n \in S_2$ and $z \in B_R(0)$ we have

$$|a_n| > 2R > 2|z| \implies |\tfrac{z}{a_n}| \le \tfrac{1}{2}.$$

Thus by the lemma for $n \in S_2$ we have

$$|E_n(\tfrac{z}{a_n}) - 1| \le 2e|\tfrac{z}{a_n}|^{n+1} \le \tfrac{e}{2^n}.$$

Applying Theorem 5.3 with $F_n(z) = E_n(\frac{z}{a_n})$ and $c_n = \frac{e}{2^n}$, we conclude that h_N (and hence f_N) converges uniformly on $B_R(0)$.

Furthermore, for $n \in S_2$, we have that $E_n(\frac{z}{a_n})$ is nonzero on $B_R(0)$, and hence by Theorem 5.3 the same is true for the limit of the h_N.

On the other hand for $N \ge N_0$, we have $g_N = 0$ precisely when $z = a_n$ for $|a_n| \le 2R$.

Conclusion. The infinite product

$$f(z) = z^m \prod_{n=1}^{\infty} E_n(\tfrac{z}{a_n})$$

converges to a holomorphic function on $B_R(0)$, with a zero of order m at zero, with all other zeros precisely at $\{a_n : |a_n| < R\}$.

Thus this function has all of the desired properties on $B_R(0)$.

However, as R was arbitrary, this (together with the uniqueness theorem) implies that f converges and has all of the desired properties on all of $\mathbb{C}$.

Finally, if h is another entire function that vanishes precisely at the sequence $\{a_n\}$, then the function $\frac{h}{f}$ is (more precisely, can be extended to) an entire function with no zeros.

Thus by Theorem 4.31, there exists an entire function g such that $\frac{h}{f} = e^g$, that is, $h = fe^g$, as needed. $\square$

To summarize: for any sequence $\{a_n\}$ such that $|a_n| \to \infty$ there exist entire functions with zeros given by $\{a_n\}$, and they are all of the form

$$f(z) = e^{g(z)} z^m \prod_{n:a_n \neq 0} E_n(\tfrac{z}{a_n})$$

for some entire function g.

Our **next goal** is a refinement of this fact (due to Hadamard) in the case that we can control the growth of f as $|z| \to \infty$.

5.3. **Functions of Finite Order.**

Definition 5.7. Let $f : \mathbb{C} \to \mathbb{C}$ be entire. If there exist $\rho, A, B > 0$ such that

$$\text{for all} \quad z \in \mathbb{C} \quad |f(z)| \leq Ae^{B|z|^\rho},$$

then we say f has **order of growth** $\leq \rho$.

We define the **order of growth** of f by

$$\rho_f = \inf\{\rho > 0 : \ f \ \text{ has order } \ \leq \rho\}.$$

Definition 5.8. Let $R > 0$ and let $f : B_R(0) \to \mathbb{C}$ be holomorphic. For $0 < r < R$ we let $n_f(r)$ denote the number of zeros of f inside $B_r(0)$.

Remark 5.9. Note that n_f is an increasing function, that is, $r_2 > r_1 \implies n_f(r_2) \geq n_f(r_1)$.

We can relate the order of an entire function to its zeros.

Theorem 5.10. *If $f : \mathbb{C} \to \mathbb{C}$ is entire and has order of growth $\leq \rho$, then*

(i) *there exists $C > 0$ such that $|n_f(r)| \leq Cr^\rho$ for all large $r > 0$,*
(ii) *if $\{z_k\} \subset \mathbb{C}\backslash\{0\}$ denote the zeros of f, then for any $s > \rho$ we have*

$$\sum_k \frac{1}{|z_k|^s} < \infty.$$

Remark 5.11. The condition $s > \rho$ in (ii) is sharp. To see this, consider $f(z) = \sin \pi z$, which has simple zeros at each $k \in \mathbb{Z}$.

As $f(z) = \frac{1}{2i}[e^{i\pi z} - e^{-i\pi z}]$, we find that $|f(z)| \leq e^{\pi|z|}$ so that f has order of growth ≤ 1.

We now note that $\sum_{n \neq 0} \frac{1}{|n|^s} < \infty$ if and only if $s > 1$. $\square$

We have some work to do before we can prove Theorem 5.10.

We begin with a lemma.

Lemma 5.12 (Mean value property). *Let $z_0 \in \mathbb{C}$ and $R > 0$, and let $f : B_R(z_0) \to \mathbb{C}$ be holomorphic. Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \quad \textit{for all} \quad 0 < r < R.$$

Proof. We use the Cauchy integral formula to write

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0}\, dz.$$

Parametrizing $\partial B_r(z_0)$ by $z(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi]$, we find

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

□

Next we derive "Jensen's formula".

Proposition 5.13 (Jensen's formula). *Let $R > 0$ and suppose $\Omega \subset \mathbb{C}$ is open, with $\overline{B_R(0)} \subset \Omega$.*

Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, satisfies $f(0) \neq 0$, and is nonzero on $\partial B_R(0)$. If $\{z_k\}_{k=1}^n$ denote the zeros of f in $B_R(0)$, counting multiplicity, then

$$\log |f(0)| = \sum_{k=1}^n \log\left(\tfrac{|z_k|}{R}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})|\, d\theta.$$

Proof. By considering the rescaled function $f_R(z) := f(\frac{z}{R})$, we see that it suffices to treat the case $R = 1$.

Define the "Blaschke product" $g : \overline{B_1(0)} \to \mathbb{C}$ by

$$g(z) = \prod_{k=1}^n \frac{z - z_k}{1 - \bar{z}_k z}.$$

We note that $g : B_1(0) \to B_1(0)$ is holomorphic, with $g : \partial B_1(0) \to \partial B_1(0)$. (See Exercise 3.2.)

Furthermore, g has the same zeros as f (counting multiplicity).

It follows that the function $z \mapsto \frac{f(z)}{g(z)}$ is (more precisely, can be extended to) a holomorphic function on $B_1(0)$ with no zeros inside $B_1(0)$.

Thus, as $B_1(0)$ is simply connected we may use Theorem 4.31 to construct a holomorphic function $h : B_1(0) \to \mathbb{C}$ such that $\frac{f}{g} = e^h$.

Note that

$$\left|\frac{f(z)}{g(z)}\right| = |e^{h(z)}| = |e^{\operatorname{Re} h(z) + i \operatorname{Im} h(z)}| = e^{\operatorname{Re} h(z)} \implies \log\left|\frac{f(z)}{g(z)}\right| = \operatorname{Re}(h(z)).$$

Thus applying the mean value formula to the h and taking the real part yields

$$\log\left|\frac{f(0)}{g(0)}\right| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|\frac{f(e^{i\theta})}{g(e^{i\theta})}\right| d\theta.$$

As $|g(e^{i\theta})| \equiv 1$, we find

$$\log|f(0)| = \log|g(0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta.$$

Noting that

$$g(0) = \prod_{k=1}^n z_k \implies \log|g(0)| = \log\Big(\prod_{k=1}^n |z_k|\Big) = \sum_{k=1}^n \log|z_k|,$$

we complete the proof. □

We next use Jensen's formula to derive a formula concerning $n_f(r)$.

Proposition 5.14. *Let $R > 0$ and suppose $\Omega \subset \mathbb{C}$ is open, with $\overline{B_R(0)} \subset \Omega$. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, satisfies $f(0) \neq 0$, and is nonzero on $\partial B_R(0)$. Then*

$$\int_0^R \frac{n_f(r)}{r}\,dr = -\log|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta.$$

Proof. Let $\{z_k\}_{k=1}^n$ denote the zeros of f in $B_R(0)$, counting multiplicity.

For each k we define

$$a_k(r) = \begin{cases} 1 & r > |z_k| \\ 0 & r \leq |z_k| \end{cases}$$

and notice that $n_f(r) = \sum_{k=1}^n a_k(r)$.

We compute

$$\int_0^R \frac{n_f(r)}{r}\,dr = \int_0^R \sum_{k=1}^n a_k(r)\frac{dr}{r} = \sum_{k=1}^n \int_0^R a_k(r)\frac{dr}{r} = \sum_{k=1}^n \int_{|z_k|}^R \frac{dr}{r} = -\sum_{k=1}^n \log\big(|\tfrac{z_k}{R}|\big).$$

Applying Jensen's formula, we complete the proof. □

Finally we are ready to prove Theorem 5.10.

Proof of Theorem 5.10. For (i) we claim it suffices to consider the case $f(0) \neq 0$.

Indeed, if f has a zero of order ℓ at $z = 0$, we define $F(z) = z^{-\ell}f(z)$. Then F is an entire function with $F(0) \neq 0$, n_f and n_F differ only by a constant, and F also has order of growth $\leq \rho$.

Fix $r > 1$. As $f(0) \neq 0$ we may use Proposition 5.14 and the growth condition to write

$$\begin{aligned}\int_r^{2r} \frac{n_f(x)}{x}\,dx &\leq \int_0^{2r} \frac{n_f(x)}{x}\,dx \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\theta})|\,d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \log|Ae^{B(2r)^\rho}|\,d\theta \leq Cr^\rho\end{aligned}$$

for some $C > 0$.

On the other hand, as n_f is increasing we can estimate

$$\int_r^{2r} \frac{n_f(x)}{x}\,dx \geq n_f(r)\int_r^{2r} \frac{dx}{x} \geq n_f(r)[\log 2r - \log r] \geq n_f(r)\log 2.$$

Rearranging yields $n_f(r) \leq \tilde{C}r^\rho$, as needed.

For part (ii) we estimate as follows:

$$\sum_{|z_k|\geq 1} |z_k|^{-s} \leq \sum_{j=0}^{\infty} \sum_{2^j \leq |z_k| \leq 2^{j+1}} |z_k|^{-s} \leq \sum_{j=0}^{\infty} 2^{-js} n_f(2^{j+1})$$
$$\leq C \sum_{j=0}^{\infty} 2^{-js} 2^{\rho(j+1)} \leq 2^{\rho} C \sum_{j=0}^{\infty} (2^{\rho - s})^j < \infty$$

since $s > \rho$. As only finitely many z_k can have $|z_k| < 1$, this estimate suffices to show part (ii). □

5.4. **Hadamard's Factorization Theorem.** We turn to Hadamard's factorization theorem, which is a refinement of Weierstrass's theorem for functions of finite order of growth.

Theorem 5.15 (Hadamard's factorization theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be entire and have order of growth ρ_f. Suppose f has a zero of order m at $z = 0$ and let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}\backslash\{0\}$ denote the remaining zeros of f. Letting k denote the unique integer such that $k \leq \rho_f < k + 1$, we have*

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} E_k(\tfrac{z}{a_n})$$

for some polynomial P of degree $\leq k$.

Proof. Let $g_N : \mathbb{C} \to \mathbb{C}$ be defined by

$$g_N(z) := z^m \prod_{n=1}^{N} E_k(\tfrac{z}{a_n}).$$

Fix $R > 0$. We use Theorem 5.3 to show that g_N converges (uniformly) in $B_R(0)$.

As $\lim_{n\to\infty} |a_n| = \infty$,

$$\text{there exists} \quad N_0 \quad \text{such that} \quad n \geq N_0 \implies |\tfrac{R}{a_n}| < \tfrac{1}{2}.$$

Thus for $n \geq N_0$ and $z \in B_R(0)$ we can use Lemma 5.6 to estimate

$$|1 - E_k(\tfrac{z}{a_n})| \leq 2e|\tfrac{z}{a_n}|^{k+1} \leq 2eR^{k+1}|a_n|^{-(k+1)}.$$

As $k + 1 > \rho_0$, we can use Theorem 5.10 to see that

$$\sum_n |a_n|^{-(k+1)} < \infty,$$

and hence Theorem 5.3 implies that g_N converges uniformly on $B_R(0)$ to the infinite product

$$g(z) = z^m \prod_{n=1}^{\infty} E_k(\tfrac{z}{a_n}),$$

which is holomorphic on $B_R(0)$, has a zero of order m at zero, and has all other zeros in $B_R(0)$ precisely at $\{a_n : |a_n| < R\}$.

As $R > 0$ was arbitrary, we can deduce that $g : \mathbb{C} \to \mathbb{C}$ is an entire function with a zero of order m at zero and all other zeros precisely at $\{a_n\}$.

Furthermore, since g and f have the same zeros, we find that $\frac{f}{g}$ is an entire function with no zeros, and hence we can use Theorem 4.31 to write $\frac{f}{g} = e^h$ for some entire function h.

To complete the proof, it remains to show that h must be a polynomial of degree at most k.

We first notice that

$$e^{\mathrm{Re}\, h(z)} = |e^{h(z)}| = \left|\frac{f(z)}{g(z)}\right|.$$

We now need the following lemma.

Lemma 5.16. *For any $s \in (\rho_f, k+1)$,*

$$\text{there exists} \quad C > 0, \quad r_j \to \infty \quad \text{such that} \quad \mathrm{Re}\,(h(z)) \leq C|z|^s \quad \text{for } |z| = r_j.$$

The proof of this lemma is a bit technical and so we save it until the next section. The idea is as follows: by proving lower bounds for the E_k and using Theorem 5.10, one can prove exponential lower bounds for $|g|$ on the order of $e^{-c|z|^s}$ (along some sequence of increasing radii). As f has order of growth $\leq s$, one can deduce the lemma.

To finish the proof, it suffices to show that the lemma implies that h is a polynomial of degree $\leq s$. (This is like the version of Liouville's theorem from Exercise 3.23.)

