The nonlinear Schrödinger equation with an inverse-square potential

Jason Murphy

Abstract. We discuss recent results on the scattering/blow-up dichotomy below the ground state threshold for the focusing nonlinear Schrödinger equation with an inverse-square potential and a nonlinearity that is mass-supercritical and energy-subcritical.

1. Introduction

Nonlinear Schrödinger equations (NLS) with power-type nonlinearities comprise a class of PDE of wide mathematical and physical interest. The most basic equations of this type take the form

$$i\partial_t u = -\Delta u + \mu |u|^p u, \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$, $p > 0$, and $\mu \in \{\pm 1\}$ (giving the defocusing and focusing cases, respectively). One may also consider such equations in the presence of an external potential, in which case $-\Delta$ is replaced by a Schrödinger operator of the form $-\Delta + V$. In this article, we consider the case of an inverse-square potential $V(x) = a|x|^{-2}$.

More precisely, we consider the Schrödinger operator

$$\mathcal{L}_a = -\Delta + a|x|^{-2}, \quad a > -(\tfrac{d-2}{2})^2, \tag{1.2}$$

in dimensions $d \geq 3$, defined via the Friedrichs extension with domain $C_c^\infty(\mathbb{R}^d \backslash \{0\})$. The restriction on a comes from the sharp Hardy inequality—it guarantees that $\mathcal{L}_a$ is a positive operator. The operator $\mathcal{L}_a$ appears in a variety of physical settings and is an interesting borderline case from the mathematical point of view.

The addition of an external potential to (1.1) often breaks symmetries of the underlying equation, resulting in new analytic challenges. The inverse-square potential breaks space-translation symmetry; however, one feature of this particular model is that it retains the scaling symmetry

$$u(t,x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x). \tag{1.3}$$

This suggests that one cannot generally rely on perturbative techniques, as the potential and the Laplacian are of equal strength at every scale.

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Two cases of (1.1) have received a great deal of attention, singled out by the fact that in these cases the equation has a conserved quantity that is invariant under the scaling symmetry of the equation. These are the mass- and energy-critical NLS, corresponding to $p = \frac{4}{d}$ and $p = \frac{4}{d-2}$ (in dimensions $d \geq 3$), respectively. The intercritical case refers to $p \in (\frac{4}{d}, \frac{4}{d-2})$.

For the NLS with an inverse-square potential, scattering in the defocusing energy-critical case was established in [**12**], while scattering/blowup dichotomies below the ground state threshold in the focusing intercritical setting were established in [**9**, **10**]. These parallel results previously obtained for the standard NLS (see e.g. [**1**, **3**–**8**]). In fact, a wealth of refined results have been established for (1.1); accordingly, there are many directions of research to be pursued for the inverse-square potential.

To treat the nonlinear equation with an inverse-square potential firstly requires a good linear theory (e.g. Strichartz estimates [**2**]), as well as a set of harmonic analysis tools adapted to the inverse-square potential (see [**12**]). With these ingredients, many of the strategies developed to treat (1.1) (e.g. the concentration compactness approach to induction on energy) can be adapted to attack problems involving the inverse-square potential.

The results discussed in this article, which appeared originally in [**9**,**10**], concern the long-time behavior of solutions to the intercritical focusing NLS with an inverse square potential:

$$i\partial_t u = \mathcal{L}_a u - |u|^p u, \tag{1.4}$$

where $u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$, $\frac{4}{d} < p < \frac{4}{d-2}$ and $d \geq 3$.

For $a \in (-(\frac{d-2}{2})^2, 0]$, equation (1.4) admits a global but non-scattering solution of the form $u(t) = e^{it} Q_a$, where Q_a (the 'ground state') solves the elliptic problem

$$-\mathcal{L}_a Q_a - Q_a + |Q_a|^p Q_a = 0. \tag{1.5}$$

The functions Q_a may be constructed as optimizers to the Gagliardo–Nirenberg inequality

$$\|f\|_{L^{p+2}}^{p+2} \leq C_a \|f\|_{L^2}^{\frac{4-(d-2)p}{2}} \|f\|_{\dot{H}^1_a}^{\frac{dp}{2}}, \tag{1.6}$$

where $\dot{H}^1_a$ is the homogeneous Sobolev space defined in terms of the operator $\mathcal{L}_a$[1]. For $a > 0$, the sharp constant C_a is equal to C_0; however, no optimizers exist[2].

The functions Q_a, which lead to global but non-scattering solutions to (1.4), provide a natural threshold below which one can prove a simple scattering/blow-up dichotomy.

THEOREM 1.1 (Scattering/blowup dichotomy, [**9**,**10**]). *Suppose* $d \geq 3$, $\frac{4}{d} < p < \frac{4}{d-2}$, *and* $a > -(\frac{d-2}{2})^2$ *are as in* (1.9) *below, and let* $u_0 \in H^1(\mathbb{R}^d)$.

There exists a unique maximal-lifespan solution u *to* (1.4) *with* $u|_{t=0} = u_0$. *If* u_0 *is below the ground state threshold, in the sense that*

$$M(u_0)^{\frac{4-p(d-2)}{dp-4}} E_a(u_0) < M(Q_{a\wedge 