We argue as follows. We expand h in a power series centered at $z = 0$:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n.$$

By the Cauchy integral formulas and parametrization of $\partial B_r(0)$ we can deduce that for any $r > 0$:

$$\tfrac{1}{2\pi} \int_0^{2\pi} h(re^{i\theta}) e^{-in\theta}\, d\theta = \begin{cases} a_n r^n & n \geq 0 \\ 0 & n < 0. \end{cases} \qquad (\textit{Check!})$$

Taking complex conjugates yields

$$\tfrac{1}{2\pi} \int_0^{2\pi} \overline{h(re^{i\theta})} e^{-in\theta}\, d\theta = 0 \quad \text{for} \quad n > 0.$$

As $\mathrm{Re}\,(h) = \frac{1}{2}(h + \bar{h})$ we add the two identities above to find

$$\tfrac{1}{\pi} \int_0^{2\pi} \mathrm{Re}\,[h(re^{i\theta})] e^{-in\theta}\, d\theta = a_n r^n \quad \text{for} \quad n > 0.$$

We can also take the real part directly in the case $n = 0$ to get

$$\tfrac{1}{\pi} \int_0^{2\pi} \mathrm{Re}\,[h(re^{i\theta})]\, d\theta = 2\mathrm{Re}\,(a_0). \quad (*)$$

As $\int_0^{2\pi} e^{-in\theta}\, d\theta = 0$ for any $n > 0$, we find:

$$\begin{aligned} a_n &= \tfrac{1}{\pi r^n} \int_0^{2\pi} \mathrm{Re}\,[h(re^{i\theta})] e^{-in\theta}\, d\theta \\ &= \tfrac{1}{\pi r^n} \int_0^{2\pi} \big\{ \mathrm{Re}\,[h(re^{i\theta})] - Cr^s \big\} e^{-in\theta}\, d\theta \end{aligned}$$

for $n > 0$, where C, s are as in the lemma.

We now choose $r = r_j$ as in Lemma 5.16 and use $(*)$ to find

$$|a_n| \leq \frac{1}{\pi r_j^n} \int_0^{2\pi} \big\{ Cr_j^s - \mathrm{Re}\,[h(r_j e^{i\theta})] \big\}\, d\theta \leq 2Cr_j^{s-n} - 2\mathrm{Re}\,(a_0) r_j^{-n}.$$

Sending $j \to \infty$ now implies $|a_n| = 0$ for $n > s$, which implies that h is a polynomial of degree $\leq s$, as was needed to show. $\square$

5.5. **Proof of a Technical Lemma.** We include here a proof of Lemma 5.16, which is a bit technical. The reader may skip this section if desired.

We use the notation introduced in the proof of Hadamard's factorization theorem.

Lemma 5.17 (More bounds for E_k)**.** *For all k we have:*

(ii) $|z| \leq \frac{1}{2} \implies |E_k(z)| \geq e^{-2|z|^{k+1}}$

(iii) $|z| \geq \frac{1}{2} \implies |E_k(z)| \geq |1-z|e^{-c|z|^k}$, *where the constant may depend on k.*

Proof. For $|z| \leq \frac{1}{2}$ we can write $\log(1-z)$ in a power series

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n},$$

with $1 - z = e^{\log(1-z)}$. Thus

$$E_k(z) = e^{\log(1-z)+z+z^2/2+\cdots+z^k/k} = e^{-\sum_{j=k+1}^{\infty} z^j/j}.$$

We now notice that since $|z| \leq \frac{1}{2}$, we have

$$\Big|\sum_{j=k+1}^{\infty} \frac{z^j}{j}\Big| \leq |z|^{k+1} \sum_{j=k+1}^{\infty} |z|^{j-k-1} \leq |z|^{k+1} \sum_{j=0}^{\infty} (\tfrac{1}{2})^j \leq 2|z|^{k+1}.$$

Thus since $|e^w| \geq e^{-|w|}$ we can estimate

$$|E_k(z)| = |e^{-\sum_{j=k+1}^{\infty} z^j/j}| \geq e^{-2|z|^{k+1}},$$

which gives (ii).

For (iii), suppose $|z| \geq \frac{1}{2}$. As $|e^w| \geq e^{-|w|}$ it suffices to show

$$e^{-|z+z^2/2+\cdots+z^k/k|} \geq e^{-c|z|^k}.$$

This follows from the fact that for $|z| \geq \frac{1}{2}$ we have

$$|z + z^2/2 + \cdots + z^k/k| \leq C_k|z|^k.$$

$\square$

Lemma 5.18. *With $\rho_f, k, \{a_n\}$ as in Theorem 5.15, we have the following estimate:*

$$\forall s \in (\rho_f, k+1) \quad \exists\, c > 0 \ : z \in \mathbb{C}\backslash \bigcup_n B_{\frac{1}{|a_n|^{k+1}}}(a_n) \implies \Big|\prod_{n=1}^{\infty} E_k(\tfrac{z}{a_n})\Big| \geq e^{-c|z|^s},$$

where c may depend on k.

Proof. We write

$$\prod_n E_k(\tfrac{z}{a_n}) = \prod_{n \in S_1} E_k(\tfrac{z}{a_n}) \prod_{n \in S_2} E_k(\tfrac{z}{a_n}),$$

where

$$S_1 = \{n : |\tfrac{z}{a_n}| \leq \tfrac{1}{2}\}, \quad S_2 = \{n : |\tfrac{z}{a_n}| > \tfrac{1}{2}\}.$$

Consider $n \in S_1$. From Lemma 5.17 we have $|w| \leq \frac{1}{2} \implies |E_k(w)| \geq e^{-c|w|^{k+1}}$. Thus for all z,

$$\Big| \prod_{n \in S_1} E_k(\tfrac{z}{a_n}) \Big| \geq \prod_{n \in S_1} e^{-c|\frac{z}{a_n}|^{k+1}} \geq e^{-c|z|^{k+1} \sum_{n \in S_1} |a_n|^{-(k+1)}}.$$

Now,

$$\begin{aligned}
\sum_{n \in S_1} |a_n|^{-(k+1)} &= \sum_{n \in S_1} |a_n|^{-s} |a_n|^{s-(k+1)} \\
&\leq C|z|^{s-(k+1)} \sum_{n \in S_1} |a_n|^{-s} \quad \text{(definition of } S_1\text{)} \\
&\leq C'|z|^{s-(k+1)} \quad \text{(Theorem 5.10)},
\end{aligned}$$

so that

$$\Big| \prod_{n \in S_1} E_k(\tfrac{z}{a_n}) \Big| \geq e^{-c|z|^s}.$$

Now take $n \in S_2$. Recall from Lemma 5.17 that $|w| > \frac{1}{2} \implies |E_k(w)| \geq |1-w|e^{-c|w|^k}$. Thus

$$\Big| \prod_{n \in S_2} E_k(\tfrac{z}{a_n}) \Big| \geq \prod_{n \in S_2} |1 - \tfrac{z}{a_n}| \prod_{n \in S_2} e^{-c|\frac{z}{a_n}|^k}.$$

Now for any z we have

$$\prod_{n \in S_2} e^{-c|\frac{z}{a_n}|^k} \geq e^{-c|z|^k \sum_{n \in S_2} |a_n|^{-k}},$$

and

$$\begin{aligned}
\sum_{n \in S_2} |a_n|^{-k} &= \sum_{n \in S_2} |a_n|^{-s} |a_n|^{s-k} \\
&\leq C|z|^{s-k} \sum_{n \in S_2} |a_n|^{-s} \quad \text{(definition of } S_2\text{)} \\
&\leq C'|z|^{s-k} \quad \text{(Theorem 5.10)},
\end{aligned}$$

so that

$$\prod_{n \in S_2} e^{-|\frac{z}{a_n}|^k} \geq e^{-c|z|^s}.$$

Finally we note that

$$z \in \mathbb{C} \backslash \bigcup_n B_{\frac{1}{|a_n|^{k+1}}}(a_n) \implies |z - a_n| \geq |a_n|^{-(k+1)} \quad \text{for all } n,$$

so

$$\begin{aligned}\prod_{n\in S_2}|1-\tfrac{z}{a_n}| &\geq \prod_{n\in S_2}|\tfrac{a_n-z}{a_n}| \geq \prod_{n\in S_2}|a_n|^{-(k+2)}\\ &\geq \prod_{n\in S_2} e^{-(k+2)\log|a_n|}\\ &\geq e^{-(k+2)\sum_{n\in S_2}\log|a_n|}\\ &\geq e^{-(k+2)n_f(2|z|)\log(2|z|)} \quad \text{(definition of } S_2)\\ &\geq e^{-(k+2)c|z|^{s'}\log(2|z|)} \quad \text{(Theorem 5.10, with } s'>\rho_f)\\ &\geq e^{-c'|z|^s},\end{aligned}$$

where we choose $\rho_f < s' < s$ such that $|z|^{s'}\log(2|z|)| \leq C|z|^s$.

This completes the proof of the lemma. □

Lemma 5.19. *With $\rho_f, k, \{a_n\}, s$ as above, there exists a sequence $\{r_j\} \in (0,\infty)$ and $c>0$ such that $r_j \to \infty$ and*

$$\left|\prod_{k=1}^{\infty} E_k(\tfrac{z}{a_n})\right| \geq e^{-c|z|^s} \quad \textit{for all } z \textit{ such that } |z| = r_j.$$

Proof. As Theorem 5.10 implies $\sum_n |a_n|^{-(k+1)} < \infty$, we may find N such that

$$\sum_{n\geq N}|a_n|^{-(k+1)} < \tfrac{1}{10}. \quad (**)$$

We now claim that for all large integers L,

$$\exists\ r \in [L, L+1] : \partial B_r(0) \cap \bigcup_{n\geq N} B_{\frac{1}{|a_n|^{k+1}}}(a_n) = \emptyset.$$

With this claim, the corollary follows from the previous lemma.

Suppose the claim is false. Then we may find a large integer L such that

$$\forall\ r \in [L, L+1]\ \exists\ z \in B_{\frac{1}{|a_n|^{k+1}}}(a_n) \quad \text{such that} \quad n \geq N \quad \text{and} \quad |z| = r.$$

But this implies that

$$[L, L+1] \subset \cup_{n\geq N}[|a_n| - |a_n|^{-(k+1)}, |a_n| + |a_n|^{-(k+1)},$$

which implies

$$2\sum_{n\geq N}|a_n|^{-(k+1)} \geq 1,$$

contradicting $(**)$. □

From Lemma 5.19 we take the logarithm to deduce Lemma 5.16.

5.6. **Exercises.**

Exercise 5.1.

(i) Construct a non-constant holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that f has infinitely many zeros inside $\mathbb{D}$.

(ii) Why does the existence of such a function not contradict the "uniqueness theorem"?

Exercise 5.2. Express the following functions as products:
(i) $f(z) = \sin(\pi z)$
(ii) $g(z) = e^z - 1$.

Exercise 5.3. Prove that
$$\frac{1}{1-z} = \prod_{k=0}^{\infty}(1 + z^{2^k}) \quad \text{for} \quad z \in \mathbb{D}.$$

Exercise 5.4. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and has finite order of growth. Suppose that there exist distinct points $z_1, z_2 \in \mathbb{C}$ such that $f(z) \notin \{z_1, z_2\}$ for any $z \in \mathbb{C}$. Prove that f is constant.

Exercise 5.5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and has finite order of growth. Suppose that $f^{(n)}(z) \neq 0$ for any non-negative integer n and any $z \in \mathbb{C}$. Show that $f(z) = e^{az+b}$ for some $a, b \in \mathbb{C}$.

Exercise 5.6. How many solutions does the equation $e^z = z$ have in $\mathbb{C}$? Your options are: zero, finitely many, countably infinitely many, or uncountably many. (Prove that your answer is correct.)

Exercise 5.7. (i) Find a sequence $\{a_n\} \subset \mathbb{C}$ such that $\sum_n a_n$ converges but $\prod_n(1+a_n)$ diverges.

(ii) Find a sequence $\{a_n\} \subset \mathbb{C}$ such that $\prod_n(1 + a_n)$ converges but $\sum_n a_n$ diverges.

6. Conformal Mappings

We start this section with a few definitions.

Definition 6.1 (Biholomorphism)**.** Let $U, V \subset \mathbb{C}$ be open. If $f : U \to V$ is holomorphic and bijective (that is, one-to-one and onto), we call f a **biholomorphism**. We call the sets U and V **biholomorphic** and write $U \sim V$.

Definition 6.2 (Automorphism)**.** If $U \subset \mathbb{C}$ is open and $f : U \to U$ is a biholomorphism, we call f an **automorphism** of U.

In this section we will address two general questions:

1. Given an open set U, can we classify the automorphisms of U?
2. Which open sets $U, V \subset \mathbb{C}$ are biholomorphic?

Two sets will show up frequently, namely the **unit disk**

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and the **upper half plane**

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

6.1. **Preliminaries.**

Proposition 6.3. *Let $U, V \subset \mathbb{C}$ be open and let $f : U \to V$ be a biholomorphism. Then $f'(z) \neq 0$ for all $z \in U$, and $f^{-1} : V \to U$ is a biholomorphism.*

Proof. Suppose toward a contradiction that $f'(z_0) = 0$ for some $z_0 \in U$.

We expand f in a power series in some open ball $\Omega \ni z_0$:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad z \in \Omega.$$

As f is injective, it is non-constant, and hence we may choose Ω possibly smaller to guarantee that $f'(z) \neq 0$ for $z \in \Omega \backslash \{z_0\}$.

Rearranging the formula above, using $a_1 = f'(z_0) = 0$, and re-indexing, we can write

$$f(z) - f(z_0) = a_k (z - z_0)^k + (z - z_0)^{k+1} \sum_{\ell=0}^{\infty} b_\ell (z - z_0)^\ell$$

where $a_k \neq 0$, $k \geq 2$, and $b_\ell := a_{\ell+k+1}$.

We next notice that

$$\lim_{\delta \to 0} \delta \sum_{\ell=0}^{\infty} |b_\ell| \delta^\ell = 0.$$

Thus we may choose $\delta > 0$ sufficiently small that

(i) $B_\delta(z_0) \subset \Omega$,
(ii) the following holds:

$$\delta^{k+1} \sum_{\ell=0}^{\infty} |b_\ell| \delta^\ell \leq \tfrac{1}{2} |a_k| \delta^k.$$

In particular, (ii) implies that there exists $\varepsilon > 0$ small enough such that

$$w \in B_\varepsilon(0) \implies$$
$$\left|(z-z_0)^{k+1}\sum_{\ell=0}^{\infty} b_\ell (z-z_0)^\ell - w\right| < |a_k(z-z_0)^k| \quad \text{for} \quad z \in \partial B_\delta(z_0). \quad (*)$$

For $w \in B_\varepsilon(0)\backslash\{0\}$ we write

$$f(z) - f(z_0) - w = \underbrace{a_k(z-z_0)^k}_{:=F(z)} + \underbrace{(z-z_0)^{k+1}\sum_{\ell=0}^{\infty} b_\ell(z-z_0)^\ell - w}_{:=G(z)}.$$

As F has k zeros in $\partial B_\delta(z_0)$ (counting multiplicity), and $(*)$ implies $|G(z)| < |F(z)|$ for $z \in \partial B_\delta(z_0)$, we can use Rouchè's theorem to conclude that

$$z \mapsto f(z) - f(z_0) - w$$

has at least two zeros in $B_\delta(z_0)$. That is, there exists $z_1, z_2 \in B_\delta(z_0)$ such that

$$f(z_1) = f(z_2) = f(z_0) + w.$$

We now claim that we must have $z_1 \neq z_2$, so that we have contradicted the injectivity of f.

We first note that $w \neq 0$ implies $z_1, z_2 \neq z_0$.

Now on the one hand we have $f'(z) \neq 0$ for $z \in B_\delta(z_0)$. On the other hand, if $z \mapsto f(z) - f(z_0) - w$ had a zero of order ≥ 2 at z then we would have $f'(z) = 0$.

Thus the zeros of $f(z) - f(z_0) - w$ must be simple, so that any two zeros must be distinct, as was needed to show.

It remains to check that f^{-1} is a biholomorphism. As f^{-1} is bijective, it suffices to verify that f^{-1} is holomorphic.

To this end, let $w, w_0 \in V$, with $w \neq w_0$. Then

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{1}{\frac{w-w_0}{f^{-1}(w)-f^{-1}(w_0)}} = \frac{1}{\frac{f(f^{-1}(w))-f(f^{-1}(w_0))}{f^{-1}(w)-f^{-1}(w_0)}}.$$

Now we would like take the limit as $w \to w_0$.

We first note that the open mapping theorem implies f^{-1} is continuous. (*Why?*)

Thus as $w \to w_0$, we have $f^{-1}(w) \to f^{-1}(w_0)$.

Moreover, since we know $f' \neq 0$ on U, we can safely take the limit above to see that

$$\tfrac{d}{dz}(f^{-1})(w_0) = \frac{1}{f'(f^{-1}(w_0))}.$$

□

Remark 6.4. We can now verify that being biholomorphic is an **equivalence relation**. That is,

(i) $U \sim U$ for all open sets U.
(ii) $U \sim V \implies V \sim U$ for all open sets, U, V.
(iii) $[U \sim V$ and $V \sim W] \implies U \sim W$ for all open sets U, V, W.

For (i), we observe that $f(z) = z$ is a biholomorphism.

For (ii), we note that if $f : U \to V$ is a biholomorphism, then the proposition above implies $f^{-1} : V \to U$ is a biholomorphism.

For (iii), we note that if $f : U \to V$ and $g : V \to W$ are biholomorphisms, then $f \circ g : U \to W$ is a biholomorphism. (*Check!*) $\square$

We next discuss the main **geometric** property of biholomorphisms. In particular they are "**conformal**", which is a synonym for "angle-preserving".

Recall that for vectors $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ we define the **inner product** of v and w by

$$\langle v, w\rangle_{\mathbb{R}^n} = v_1 w_1 + \cdots + v_n w_n.$$

The **length** of a vector $v \in \mathbb{R}^n$ is given by $|v| = \sqrt{\langle v, v\rangle_{\mathbb{R}^n}}$. The **angle** $\theta \in [0, \pi]$ between vectors $v, w \in \mathbb{R}^n$ is given by the formula

$$\cos\theta = \frac{\langle v, w\rangle_{\mathbb{R}^n}}{|v|\,|w|}.$$

If $M = (m_{jk})$ is an $n \times n$ matrix (with real or complex entries) and $v, w \in \mathbb{R}^n$, then we have

$$\langle Mv, w\rangle_{\mathbb{R}^n} = \langle v, M^t w\rangle_{\mathbb{R}^n}, \quad (*)$$

where M^t is the **transpose** of M, whose $(j,k)^{th}$ entry is m_{kj}.

Definition 6.5 (Angle)**.** Let $\gamma_j : (-1,1) \to \mathbb{R}^n$ parametrize smooth curves for $j = 1, 2$. Suppose that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_j'(0) \neq 0$ for $j = 1, 2$. We define the **angle** $\theta \in [0, \pi]$ between γ_1 and γ_2 by the formula

$$\cos\theta = \frac{\langle \gamma_1'(0), \gamma_2'(0)\rangle_{\mathbb{R}^n}}{|\gamma_1'(0)|\,|\gamma_2'(0)|}.$$

We extend this notion to curves in $\mathbb{C}$ via the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$.

The next proposition shows that biholomorphisms preserve angles.

Proposition 6.6. *Let $\gamma_j : (-1,1) \to \mathbb{C}$ parametrize smooth curves for $j = 1, 2$, with $\gamma_1(0) = \gamma_2(0) = z_0 \in \mathbb{C}$ and $\gamma_j'(0) \neq 0$ for $j = 1, 2$. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at z_0 and $f'(z_0) \neq 0$. Then the angle between γ_1 and γ_2 equals the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$.*

Proof. By the chain rule we have

$$(f \circ \gamma_j)'(0) = f'(z_0)\gamma_j'(0).$$

We use polar coordinates to write

$$f'(z_0) = |f'(z_0)|(\cos\theta + i \sin\theta)$$

for some $\theta \in [0, 2\pi]$.

Under the identification of $\mathbb{C}$ with $\mathbb{R}^2$, we may identify $\gamma_j'(0)$ with an element of $\mathbb{R}^2$ and $f'(z_0)$ with the 2×2 real matrix given by

$$\begin{pmatrix} \mathrm{Re}\,[f'(z_0)] & -\mathrm{Im}\,[f'(z_0)] \\ \mathrm{Im}\,[f'(z_0)] & \mathrm{Re}\,[f'(z_0)] \end{pmatrix} = |f'(z_0)| \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{:=M}.$$

As $\cos^2\theta + \sin^2\theta = 1$, we can compute that

$$M^t M = M M^t = Id, \quad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus using $(*)$ we deduce

$$\langle Mv, Mw\rangle_{\mathbb{R}^2} = \langle v, w\rangle_{\mathbb{R}^2}, \qquad |Mv| = |v|$$

for all $v, w \in \mathbb{R}^2$.

We can now compute

$$\begin{aligned}\frac{\langle (f\circ\gamma_1)'(0), (f\circ\gamma_2)'(0)\rangle_{\mathbb{R}^2}}{|(f\circ\gamma_1)'(0)|\,|(f\circ\gamma_2)'(0)|} &= \frac{\langle |f'(z_0)|M\gamma_1'(0), |f'(z_0)|M\gamma_2'(0)\rangle_{\mathbb{R}^2}}{|f'(z_0)\gamma_1'(0)|\,|f'(z_0)\gamma_2'(0)|} \\ &= \frac{|f'(z_0)|^2}{|f'(z_0)|^2}\frac{\langle M\gamma_1'(0), M\gamma_2'(0)\rangle_{\mathbb{R}^2}}{|M\gamma_1'(0)|\,|M\gamma_2'(0)|} \\ &= \frac{\langle \gamma_1'(0), \gamma_2'(0)\rangle_{\mathbb{R}^2}}{|\gamma_1'(0)|\,|\gamma_2'(0)|},\end{aligned}$$

which completes the proof. □

6.2. **Some Examples.**

Example 6.1 (Translation, dilation, rotation)**.** For any $z_0, \lambda \in \mathbb{C}$ the map $z \mapsto z_0 + \lambda z$ is a conformal map from $\mathbb{C}$ to $\mathbb{C}$.

The special case $z \mapsto e^{i\theta}z$ for some $\theta \in \mathbb{R}$ is called a **rotation**.

Example 6.2. For $n \in \mathbb{N}$ define the sector

$$S_n = \{z \in \mathbb{C} : 0 < \arg(z) < \tfrac{\pi}{n}\}.$$

The function $z \mapsto z^n$ is a conformal map from S_n to $\mathbb{H}$. Its inverse is given by $z \mapsto z^{1/n}$ (defined in terms of the principal branch of the logarithm).

Example 6.3. The map $z \mapsto \log z$ is a conformal map from $\mathbb{H}$ to the strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$.

This follows from the fact that if $z = re^{i\theta}$ with $\theta \in (0, \pi)$ then $\log z = \log r + i\theta$.

The inverse is given by $z \mapsto e^z$.

Example 6.4. The map $z \mapsto \log z$ is also a conformal map from the half-disk $\{z \in \mathbb{D} : \operatorname{Im} z > 0\}$ to the half-strip $\{z \in \mathbb{C} : \operatorname{Re} z < 0,\ 0 < \operatorname{Im} z < \pi\}$.

Example 6.5. The map $z \mapsto \sin z$ is a conformal map from the half-strip

$$\Omega := \{z \in \mathbb{C} : -\tfrac{\pi}{2} < \operatorname{Re} z < \tfrac{\pi}{2},\ \operatorname{Im} z > 0\}$$

to $\mathbb{H}$.

To see this, we first use the identity

$$\sin z = -\tfrac{1}{2}\left[ie^{iz} + \tfrac{1}{ie^{iz}}\right]$$

to write $\sin z = h(ig(z))$, where $g(z) = e^{iz}$ and $h(z) = -\frac{1}{2}(z + \frac{1}{z})$.

It then suffices to note the following:

- g is a conformal map from Ω to $\{z \in \mathbb{D} : \operatorname{Re} z > 0\}$,
- $z \mapsto iz$ rotates $\{z \in \mathbb{D} : \operatorname{Re} z > 0\}$ to $\{z \in \mathbb{D} : \operatorname{Im} z > 0\}$,
- h is a conformal map from $\{z \in \mathbb{D} : \operatorname{Im} z > 0\}$ to $\mathbb{H}$. (*Check!*)

6.3. **Introduction to Groups.** We next discuss the notion of groups, which arise in the study of automorphisms.

Definition 6.7 (Group)**.** A **group** is a set G, together with a function $b : G \times G \to G$ such that

(i) $b(b(x, y), z) = b(x, b(y, z)$ for all $x, y, z \in G$,
(ii) there exists (unique) $e \in G$ such that $b(e, x) = b(x, e) = x$ for all $x \in G$,
(iii) for all $x \in G$ there exists (unique) $y \in G$ such that $b(x, y) = b(y, x) = e$.

We call b the **group operation**.
We call the element e in (ii) the **identity** element.
We call the element y in (iii) the **inverse** of x and write $y = x^{-1}$.
To simplify notation one usually writes $b(x, y)$ as xy (or $x \cdot y$, or $x + y$, or $x \circ y$, or ...).
A **subgroup** of a group (G, b) is a subset $A \subset G$ such that (A, b) forms a group. We write $A \leq G$.
A **normal subgroup** of a group G is a subgroup A such that

$$\text{for all} \quad a \in A,\ g \in G, \quad gag^{-1} \in A.$$

Suppose (G_1, b_1) and (G_2, b_2) are groups. We say G_1 is **isomorphic** to G_2 if there exists a bijection $\varphi : G_1 \to G_2$ such that

$$\varphi(b_1(x, y)) = b_2(\varphi(x), \varphi(y)) \quad \text{for all} \quad x, y \in G_1.$$

We write $G_1 \cong G_2$. Being isomorphic is an equivalence relation, as one can check. □

Example 6.6 (Matrix groups)**.** Throughout this example we let F denote either $\mathbb{C}$ or $\mathbb{R}$.
Let $\mathcal{M}_2(F)$ denote the set of all 2×2 matrices with entries in F:

$$\mathcal{M}_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F \right\}$$

Recall that for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define

$$\det M := ad - bc,$$

and M is invertible if and only if $\det M \neq 0$.
The set $\mathcal{M}_2(F)$ under matrix multiplication does **not** form a group, since not all matrices are invertible.
Recalling that $\det(M_1 M_2) = \det M_1 \cdot \det M_2$, it follows that the set

$$GL_2(F) = \{M \in \mathcal{M}_2(F) : \det M \neq 0\},$$

forms a group under matrix multiplication. The identity element is given by

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We call $GL_2(F)$ the **general linear group** of 2×2 matrices with entries in F.
We define the **special linear group** $SL_2(F) \leq GL_2(F)$ by

$$SL_2(F) = \{M \in \mathcal{M}_2(F) : \det M = 1\}.$$

Example 6.7 (Quotient groups)**.** Suppose G is a group, with the operation denoted by $b(x,y) = xy$.

Suppose that $A \leq G$ is a normal subgroup.

We define a relation $\sim$ on G as follows:

$$x \sim y \quad \text{if} \quad xy^{-1},\ x^{-1}y \in A.$$

Because A is a subgroup, $\sim$ defines an equivalence relation on G. Indeed:

- $x \sim x$ for any $x \in G$, since $xx^{-1} = x^{-1}x = e \in A$.
- if $x \sim y$, then $y \sim x$, since $yx^{-1} = (xy^{-1})^{-1} \in A$ and $y^{-1}x = (x^{-1}y)^{-1} \in A$,
- if $x \sim y$ and $y \sim z$, then $xz^{-1} = (xy^{-1})(yz^{-1}) \in A$ and $x^{-1}z = (x^{-1}y)(y^{-1}z) \in A$.

For any $x \in G$ we define the **equivalence class** of x by

$$[x] = \{y \in G : y \sim x\}.$$

Because A is a *normal* subgroup, we can show the following:

$$[x_1 \sim x_2 \text{ and } y_1 \sim y_2] \implies x_1y_1 \sim x_2y_2.$$

Indeed, we have

$$(x_1y_1)(x_2y_2)^{-1} = x_1y_1y_2^{-1}x_2^{-1} = x_1(y_1y_2^{-1}x_2^{-1}x_1)x_1^{-1} \in A$$

and

$$(x_1y_1)^{-1}(x_2y_2) = y_1^{-1}x_1^{-1}x_2y_2 = y_1^{-1}(x_1^{-1}x_2y_2y_1^{-1})y_1 \in A.$$

In particular we see that

$$[x_1y_1] = [x_2y_2] \quad \text{provided} \quad x_1 \sim x_2 \quad \text{and} \quad y_1 \sim y_2. \quad (*)$$

Thus we may define the **quotient group** $G/A := \{[x] : x \in G\}$, where we define the group operation by $[x][y] = [xy]$. (Note $(*)$ implies that this operation is well-defined.)

One can check that G/A forms a group, with the identity given by $[e]$ and inverses given by $[x]^{-1} = [x^{-1}]$.

Example 6.8 (Projective groups)**.** As before we let F denote either $\mathbb{C}$ or $\mathbb{R}$. Let

$$A = \{\lambda\, Id : \lambda \in F\backslash\{0\}\ \} \subset GL_2(F),$$

where Id is the identity matrix.

As one can check, A forms a normal subgroup of $GL_2(F)$.

Thus we can define the **projective linear group** by

$$PGL_2(F) := GL_2(F)/A.$$

Similarly the set $\{\pm Id\}$ forms a normal subgroup of $SL_2(F)$.

Thus we can define the **projective special linear group** by

$$PSL_2(F) = SL_2(F)/\{\pm Id\}.$$

Example 6.9 (Automorphism groups)**.** Let $U \subset \mathbb{C}$ be an open set. The set of automorphisms of U forms a group under composition, which we denote by $\mathrm{Aut}(U)$. Indeed, we have the following:

- if $f, g \in \mathrm{Aut}(U)$ then $f \circ g \in \mathrm{Aut}(U)$,
- $[f \circ g] \circ h = f \circ [g \circ h]$ for $f, g, h \in \mathrm{Aut}(U)$,
- the identity element is given by the function $e(z) = z$,
- if $f \in \mathrm{Aut}(U)$ then $f^{-1} \in \mathrm{Aut}(U)$ (cf. Proposition 6.3).

6.4. **Möbius Transformations.** In this section we introduce an important class of conformal mappings called Möbius transformations (also known as fractional linear transformations).

Recall that we identified the extended complex plane $\mathbb{C}\cup\{\infty\}$ with the Riemann sphere

$$\mathbb{S} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z - \tfrac{1}{2})^2 = \tfrac{1}{4}\}$$

via the stereographic projection map $\Phi : \mathbb{S} \to \mathbb{C} \cup \{\infty\}$ given by

$$\Phi((x,y,z)) = \tfrac{x}{1-z} + i\tfrac{y}{1-z}, \quad \Phi^{-1}(x+iy) = \left(\tfrac{x}{1+x^2+y^2}, \tfrac{y}{1+x^2+y^2}, \tfrac{x^2+y^2}{1+x^2+y^2}\right).$$

The north pole $(0,0,1)$ corresponds to ∞, since $|x+iy| \to \infty \iff \Phi^{-1}(x+iy) \to (0,0,1)$.

The set $\{(x,y,z) \in \mathbb{S} : z < \frac{1}{2}\}$ corresponds to $\mathbb{D}$.

The set $\{(x,y,z) \in \mathbb{S} : y > 0\}$ corresponds to $\mathbb{H}$.

A computation (in the spirit of the proof of Proposition 6.6) shows that Φ (and similarly Φ^{-1}) is conformal, that is, it preserves angles between curves. (See Exercise 6.5.)

Definition 6.8 (Lift)**.** Suppose $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. We define the **lift of** f **to** $\mathbb{S}$ to by

$$\Phi^{-1} \circ f \circ \Phi : \mathbb{S} \to \mathbb{S}.$$

Definition 6.9 (Möbius transformations)**.** For any

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}),$$

we define the **Möbius transformation**

$$f_M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \quad \text{by} \quad f_M(z) = \frac{az+b}{cz+d}.$$

Proposition 6.10. *The set of Möbius transformations forms a group under composition. Moreover for $F = \mathbb{R}$ or $\mathbb{C}$ we have the following:*

$$\{f_M : M \in GL_2(F)\} \cong PGL_2(F),$$

$$\{f_M : M \in SL_2(F)\} \cong PSL_2(F).$$

Proof. We consider the case of $GL_2(F)$, as the case of $SL_2(F)$ is similar.

A direct computation shows

$$f_M \circ f_N = f_{MN} \quad \text{for} \quad M, N \in GL_2(F). \quad (\textit{Check!})$$

In particular, for any $M \in GL_2(F)$ we have

$$f_M \circ f_{Id} = f_{Id} \circ f_M = f_M \quad \text{and} \quad f_M \circ f_{M^{-1}} = f_{Id}.$$

Furthermore

$$(f_L \circ f_M) \circ f_N = f_{LM} \circ f_N = f_{LMN} = f_L \circ f_{MN} = f_L \circ (f_M \circ f_N)$$

for $L, M, N \in GL_2(F)$.

It follows that $G := \{f_M : M \in GL_2(F)\}$ forms a group under composition.

We now define

$$\varphi : G \to PGL_2(F) \quad \text{by} \quad \varphi(f_M) = [M].$$

We first observe that f is onto.

Next suppose $\varphi(f_M) = \varphi(f_N)$ for some $N, M \in GL_2(F)$. Then $N = \lambda M$ for some $\lambda \in F \backslash \{0\}$. As

$$\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az + b}{cz + d},$$

we find that $f_N = f_M$, so that φ is one-to-one.

Thus φ is a bijection. Moreover,

$$\varphi(f_N \circ f_M) = \varphi(f_{NM}) = [NM] = [N][M] = \varphi(f_N)\varphi(f_M),$$

so that φ is an isomorphism. $\square$

Lemma 6.11. *For all distinct* $\{\alpha, \beta, \gamma\} \subset \mathbb{C} \cup \{\infty\}$, *there exists* $M \in GL_2(\mathbb{C})$ *such that*

$$f_M(\alpha) = 1, \quad f_M(\beta) = 0, \quad f_M(\gamma) = \infty.$$

Proof. If $\{\alpha, \beta, \gamma\} \subset \mathbb{C}$ then we can take

$$f_M(z) = \frac{z - \beta}{z - \gamma} \cdot \frac{\alpha - \gamma}{\alpha - \beta}.$$

If $\alpha, \beta, \gamma = \infty$ we instead take

$$f_M(z) = \frac{z - \beta}{z - \gamma}, \ \frac{\alpha - \gamma}{z - \gamma}, \ \text{or} \ \frac{z - \beta}{\alpha - \beta},$$

respectively. $\square$

Proposition 6.12. *The sets* $\mathbb{D}$ *and* $\mathbb{H}$ *are biholomorphic. Consequently* $Aut(\mathbb{D}) \cong Aut(\mathbb{H})$.

Proof. We need to construct a biholomorphism $F : \mathbb{D} \to \mathbb{H}$. We will use a Möbius transformation.

We can think in terms of the lift of F. As a map on $\mathbb{S}$, we want a 90° rotation about the x-axis.

Thus as a map on $\mathbb{C} \cup \{\infty\}$, we want

$$1 \mapsto 1, \quad -i \mapsto 0, \quad i \mapsto \infty.$$

Thus, as in Lemma 6.11 we define

$$F(z) = \frac{z + i}{z - i} \cdot \frac{1 - i}{1 + i} = -i\frac{z + i}{z - i}.$$

This function defines a biholomorphism from $\mathbb{D}$ to $\mathbb{H}$, as one should check.

We now define $\varphi : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ by

$$\varphi(f) = F \circ f \circ F^{-1} \quad \text{for} \quad f \in \mathrm{Aut}(\mathbb{D}).$$

One can check that φ is one-to-one and onto, and moreover

$$\varphi(f) \circ \varphi(g) = F \circ f \circ F^{-1} \circ F \circ g \circ F^{-1} = F \circ (f \circ g) \circ F^{-1} = \varphi(f \circ g),$$

so that φ is an isomorphism. $\square$

6.5. **Automorphisms of $\mathbb{D}$ and $\mathbb{H}$.** We will now investigate $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$.

We have actually already encountered some elements of $\mathrm{Aut}(\mathbb{D})$, namely the **Blaschke factors**

$$\psi_\alpha(z) := \frac{z-\alpha}{\bar{\alpha}z-1} \quad \text{for} \quad \alpha \in \mathbb{D}.$$

Indeed, in Exercise 3.2 you showed that each ψ_α is an automorphism of $\mathbb{D}$. In fact,

$$\psi_\alpha^{-1} = \psi_\alpha. \quad (\textit{Check!})$$

Note that Blaschke factors are instances of Möbius transformations, with

$$\psi_\alpha = f_{M_\alpha}, \quad M_\alpha = \begin{pmatrix} 1 & -\alpha \\ \bar{\alpha} & -1 \end{pmatrix} \in GL_2(\mathbb{C}).$$

As we will see, the Blaschke factors turn out to give (essentially) *all* automorphisms of $\mathbb{D}$!

To see this we will use the following lemma.

Lemma 6.13 (Schwarz lemma)**.** *Suppose $f : \mathbb{D} \to \overline{\mathbb{D}}$ is holomorphic and $f(0) = 0$. Then*

(i) $|f(z)| \le |z|$ *for* $z \in \mathbb{D}$,
(ii) *if there exists* $z_0 \in \mathbb{D}\backslash\{0\}$ *such that* $|f(z_0)| = |z_0|$, *then f is a rotation,*
(iii) $|f'(0)| \le 1$, *with equality if and only if f is a rotation.*

Proof. We expand f in a power series centered at 0:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \quad \text{for} \quad z \in \mathbb{D}.$$

As $f(0) = a_0 = 0$, we find that $z \mapsto \frac{f(z)}{z}$ is (more precisely, can be extended to) a holomorphic function on $\mathbb{D}$.

Now fix $0 < r < 1$. By the maximum principle and the fact that $|f(z)| \le 1$, we find

$$\max_{z \in B_r(0)} \left|\tfrac{f(z)}{z}\right| = \max_{z \in \partial B_r(0)} \left|\tfrac{f(z)}{z}\right| \le \tfrac{1}{r}.$$

Sending $r \to 1$ we deduce that

$$|f(z)| \le |z| \quad \text{for} \quad z \in \mathbb{D},$$

which gives (i).

For (ii) we note that if $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}\backslash\{0\}$ then $z \mapsto \frac{f(z)}{z}$ attains its maximum in $\mathbb{D}$ and hence is constant.

Thus $f(z) = cz$, and since $|f(z_0)| = |z_0|$ we must have $|c| = 1$, so that f is a rotation.

For (iii), we write $g(z) = \frac{f(z)}{z}$ and note

$$g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Thus $|f'(0)| \le 1$, and if equality holds then then g attains its maximum in $\mathbb{D}$ and hence f is a rotation, as before. □

Theorem 6.14 (Automorphisms of $\mathbb{D}$)**.**

$$Aut(\mathbb{D}) = \{e^{i\theta}\psi_\alpha : \theta \in \mathbb{R},\ \alpha \in \mathbb{D}\}.$$

In particular if $f \in Aut(\mathbb{D})$ and $f(0) = 0$ then f is a rotation.

Remark 6.15. One can actually show that $\mathrm{Aut}(\mathbb{D})$ is isomorphic to a matrix group called $PSU(1,1)$, but the proof is a bit technical and we do not pursue it here.

Proof of Theorem 6.14. Suppose $f \in \mathrm{Aut}(\mathbb{D})$.

Choose $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$ and consider $g = f \circ \psi_\alpha \in \mathrm{Aut}(\mathbb{D})$.

We have $g(0) = f(\psi_\alpha(0)) = f(\alpha) = 0$, and so the Schwarz lemma implies

$$|g(z)| \leq |z| \quad \text{for} \quad z \in \mathbb{D}.$$

On the other hand, $g^{-1}(0) = \psi_\alpha^{-1}(f^{-1}(0)) = \psi_\alpha(\alpha) = 0$, and so the Schwarz lemma implies

$$|g^{-1}(z)| \leq |z| \quad \text{for} \quad z \in \mathbb{D}.$$

In particular

$$|z| \leq |g^{-1}(g(z))| \leq |g(z)| \quad \text{for} \quad z \in \mathbb{D}.$$

Thus $|g(z)| = |z|$ for $z \in \mathbb{D}$, and hence the Schwarz lemma implies that g is a rotation:

$$g(z) = f \circ \psi_\alpha(z) = e^{i\theta} z.$$

In particular

$$f(z) = f \circ \psi_\alpha(\psi_\alpha^{-1}(z)) = e^{i\theta}\psi_\alpha^{-1}(z) = e^{i\theta}\psi_\alpha(z).$$

The result follows. □

We next consider $\mathrm{Aut}(\mathbb{H})$.

Theorem 6.16 (Automorphisms of $\mathbb{H}$)**.**

$$Aut(\mathbb{H}) = \{f_M : M \in SL_2(\mathbb{R})\}.$$

In fact $Aut(\mathbb{H}) \cong PSL_2(\mathbb{R})$.

Proof. First let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Then f_M holomorphic on $\mathbb{H}$, and we can write

$$f_M(z) = \frac{az+b}{cz+d} \cdot \frac{c\bar{z}+d}{c\bar{z}+d} = \frac{ac|z|^2 + bd + adz + bc\bar{z}}{|cz+d|^2}.$$

Thus for $z \in \mathbb{H}$ we have

$$\mathrm{Im}\,[f_M(z)] = \frac{ad-bc}{|cz+d|^2}\,\mathrm{Im}\,z = \frac{\mathrm{Im}\,z}{|cz+d|^2} > 0.$$

In particular we can deduce that $f_M \in \mathrm{Aut}(\mathbb{H})$.

Next let $f \in \mathrm{Aut}(\mathbb{H})$ and choose $\beta \in \mathbb{H}$ such that $f(\beta) = i$.

We claim that there exists $M_\beta \in SL_2(\mathbb{R})$ such that $f_{M_\beta}(i) = \beta$. Indeed we can take

$$M_\beta = \tfrac{1}{\sqrt{\mathrm{Im}\,\beta}} \begin{pmatrix} \mathrm{Re}\,\beta & -\mathrm{Im}\,\beta \\ 1 & 0 \end{pmatrix}.$$

Thus $f \circ f_{M_\beta} \in \mathrm{Aut}(\mathbb{H})$ with $f \circ f_{M_\beta}(i) = i$.

Now recall from Proposition 6.12 that there exists a biholomorphism $F : \mathbb{D} \to \mathbb{H}$ with $F(0) = i$. In particular,

$$F = f_A, \quad \text{with} \quad A = \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}.$$

Now consider the function

$$g = f_{A^{-1}} \circ f \circ f_{M_\beta} \circ f_A.$$

Then $g \in \mathrm{Aut}(\mathbb{D})$ with $g(0) = 0$, and hence

$$g = f_{A^{-1}} \circ f \circ f_{M_\beta} \circ f_A = e^{2i\theta} \quad (*)$$

for some $\theta \in \mathbb{R}$.

An explicit computation shows that $(*)$ implies

$$f \circ f_{M_\beta} = f_{M_\theta}, \quad \text{where} \quad M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SL_2(\mathbb{R}). \quad (\textit{Check!})$$

Thus

$$f = f_{M_\theta} \circ f_{M_\beta}^{-1} = f_{M_\theta M_\beta^{-1}} \in \{f_M : M \in SL_2(\mathbb{R})\},$$

as needed.

We now define $\varphi : \mathrm{Aut}(\mathbb{H}) \to PSL_2(\mathbb{R})$ by

$$\varphi(f_M) = [M] \quad \text{for} \quad M \in SL_2(\mathbb{R}).$$

It is clear that φ is onto. Next if $\varphi(f_M) = \varphi(f_N)$ for some $N, M \in SL_2(\mathbb{R})$, then $N = \pm M$ and hence $f_M = f_N$. Thus φ is one-to-one.

Thus φ is a bijection. Moreover,

$$\varphi(f_N \circ f_M) = \varphi(f_{NM}) = [NM] = [N][M] = \varphi(f_N)\varphi(f_M),$$

so that φ is an isomorphism. □

6.6. **Normal Families.** We now turn to the second main question of this section, namely which subsets of $\mathbb{C}$ are biholomorphic. We will eventually construct biholomorphisms as limits of sequences of functions. In this section we develop some tools related to taking such limits.

Definition 6.17. Let $\Omega \subset \mathbb{C}$ be open and let $\mathcal{F}$ be a collection of functions $f : \Omega \to \mathbb{C}$.

- We call $\mathcal{F}$ a **normal family** if every sequence in $\mathcal{F}$ has a subsequence that converges locally uniformly.
- We call $\mathcal{F}$ **locally uniformly bounded** if

$$\text{for all compact } K \subset \Omega \quad \text{there exists} \quad B > 0 \quad \text{such that}$$
$$\text{for all} \quad f \in \mathcal{F},\ z \in K, \quad \text{we have} \quad |f(z)| \le B.$$

- We call $\mathcal{F}$ **locally uniformly equicontinuous** if

$$\text{for all compact } K \subset \Omega \quad \text{and for all} \quad \varepsilon > 0, \quad \text{there exists} \quad \delta > 0$$
$$\text{such that for all} \quad f \in \mathcal{F},\ z, w \in K, \quad |z - w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Remark 6.18. Recall that $\mathbb{R}$ is **separable**, that is, it has a countable dense subset. This means that there exists a countable set $S \subset \mathbb{R}$ such that

$$\text{for all} \quad x \in \mathbb{R},\ \varepsilon > 0 \quad \text{there exists} \quad y \in S \quad \text{such that} \quad |x - y| < \varepsilon.$$

Indeed one can take $S = \mathbb{Q}$ (the rationals).

One can similarly show that $\mathbb{R}^n$ is separable for $n \ge 2$. As $\mathbb{C}$ inherits its metric space structure from $\mathbb{R}^2$, it follows that $\mathbb{C}$ is separable.

Lemma 6.19 (Arzelá–Ascoli theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and $\mathcal{F}$ a family of functions $f : \Omega \to \mathbb{C}$. If $\mathcal{F}$ is locally uniformly bounded and locally uniformly equicontinuous, then $\mathcal{F}$ is a normal family.*

Proof. We use a "diagonalization" argument.

Let $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ and let $K \subset \Omega$ be compact. Let $\{w_j\}_{j=1}^\infty$ be a dense subset of K.

As the sequence $\{f_n(w_1)\}$ is uniformly bounded, there exists a subsequence $\{f_n^1\}$ such that $f_n^1(w_1)$ converges.

Similarly, from the sequence $\{f_n^1\}$ we can extract a sequence $\{f_n^2\}$ such that $f_n^2(w_2)$ converges. Note that $f_n^2(w_1)$ also converges.

Proceeding inductively, we can construct subsequences $\{f_n^k\}$ such that $f_n^k(w_j)$ converges for $j = 1, \dots, k$.

Now consider the diagonal sequence $g_n = f_n^n$. By construction $g_n(w_j)$ converges for all j.

We will now show that in fact g_n converges uniformly on K.

Let $\varepsilon > 0$ and (by equicontinuity) choose $\delta > 0$ so that

$$\text{for all} \quad f \in \mathcal{F},\ z, w \in K, \quad \text{we have} \quad |z - w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

By the denseness of $\{w_j\}$ and compactness of K,

$$\text{there exists} \quad J \in \mathbb{N} \quad \text{such that} \quad K \subset \bigcup_{j=1}^J B_\delta(w_j).$$

We may now find $N \in \mathbb{N}$ large enough that

$$n, m > N \implies |g_n(w_j) - g_m(w_j)| < \varepsilon \quad \text{for all} \quad j = 1, \dots, J.$$

Now let $z \in K$. Then there exists $j \in \{1, \dots, J\}$ such that $z \in B_\delta(w_j)$. Thus for $n, m > N$ we have

$$\begin{aligned} |g_n(z) - g_m(z)| &\le |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| \\ &\qquad + |g_m(w_j) - g_m(z)| \\ &< 3\varepsilon. \end{aligned}$$

It follows that $\{g_n\}$ is uniformly Cauchy on K, and thus converges uniformly on K (see Exercise 6.11).

We have shown: for any compact set K, $\{f_n\}$ has a subsequence that converges uniformly on K.

However, we need to find *one* subsequence that converges uniformly on every compact set.

To this end, for each ℓ we define

$$K_\ell = \{z \in \Omega : |z| \le \ell \quad \text{and} \quad \inf_{w \in \mathbb{C} \setminus \{\Omega\}} |z - w| \ge \tfrac{1}{\ell}\}.$$

Then each K_ℓ is compact, $K_\ell \subset K_{\ell+1}$, and $\Omega = \cup_\ell K_\ell$.

Now let $\{f_n^1\}$ be a subsequence of $\{f_n\}$ that converges uniformly on K_1; let $\{f_n^2\}$ be a subsequence of $\{f_n^1\}$ that converges uniformly on K_2, and so on.

Now consider the diagonal sequence $g_n = f_n^n$. Then $\{g_n\}$ converges uniformly on each K_ℓ.

Since any compact $K \subset \Omega$ is contained in some K_ℓ, it follows that $\{g_n\}$ converges uniformly on every compact subset of Ω. $\square$

The next result tells us that for a family of holomorphic functions, boundedness implies equicontinuity "for free".

Theorem 6.20 (Montel's theorem)**.** *Suppose $\mathcal{F}$ is a family of holomorphic functions that is locally uniformly bounded. Then $\mathcal{F}$ is locally uniformly equicontinuous, and hence (by Arzelá–Ascoli) $\mathcal{F}$ is a normal family.*

Proof. Let $K \subset \Omega$ be compact. By compactness, we may find $r > 0$ such that $B_{3r}(z) \subset \Omega$ for all $z \in K$ (see Exercise 6.12).

We next define the set

$$S_r = \{\alpha \in \Omega : \inf_{\beta \in K} |\alpha - \beta| \leq 2r\}$$

and note that S is compact (*why?*). Thus by assumption there exists $A_r > 0$ such that

$$|f(\alpha)| \leq A_r \quad \text{for} \quad \alpha \in S_r,\ f \in \mathcal{F}$$

Now let $z, w \in K$ with $|z - w| < r$.

Using the Cauchy integral formula and the fact that $\partial B_{2r}(w) \subset S_r$, we find that for $f \in \mathcal{F}$ we have

$$\begin{aligned} |f(z) - f(w)| &= \left| \tfrac{1}{2\pi i} \int_{\partial B_{2r}(w)} f(\alpha) \left[\frac{1}{\alpha - z} - \frac{1}{\alpha - w} \right] d\alpha \right| \\ &\leq \frac{1}{2\pi} \int_{\partial B_{2r}(w)} |f(\alpha)| \, \frac{|z - w|}{|\alpha - z| \, |\alpha - w|} \, d\alpha \\ &\leq \frac{4\pi r}{2\pi} \frac{|z - w|}{2r \cdot r} \sup_{\alpha \in \partial B_{2r}(w)} |f(\alpha)| \\ &\leq \frac{A_r}{r} |z - w|. \end{aligned}$$

Hence given $\varepsilon > 0$, we may choose $\delta < \min\{r, \frac{\varepsilon r}{A_r}\}$ and it follows that for $z, w \in K$ we have

$$|z - w| < \delta \implies |f(z) - f(w)| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

As K was arbitrary, we conclude that $\mathcal{F}$ is locally uniformly equicontinuous, as needed. □

6.7. **The Riemann Mapping Theorem.** We turn to the main result in our study of conformal mappings.

Theorem 6.21 (Riemann mapping theorem)**.** *Let $\emptyset \neq \Omega \subsetneq \mathbb{C}$ be simply connected and $z_0 \in \Omega$. Then there exists a unique biholomorphism $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.*

As a consequence, if $\emptyset \neq U, V \subsetneq \mathbb{C}$ are simply connected, then $U \sim V$.

Remark 6.22. The uniqueness statement follows immediately, since if F and $\tilde{F}$ are two such biholomorphisms, then $g = F \circ \tilde{F}^{-1} \in \mathrm{Aut}(\mathbb{D})$ with $g(0) = 0$, so that $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. As $g'(0) > 0$ we must have $g(z) = z$, i.e. $F = \tilde{F}$.

Also, to see that any simply connected $U, V \subsetneq \mathbb{C}$ are biholomorphic, we simply recall that $U \sim \mathbb{D}$ and $\mathbb{D} \sim V$ implies $U \sim V$. □

As mentioned above, we will construct the biholomorphism as a limit of functions.

As such, the following lemma will be useful.

Lemma 6.23. *Let $\Omega \subset \mathbb{C}$ be open and connected. Suppose $\{f_n\}$ is a sequence of injective functions $f_n : \Omega \to \mathbb{C}$ that converge locally uniformly to the function $f : \Omega \to \mathbb{C}$. Then f is either injective or constant.*

Proof. First note that as f is the locally uniform limit of holomorphic functions, it is holomorphic.

Suppose that f is not injective, so that there exist distinct $z_1, z_2 \in \Omega$ such that $f(z_1) = f(z_2)$.

We will show that f is constant.

Define $g_n(z) = f_n(z) - f_n(z_1)$.

As each f_n is injective, we see that each g_n has exactly one zero in Ω at $z = z_1$.

We also note that $\{g_n\}$ converges locally uniformly on Ω to $g(z) := f(z) - f(z_1)$.

Suppose g is not identically zero. Then g has an isolated zero at z_2 (since Ω is connected).

For a sufficiently small circle γ around z_2, we can guarantee that g does not vanish on γ and $z_1 \notin \gamma \cup \operatorname{interior}(\gamma)$.

Using the argument principle and the fact that $\frac{1}{g_n} \to \frac{1}{g}$ and $g_n' \to g'$ uniformly on γ, we deduce

$$0 \equiv \frac{1}{2\pi i} \int_\gamma \frac{g_n'(z)}{g_n(z)}\,dz \to \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)}\,dz = 1,$$

a contradiction. Thus g must be identically zero, that is, $f(z) \equiv f(z_1)$. □

Proof of Theorem 6.21. We proceed in three main steps.

Step 1. We show that there exists an open set $U \subset \mathbb{D}$ such that $\Omega \sim U$ and $0 \in U$.

To this end, pick $\alpha \in \mathbb{C}\backslash\Omega$. As the holomorphic function $z \mapsto z - \alpha$ is nonzero on Ω we may define a holomorphic function $f : \Omega \to \mathbb{C}$ such that

$$e^{f(z)} = z - \alpha. \quad (*)$$

In particular f is injective.

We now fix $w \in \Omega$. We claim that there exists $\varepsilon > 0$ such that

$$|f(z) - (f(w) + 2\pi i)| > \varepsilon \quad \text{for all} \quad z \in \Omega.$$

Indeed, otherwise we may find $\{z_n\} \subset \Omega$ such that $f(z_n) \to f(w) + 2\pi i$.

But then

$$e^{f(z_n)} = z_n - \alpha \to e^{f(w)} = w - \alpha, \quad \text{so that} \quad z_n \to w.$$

However $f(z_n) \to f(w) + 2\pi i \neq f(w)$, so this contradicts the continuity of f.

It follows that the function $F : \Omega \to \mathbb{C}$

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}$$

is a holomorphic, injective, and bounded function.

In particular F is a biholomorphism onto its (open) image.

As F is bounded, we may translate and rescale F so that $F(\Omega) \subset \mathbb{D}$ and $0 \in F(\Omega)$.

Step 2. By Step 1, we may assume without loss of generality that $\Omega \subset \mathbb{D}$ is an open set with $0 \in \Omega$. Define

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \,|\, f \text{ is holomorphic, injective, and } f(0) = 0\}.$$

In this step we find $f \in \mathcal{F}$ that maximizes $|f'(0)|$.

Note that $\mathcal{F} \neq \emptyset$, since it contains the function $f(z) = z$.

It is easy to see that $\mathcal{F}$ is uniformly bounded, since $|f(z)| \leq 1$ for all $f \in \mathcal{F}$ and $z \in \Omega$.

In fact, by the Cauchy integral formulas we can deduce that

$$s := \sup_{f \in \mathcal{F}} |f'(0)| < \infty.$$

We now choose a sequence $\{f_n\} \subset \mathcal{F}$ such that $|f_n'(0)| \to s$ as $n \to \infty$.

By Montel's theorem, this sequence converges locally uniformly along a subsequence to a holomorphic function $f : \Omega \to \mathbb{C}$ with $|f'(0)| = s$.

Note that since $z \mapsto z$ belongs to $\mathcal{F}$, we must have $s \geq 1$.

Thus by the lemma we find that f is non-constant and hence injective.

By continuity we find $\sup |f| \leq 1$, and since f is non-constant the maximum principle implies $\sup |f| < 1$.

Finally, since $f(0) = 0$, we conclude that $f \in \mathcal{F}$ with $|f'(0)| = s$.

Step 3. We show that $f : \Omega \to \mathbb{D}$ is a biholomorphism.

It suffices to show that f is onto.

Suppose toward a contradiction that

$$\text{there exists} \quad \alpha \in \mathbb{D} \quad \text{such that} \quad f(z) \neq \alpha \quad \text{for all} \quad z \in \Omega.$$

Consider $\psi_\alpha \in \operatorname{Aut}(\mathbb{D})$ and define the set

$$A = \psi_\alpha \circ f(\Omega).$$

As Ω is simply connected, so is A. (This follows from the continuity of $\psi_\alpha \circ f$ and the open mapping theorem.)

Furthermore $\alpha \notin f(\Omega) \implies 0 \notin A$, so that we may define a branch of the logarithm $\log_A$ on A.

Now consider the square root function $g : A \to \mathbb{C}$ given by

$$g(z) = e^{\frac{1}{2}\log_A(z)}.$$

Now define $F : \Omega \to \mathbb{C}$ by

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f.$$

We now notice that $F \in \mathcal{F}$. Indeed, F is holomorphic and satisfies

$$F(0) = \psi_{g(\alpha)} \circ g \circ \psi_\alpha(0) = \psi_{g(\alpha)} \circ g(\alpha) = 0.$$

Moreover, $F(\Omega) \subset \mathbb{D}$ since this is true for each function in the composition. (Note for instance that $|g(z)|^2 = |z| < 1$ for $z \in \mathbb{D}$.)

Finally, we note that F is injective as well since each function in the composition is.

We now define $h(z) = z^2$ and recall $\psi_\alpha^{-1} = \psi_\alpha$. Then

$$f = \Phi \circ F, \quad \text{with} \quad \Phi = \psi_\alpha \circ h \circ \psi_{g(\alpha)}.$$

Now $\Phi : \mathbb{D} \to \mathbb{D}$ is holomorphic with $\Phi(0) = 0$, but it is not injective because h is not.

Thus by the Schwarz lemma we conclude $\Phi'(0) < 1$.

However,

$$f'(0) = \Phi'(F(0))F'(0) = \Phi'(0)F'(0) \implies |f'(0)| < |F'(0)|,$$

contradicting the fact that f maximizes $|\varphi'(0)|$ for $\varphi \in \mathcal{F}$.

We conclude f is onto, as needed.

To complete the proof, we simply note that we may multiply f by some $e^{i\theta}$ to guarantee that $f'(0) > 0$. $\square$

6.8. **Exercises.**

Exercise 6.1. (i) Let (X, d) and $(Y, \tilde{d})$ be metric spaces and $f : X \to Y$. Show that f is continuous on X **if and only if**

$$\text{for all open} \quad U \subset Y, \quad f^{-1}(U) \quad \text{is open in } X,$$

where

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

(ii) Suppose $U, V \subset \mathbb{C}$ are open and non-empty and $f : U \to V$ is a biholomorphism. Show that the inverse $f^{-1} : V \to U$ is continuous.
Remark: For (i) use the definition of continuity from Definition 1.10.

Exercise 6.2. Suppose $U, V \subset \mathbb{C}$ are open and biholomorphic.
(i) Show that $\mathrm{Aut}(U)$ and $\mathrm{Aut}(V)$ are isomorphic.
(ii) Show that if U is simply connected, then so is V.

Exercise 6.3. A (real) $n \times n$ matrix M is **orthogonal** if $MM^t = Id$, where ^{t} denotes transpose and Id is the $n \times n$ identity matrix. If M is orthogonal and $\det M = 1$, we call M a **rotation**.

(i) Show that the set of orthogonal matrices forms a subgroup of $GL_2(\mathbb{R})$ and that the set of rotations forms a subgroup of $SL_2(\mathbb{R})$. (These groups are known as the orthogonal group, denoted $O(n)$, and the special orthogonal group, denoted $SO(n)$, respectively.)

(ii) Show that M is orthogonal **if and only if** $\langle Mv, Mw \rangle_{\mathbb{R}^n} = \langle v, w \rangle_{\mathbb{R}^n}$ for all $v, w \in \mathbb{R}^n$.

(iii) Show that if M is orthogonal then the angle between $v, w \in \mathbb{R}^n$ equals the angle between Mv and Mw.

Remark. A few useful facts: $\det A^t = \det A$, $\det AB = \det A \cdot \det B$, $(AB)^t = B^t A^t$.

Exercise 6.4. Show that $h(z) = -\frac{1}{2}(z + \frac{1}{z})$ is a holomorphic injective function from $\{z \in \mathbb{D} : \mathrm{Im}\, z > 0\}$ to $\mathbb{H}$.

Exercise 6.5. Consider the inverse of the stereographic projection map:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(x, y) = \left(\tfrac{x}{1+x^2+y^2}, \tfrac{y}{1+x^2+y^2}, \tfrac{x^2+y^2}{1+x^2+y^2}\right).$$

For $(x, y) \in \mathbb{R}^2$ let $M(x, y)$ denote the 3×2 matrix of partial derivatives of f at (x, y). Show that for all (x, y), one can write

$$[M(x, y)]^t M(x, y) = g(x, y) Id,$$

where ^{t} denotes transpose, Id is the 2×2 identity matrix, and $g : \mathbb{R}^2 \to \mathbb{R}$ is a strictly positive function.

Remark. This computation implies that stereographic projection is conformal. (*Why?*)

Exercise 6.6. For each pair of sets U, V below, find a Möbius transformation taking U to V.
(i) $U = \{z \in \mathbb{C} : |z| > 1\}, \quad V = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$,
(ii) $U = \mathbb{D}, \quad V = \{z \in \mathbb{C} : \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4}\}$.

Exercise 6.7. Let $M, N \in GL_2(\mathbb{C})$. Show that $f_{MN} = f_M \circ f_N$, where f_M is the Möbius transformation associated with M.

Exercise 6.8. Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and satisfy $|f(z)| \leq 1$ for $z \in \mathbb{H}$ and $f(i) = 0$. Show that

$$|f(z)| \leq \left| \frac{z-i}{z+i} \right| \quad \text{for} \quad z \in \mathbb{H}.$$

Exercise 6.9. A **fixed point** of a function f is a point z such that $f(z) = z$.

(i) Show that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and has two distinct fixed points, then $f(z) = z$ for $z \in \mathbb{D}$.

(ii) True or false: every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ must have a fixed point. (Prove that your answer is correct.)

Exercise 6.10. Let

$$A = \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}, \quad M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that if $f_{A^{-1}} \circ g \circ f_A = e^{-2i\theta}$ then $g = f_M$.

Exercise 6.11. Suppose $\{f_n\}$ is a sequence of functions $f_n : \Omega \to \mathbb{C}$. Show that if $\{f_n\}$ is uniformly Cauchy on Ω, then $\{f_n\}$ converges uniformly on Ω.

Remark. Here uniformly Cauchy means that for any $\varepsilon > 0$ there exists N such that for any $n, m \geq N$ and any $z \in \Omega$ we have $|f_n(z) - f_m(z)| < \varepsilon$.

Exercise 6.12. Let $\Omega \subset \mathbb{C}$ be open and $K \subset \Omega$ be compact. Show that there exists $r > 0$ such that for all $z \in K$ we have $B_r(z) \subset \Omega$.

Remark. As Ω is open, it follows that for each $z \in K$ there exists $r > 0$ such that $B_r(z) \subset \Omega$. The point is that if K is compact, we can find a single $r > 0$ that works for all $z \in K$. The proof of this fact is essentially included in the proof of Theorem 1.8; see (1.2) therein.

Exercise 6.13. Let $\{f_n\}$ be a sequence of functions $f_n : \Omega \to \mathbb{C}$ and $\{w_j\}_{j=1}^{\infty} \subset \Omega$. Suppose that for each $k \geq 1$ we have a subsequence $\{f_n^k\}$ of $\{f_n\}$ such that

$$\{f_n^{k+1}\} \quad \text{is a subsequence of} \quad \{f_n^k\}$$

and

$$\{f_n^k(w_j)\} \quad \text{converges for} \quad j = 1, \ldots, k.$$

Define the subsequence $\{g_n\}$ by $g_n = f_n^n$. Show that $\{g_n(w_j)\}$ converges for all j.

Exercise 6.14. Suppose that $\{K_\ell\}$ is a sequence of compact sets such that $K_\ell \subset K_{\ell+1}$ for each ℓ. Suppose that $\{f_n\}$ is a sequence of a functions and that for each $\ell \geq 1$ we have a subsequence $\{f_n^\ell\}$ of $\{f_n\}$ such that

$$\{f_n^{\ell+1}\} \quad \text{is a subsequence of} \quad \{f_n^\ell\}$$

and

$$\{f_n^\ell\} \quad \text{converges uniformly on } K_\ell.$$

Define the subsequence $\{g_n\}$ by $g_n = f_n^n$. Show that $\{g_n\}$ converges uniformly on each K_ℓ.

7. The Prime Number Theorem

We next discuss an application of complex analysis to number theory. Our main reference for this section is Chapter XIV in Gamelin's *Complex Analysis*.

7.1. Preliminaries.

Definition 7.1 (Prime)**.** Let $p \in \mathbb{N}$. We call p **prime** if $p > 1$ and p has no positive divisors other than 1 and p.

Convention. We use n to refer to arbitrary natural numbers, while p always refers to primes.

We recall (without proof) the following essential fact about prime numbers.

Theorem 7.2 (Fundamental theorem of arithmetic)**.** *Every $n > 1$ can be written uniquely as a product of powers of primes.*

The prime number theorem addresses the question of the asymptotic distribution of primes. For this question even to make sense, we first need the following:

Theorem 7.3 (Euclid, 300 BC)**.** *There are infinitely many primes.*

Proof. Suppose there were only finitely many, say $p_1, \dots, p_n$. Now some prime p_j must divide $p_1 \cdots p_n + 1$. But since p_j also divides $p_1 \cdots p_n$ we now deduce that p_j divides 1, a contradiction. □

Definition 7.4. We define $\quad \pi(n) = \#\{p : p \leq n\}$.

Definition 7.5 (Asymptotic notation)**.** We write $f(n) \sim g(n)$ as $n \to \infty$ to denote

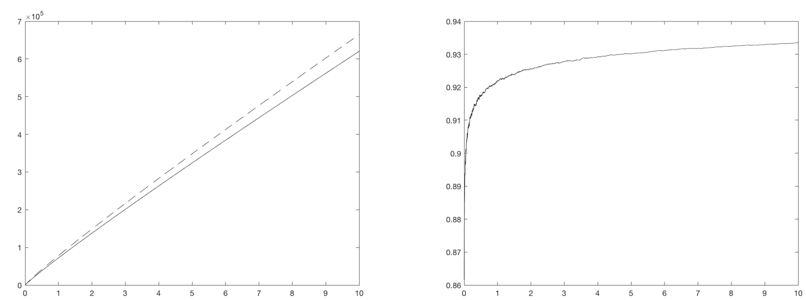
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$

The goal of this section is to prove the following:

Theorem 7.6 (Prime number theorem, Hadamard/de la Vallée Poussin, 1896)**.**

$$\pi(n) \sim \frac{n}{\log n} \quad \textit{as} \quad n \to \infty.$$

The following figure depicts $n/\log n$ (solid line) versus $\pi(n)$ (dashed line) for n between $1{,}000$ and $10{,}000{,}000$ (left), along with the ratio of $\frac{n}{\log n} \div \pi(n)$ (right).



The proof will rely on the analysis of some special functions, the first of which is the following:

Definition 7.7. We define $\quad \vartheta(x) = \sum_{p \leq x} \log p \quad$ for $x > 0$.

The next proposition makes the role of ϑ clear:

Proposition 7.8. *The prime number theorem holds if and only if*

$$\vartheta(n) \sim n \quad \textit{as} \quad n \to \infty.$$

Proof. We first note that

$$0 \le \vartheta(n) \le \pi(n) \log n \quad \text{for} \quad n \ge 1.$$

Next we fix $0 < \varepsilon < 1$. Then

$$\begin{aligned}\vartheta(n) &\ge \sum_{n^{1-\varepsilon} < p \le n} \log p \\ &\ge (1-\varepsilon) \log n \big[\pi(n) - \pi(n^{1-\varepsilon})\big] \\ &\ge (1-\varepsilon) \log n \big[\pi(n) - n^{1-\varepsilon}\big].\end{aligned}$$

Combining the two estimates above we deduce

$$\frac{\vartheta(n)}{n} \le \pi(n) \frac{\log n}{n} \le \frac{1}{1-\varepsilon} \frac{\vartheta(n)}{n} + \frac{\log n}{n^{\varepsilon}} \quad \text{for} \quad n \ge 1.$$

As

$$\lim_{n\to\infty} \frac{\log n}{n^{\varepsilon}} = 0$$

and $\varepsilon > 0$ was arbitrary, we deduce that

$$\vartheta(n) \sim n \quad \text{if and only if} \quad \pi(n) \sim \frac{n}{\log n}.$$

□

As a warmup, let's prove the following bound (due to Chebyshev):

Lemma 7.9. *For all $x \ge 1$ we have $\vartheta(x) \le (4 \log 2) x$.*

Proof. We consider the binomial coefficient $b_n := \begin{pmatrix} 2n \\ n \end{pmatrix}$ and claim the following:

(i) $b_n < 2^{2n}$
(ii) the product $\prod_{n<p<2n} p$ divides b_n (and hence is less than 2^{2n}).

For (i) we recall that b_n counts the number of subsets of $(1, \dots, 2n)$ with n elements, while 2^{2n} counts the total number of subsets of $(1, \dots, 2n)$.

For (ii) we argue as follows. Since

$$b_n = \frac{(2n)!}{n!n!} = \frac{(n+1)\cdots(2n)}{1\cdots n}$$

is an integer, we know that $1 \cdots n$ divides $(n+1)\cdots(2n)$. However, $1\cdots n$ cannot divide any prime between n and $2n$, and hence we deduce that

$$\frac{(n+1)\cdots(2n)}{1\cdots n \cdot \prod_{n<p<2n} p}$$

is an integer, as needed.

Thus we have

$$\sum_{n<p<2n} \log p = \log\Bigg(\prod_{n<p<2n} p\Bigg) \le \log(2^{2n}) \le 2n \log 2,$$

and so

$$\vartheta(2^m) = \sum_{k=1}^{m} \sum_{2^{k-1}<p<2^k} \log p \leq 2\log 2 \sum_{k=1}^{m} 2^{k-1} \leq 2^{m+1}\log 2.$$

Now for $x > 0$ we choose m so that $2^{m-1} < x \leq 2^m$. Then

$$\vartheta(x) \leq \vartheta(2^m) \leq 2^{m+1}\log 2 \leq (4\log 2)x,$$

which completes the proof. □

That is all we will say about ϑ for the moment. We turn now to our next special function.

7.2. **The Riemann Zeta Function.**

Definition 7.10. For $s > 1$ we define $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Proposition 7.11. *The series defining ζ converges on $\{s \in \mathbb{C} : \mathrm{Re}\, s > 1\}$ and defines a holomorphic function there.*

Proof. We first note that if $\varepsilon > 0$ and $S_\varepsilon = \{s \in \mathbb{C} : \mathrm{Re}\, s \geq 1+\varepsilon\}$, then the series $\sum n^{-s}$ converges absolutely uniformly for $s \in S_\varepsilon$. Indeed, if $s = \sigma + it \in S_\varepsilon$, we have

$$|n^{-s}| = n^{-\sigma}|n^{-it}| = n^{-\sigma}|e^{-it\log n}| = n^{-\sigma},$$

and hence we can use the comparison test with the series $\sum n^{-(1+\varepsilon)}$.

Thus ζ is the locally uniform limit of the holomorphic functions $f_N(s) := \sum_{n=1}^{N} n^{-s}$ on the set $\{s \in \mathbb{C} : \mathrm{Re}\, s > 1\}$, which implies the result. □

The next lemma demonstrates a clear connection between ζ and the primes.

Lemma 7.12. *For $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$, we have*

$$\frac{1}{\zeta(s)} = \prod_p (1 - \tfrac{1}{p^s}).$$

In particular $\zeta(s) \neq 0$ if $\mathrm{Re}\, s > 1$.

Proof. We first note that $\sum_p \frac{1}{p^s}$ converges absolutely (locally uniformly) on $\{s \in \mathbb{C} : \mathrm{Re}\, s > 1\}$.

Thus the product above converges.

We now claim that for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$ we have

$$\prod_p \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s), \quad (*)$$

which implies the result.

To prove $(*)$ we first note that for p prime and $\mathrm{Re}\, s > 1$ we have

$$\frac{1}{1-p^{-s}} = 1 + p^{-s} + p^{-2s} + \cdots.$$

If we apply this to the first m primes, say $p_1, \ldots, p_m$, then multiply, we find

$$\prod_{\ell=1}^{m} \frac{1}{1-p_\ell^{-s}} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} (p_1^{k_1} \cdots p_m^{k_m})^{-s}.$$

By the fundamental theorem of artithmetic, every $n > 1$ can be written uniquely as a product of powers of primes.

Thus each summand n^{-s} appears at most once in the sum above, and as we send $m \to \infty$ we will eventually cover each n^{-s}. This proves $(*)$. $\square$

The next result lets us extend ζ beyond the line $\{s \in \mathbb{C} : \operatorname{Re} s = 1\}$.

Lemma 7.13. *The function*

$$s \mapsto \zeta(s) - \frac{1}{s-1}$$

has an analytic continuation to the set $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$*. In particular,* ζ *has a meromorphic continuation to* $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ *with a single simple pole at* $s = 1$ *with* $\operatorname{res}_1 \zeta = 1$.

Proof. First for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we can write

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} n^{-s} - \int_1^{\infty} x^{-s}\,dx = \sum_{n=1}^{\infty} \left(\int_n^{n+1} [n^{-s} - x^{-s}]\,dx \right).$$

We now claim that the series on the right actually converges absolutely (locally uniformly) whenever $\operatorname{Re} s > 0$, which implies the result.

Indeed, we can write

$$\int_n^{n+1} [n^{-s} - x^{-s}]\,dx = \int_n^{n+1} \int_n^x s u^{-(s+1)}\,du\,dx,$$

so that

$$\left| \int_n^{n+1} [n^{-s} - x^{-s}]\,dx \right| \leq |s| \max_{u \in [n,n+1]} |u^{-(s+1)}|.$$

As

$$|u^{-(s+1)}| = u^{-(\operatorname{Re} s+1)} \leq n^{-(\operatorname{Re} s+1)}$$

for $u \in [n, n+1]$, the claim follows by comparison with the series $\sum_n n^{-(\operatorname{Re} s+1)}$. $\square$

Remark 7.14. One can actually show that ζ has a meromorphic continuation into all of $\mathbb{C}$, with no other singularities than the pole at $s = 1$. We will not pursue this direction.

We next study the zeros of the ζ function, which will also lead to our final special function.

We begin by using Lemma 7.12 to write

$$\log\left(\tfrac{1}{\zeta(s)}\right) = \log\left(\prod_p 1 - \tfrac{1}{p^s} \right) = \sum_p \log(1 - p^{-s}).$$

As

$$p^{-s} = e^{-s \log p} \implies \tfrac{d}{ds}\left(p^{-s}\right) = -\log p \cdot e^{-s\log p} = -\log p \cdot p^{-s},$$

we find

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s - 1}. \quad (*)$$

We split the sum into two pieces:

$$\sum_p \frac{\log p}{p^s - 1} = \sum_p \frac{\log p}{p^s} + \sum_p \frac{\log p}{p^s(p^s - 1)}$$

and define

$$\Phi(s) := \sum_p \frac{\log p}{p^s}.$$

Note that Φ converges absolutely and defines a holomorphic function on $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$.

In fact, using Lemma 7.13 and $(*)$ we can say more:

Lemma 7.15. *The function Φ has a meromorphic continuation to $\{s \in \mathbb{C} : \mathit{Re}\, s > \frac{1}{2}\}$, with simple poles precisely at the poles and zeros of ζ.*

Proof. We first rewrite $(*)$ as

$$\Phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \sum_p \frac{\log p}{p^s(p^s - 1)}.$$

As ζ is meromorphic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ and the function

$$s \mapsto \sum_p \frac{\log p}{p^s(p^s - 1)}$$

defines a holomorphic function on $\{s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2}\}$, we deduce that Φ has a meromorphic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2}\}$.

Furthermore, the formula above also shows that Φ has simple poles at the poles and zeros of ζ. (*Why?*) $\square$

Remark 7.16. We can now see that Φ has a simple pole at $s = 1$, since ζ does. Moreover, the formula above implies that $\operatorname{res}_1 \Phi = \operatorname{res}_1 \zeta = 1$.

Finally we record a result that will be crucial for the proof of the prime number theorem.

Proposition 7.17. *The function ζ has no zeros on the line $\ell := \{s \in \mathbb{C} : \mathit{Re}\, s = 1\}$. Thus Φ has no poles on ℓ other than the pole at $s = 1$.*

Proof. A fair warning: the proof is rather mysterious.

Suppose toward a contradiction that $\zeta(1 + it) = 0$ for some $t \neq 0$.

In this case we find

$$|\zeta(\sigma + it)|^4 \leq C(\sigma - 1)^4 \quad \text{as} \quad \sigma \to 1^+.$$

As ζ has a simple pole at $s = 1$, we also have

$$|\zeta(\sigma)|^3 \leq C'(\sigma - 1)^{-3} \quad \text{as} \quad \sigma \to 1^+.$$

As ζ is holomorphic at $s = \sigma + 2it_0$, we have that $\zeta(\sigma + 2it_0)$ stays bounded as $\sigma \to 1^+$.

Thus we deduce

$$|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \to 0 \quad \text{as} \quad \sigma \to 1^+. \quad (*)$$

On the other hand, using the formula above and using the power series for log we have the following for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$:

$$\log\left(\tfrac{1}{\zeta(s)}\right) = \sum_p \log(1 - p^{-s}) = -\sum_p \sum_{m=1}^\infty \frac{p^{-ms}}{m} = -\sum_{n=1}^\infty c_n n^{-s},$$

where $c_n = \frac{1}{m}$ if $n = p^m$ and $c_n = 0$ otherwise.

Thus for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we have

$$\log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-s} \quad \text{for some} \quad c_n \geq 0.$$

We now let $s = \sigma + it$ for $\sigma > 1$ and note that

$$\operatorname{Re} n^{-s} = \operatorname{Re}(n^{-\sigma} e^{it \log n}) = n^{-\sigma} \cos(t \log n).$$

Thus

$$\begin{aligned}
&\log|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \\
&\quad = 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\
&\quad = 3\text{Re } \log\zeta(\sigma) + 4\text{Re } \log\zeta(\sigma+it) + \text{Re } \log\zeta(\sigma+2it) \\
&\quad = \sum_{n=1}^{\infty} c_n n^{-\sigma}[3 + 4\cos(t\log n) + \cos(2t\log n)] \\
&\quad = \sum_{n=1}^{\infty} 2c_n n^{-\sigma}\big[1 + \cos(t\log n)\big]^2. \qquad (\textit{Check!})
\end{aligned}$$

In particular

$$\log|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 0,$$

which contradicts $(*)$ since $\log x$ is negative for $x \in (0,1)$. $\square$

Remark 7.18. We now know that ζ has no zeros on $\{s \in \mathbb{C} : \operatorname{Re} s \geq 1\}$.

One can show (via the so-called "functional equation" for ζ) that the only zeros of ζ in $\{s \in \mathbb{C} : \operatorname{Re} s \leq 0\}$ are the "trivial zeros" at the negative even integers.

Thus all "non-trivial" zeros of ζ lie in the "critical strip" $S := \{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}$.

It is known that ζ has infinitely many zeros in S, and in fact their asymptotic distribution is known.

The **Riemann hypothesis** states that all of the zeros of ζ in S lie on $\{s \in \mathbb{C} : \operatorname{Re} s = \frac{1}{2}\}$.

The Riemann hypothesis is one of the most famous open problems in mathematics; solving it will earn you a million dollar prize (not to mention international fame!).

7.3. **Laplace Transforms.** We next introduce an analytic tool that will play an important role in the proof of the prime number theorem.

Definition 7.19 (Laplace transform)**.** Let $h : [0,\infty) \to \mathbb{R}$ be piecewise continuous and have order of growth $\leq \rho$. The **Laplace transform of** h is the function

$$(\mathcal{L}h)(s) = \int_0^{\infty} e^{-st} h(t)\, dt,$$

which defines a holomorphic function on $\{s \in \mathbb{C} : \operatorname{Re} s > \rho\}$.

Laplace transforms show up in a variety of settings. They are frequently applied in the context of ODEs and electrical engineering, for example.

We have a specific goal in mind, so we will not pursue the general theory. Instead we will only prove the following:

Proposition 7.20. *Suppose $h : [0,\infty) \to \mathbb{R}$ is a bounded piecewise continuous function. Suppose $\mathcal{L}h$ has an analytic continuation across the imaginary axis. Then*

$$\lim_{T\to\infty} \int_0^T h(t)\,dt = \lim_{x\to 0+} (\mathcal{L}h)(x).$$

Proof. Let g denote the analytic continuation of $\mathcal{L}h$ and let $\varepsilon > 0$.

For $T > 0$ we define

$$g_T(z) = \int_0^T e^{-zs} h(s)\,ds$$

and note that g_T is an entire function.

To prove the theorem it suffices to show

$$|g(0) - g_T(0)| < 4\varepsilon \quad \text{for all} \quad T \quad \text{sufficiently large.} \quad (*)$$

Let M denote an upper bound for h, and choose $R > 0$ large enough that

$$\tfrac{M}{R} < \varepsilon.$$

We next choose $\delta > 0$ small enough that g is holomorphic in an open set containing

$$\Omega_\delta = \{z \in B_R(0) : \operatorname{Re} z > -\delta\}.$$

By the Cauchy integral formula, we have

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_{\partial\Omega_\delta} [g(z) - g_T(z)] \underbrace{e^{zT}\big(1 + \tfrac{z^2}{R^2}\big)}_{:=F(z)} \tfrac{dz}{z}.$$

We write

$$\partial\Omega_\delta = \gamma_1 \cup \gamma_2 \cup \gamma_3,$$

where

- $\gamma_1 = \{z \in \partial\Omega_\delta : \operatorname{Re} z > 0\}$,
- $\gamma_2 = \{z \in \partial\Omega_\delta : \operatorname{Re} z = -\delta\}$,
- $\gamma_3 = \partial\Omega_\delta \backslash (\gamma_1 \cup \gamma_2)$ (the small arcs).

For $z = x + iy \in \gamma_1$ we have

$$|g(z) - g_T(z)| \leq \left| \int_T^\infty e^{-sz} h(s)\,ds \right| \leq M \int_T^\infty e^{-sx}\,ds \leq \frac{Me^{-xT}}{x}.$$

As $|1 + \frac{z^2}{R^2}| = 2\frac{|x|}{R}$ whenever $|z| = R$, we deduce

$$\left| \frac{1}{2\pi i} \int_{\gamma_1} [g(z) - g_T(z)] F(z)\,\tfrac{dz}{z} \right| \leq \frac{\pi R}{2\pi} \frac{Me^{-xT}}{x} \frac{e^{xT}}{R} \frac{2x}{R} \leq \frac{M}{R} < \varepsilon.$$

To proceed, we treat g and g_T separately.

As g_T is entire, we may write

$$\frac{1}{2\pi i} \int_{\gamma_2 \cup \gamma_3} g_T(z) F(z)\,\tfrac{dz}{z} = \frac{1}{2\pi i} \int_{\gamma_4} g_T(z) F(z)\,\tfrac{dz}{z},$$

where

$$\gamma_4 = \{z \in B_R(0) : \operatorname{Re} z < 0\}.$$

For $z = x + iy \in \gamma_4$ we estimate

$$|g_T(z)| = \left| \int_0^T e^{-sz} h(s)\,ds \right| \leq M \int_0^T e^{-sx}\,ds \leq \frac{Me^{-xT}}{|x|}.$$

Thus as above we have

$$\left|\frac{1}{2\pi i}\int_{\gamma_4} g_T(z)F(z)\,\tfrac{dz}{z}\right| \leq \frac{\pi R}{2\pi}\frac{Me^{-xT}}{|x|}\frac{e^{xT}}{R}\frac{2|x|}{R} \leq \frac{M}{R} < \varepsilon.$$

We next note that for $z \in \gamma_3$ we have

$$|e^{zT}| = e^{RT\cos\arg z} \leq 1$$

and that the length of γ_3 tends to zero as $\delta \to 0$. Thus for δ small enough we find

$$\left|\frac{1}{2\pi i}\int_{\gamma_3} g(z)F(z)\,\tfrac{dz}{z}\right| < \varepsilon \quad \text{for any} \quad T > 0.$$

Finally for $z \in \gamma_2$ we have $|e^{zT}| = e^{-\delta T}$, thus for all T sufficiently large we get

$$\left|\frac{1}{2\pi i}\int_{\gamma_2} g(z)F(z)\,\tfrac{dz}{z}\right| < \varepsilon.$$

Collecting the estimates above gives $(*)$, as needed. □

7.4. **Proof of the Prime Number Theorem.** We turn to the proof of the prime number theorem. Recall from Proposition 7.8 that it suffices to show

$$\vartheta(n) \sim n \quad \text{as} \quad n \to \infty, \quad \text{where} \quad \vartheta(x) = \sum_{p \leq x} \log p.$$

We begin with a lemma that brings together some ideas from the previous sections:

Lemma 7.21. *For $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$ we have*

$$\big(\mathcal{L}\vartheta(e^t)\big)(s) = \frac{1}{s}\Phi(s).$$

Proof. By Lemma 7.9, the function $t \mapsto \vartheta(e^t)$ has order of growth ≤ 1. Thus its Laplace transform defines a holomorphic function on $\{s \in \mathbb{C} : \mathrm{Re}\, s > 1\}$.

Let p_n denote the n^{th} prime. Then $\vartheta(e^t)$ is constant for $\log p_n < t < \log p_{n+1}$, so that

$$\int_{\log p_n}^{\log p_{n+1}} e^{-st}\vartheta(e^t)\,dt = \vartheta(p_n)\frac{e^{-st}}{-s}\bigg|_{t=\log p_n}^{\log p_{n+1}} = \frac{1}{s}\vartheta(p_n)\big(p_n^{-s} - p_{n+1}^{-s}\big).$$

Summing over n and using that

$$\vartheta(p_n) - \vartheta(p_{n-1}) = \log p_n,$$

we deduce

$$\begin{aligned}\int_0^\infty e^{-st}\vartheta(e^t)\,dt &= \frac{1}{s}\sum_n \vartheta(p_n)(p_n^{-s} - p_{n+1}^{-s}) \\ &= \frac{1}{s}\sum_n \big[\vartheta(p_n) - \vartheta(p_{n-1})\big]p_n^{-s} \\ &= \frac{1}{s}\sum_p \frac{\log p}{p^s},\end{aligned}$$

which completes the proof. □

We next consider the function

$$h(t) = \vartheta(e^t)e^{-t} - 1.$$

Lemma 7.22. *The Laplace transform of h is given by the formula*

$$(\mathcal{L}h)(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$$

and has an analytic continuation across the imaginary axis.

Proof. By the previous lemma, we can compute that for $\operatorname{Re} s > 0$ we have

$$\begin{aligned}(\mathcal{L}h)(s) &= \int_0^\infty e^{-st}[\vartheta(e^t)e^{-t} - 1]\,dt \\ &= \int_0^\infty e^{-(s+1)t}\vartheta(e^t)\,dt - \int_0^\infty e^{-st}\,dt \\ &= \frac{\Phi(s+1)}{s+1} - \frac{1}{s}.\end{aligned}$$

We know that Φ is holomorphic for $\operatorname{Re} s > 1$, and by Lemma 7.15 we know that Φ has a mermomorphic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2}\}$.

Moreover, from Proposition 7.17 we know that the only pole of Φ on the line $\{s \in \mathbb{C} : \operatorname{Re} s = 1\}$ is the simple pole at $s = 1$, with $\operatorname{res}_1 \Phi = 1$.

We conclude that the function

$$s \mapsto \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$$

can be continued analytically across the imaginary axis, which completes the proof. □

We now have all of the ingredients we need to prove the prime number theorem.

Proof of Theorem 7.6. We first use Lemma 7.22 and Proposition 7.20 to conclude that

$$\lim_{T\to\infty} \int_0^T [\vartheta(e^t)e^{-t} - 1]\,dt$$

exists. Changing variables via $x = e^t$, this implies

$$\lim_{R\to\infty} \int_1^R \big[\tfrac{\vartheta(x)}{x} - 1\big]\,\tfrac{dx}{x} \quad \text{exists.} \quad (*)$$

We will now use $(*)$ to show that $\vartheta(x) \sim x$ as $x \to \infty$, which by Proposition 7.8 is equivalent to the prime number theorem.

First we suppose toward a contradiction that $\vartheta(x) > (1+\varepsilon)x$ for some $\varepsilon > 0$ and for arbitrarily large x.

As ϑ is increasing, for any such x we have

$$\begin{aligned}\int_x^{(1+\varepsilon)x} \big[\tfrac{\vartheta(t)}{t} - 1\big]\,\tfrac{dt}{t} &\geq \int_x^{(1+\varepsilon)} \big[\tfrac{\vartheta(x)}{t} - 1\big]\,\tfrac{dt}{t} \\ &\geq \int_x^{(1+\varepsilon)x} \big[(1+\varepsilon)\tfrac{x}{t} - 1\big]\,\tfrac{dt}{t} \\ &\geq \int_1^{1+\varepsilon} \big[\tfrac{1+\varepsilon}{r} - 1\big]\,\tfrac{dr}{r} \\ &\geq \varepsilon - \log(1+\varepsilon) > 0.\end{aligned}$$

Thus as $R \to \infty$ we have that

$$\int_1^R \left[\frac{\vartheta(t)}{t} - 1\right] \frac{dt}{t}$$

increases by $\varepsilon - \log(1+\varepsilon)$ over infinitely many disjoint intervals of the form $(x, (1+\varepsilon)x)$, which contradicts $(*)$.

Arguing similarly, we find that we cannot have $\vartheta(x) < (1-\varepsilon)x$ for some $\varepsilon > 0$ for arbitrarily large values of x.

We conclude that $\vartheta(x) \sim x$ as $x \to \infty$, which completes the proof. □

7.5. **Exercises.**

Exercise 7.1. Show that

$$\int_0^1 x^s \, dx = \frac{1}{s+1}$$

for $s \in \mathbb{C}$ such that $\operatorname{Re} s > -1$.

Hint. Recall Exercise 3.21.

Exercise 7.2. Suppose F is a meromorphic function on $\mathbb{C}$ and define $f(z) = \frac{F'(z)}{F(z)}$. Prove the following:

(i) If F has a zero of order n at z_0, then f has a simple pole at z_0 and $\operatorname{res}_{z_0} f = n$.

(ii) If F has a pole of order m at z_0, then f has a simple pole at z_0 and $\operatorname{res}_{z_0} f = -m$.

Exercise 7.3. For $s > 0$ define the **gamma function** by the convergent integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt.$$

(i) Show that the integral defining Γ defines a holomorphic function on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

(ii) Show that Γ has a meromorphic continuation to all of $\mathbb{C}$. Determine its poles, their orders, and their residues.

Hints. For $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ write

$$\Gamma(s) = \int_0^1 e^{-t} t^{s-1} \, dt + \int_1^\infty e^{-t} t^{s-1} \, dt.$$

Note that the second integral defines an entire function of s. Express the first integral as a series by writing e^{-t} in a power series and integrating term by term. Show that the series defines a meromorphic function on $\mathbb{C}$ and determine its poles, their orders, and their residues.

Remark. The gamma function shows up all over the place. One important fact you may try to prove is that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. To prove this, show that (i) $\Gamma(1) = 1$ and (ii) $\Gamma(s+1) = s\Gamma(s)$ for $\operatorname{Re} s > 0$. For (ii), integrate by parts in the integral formula for Γ.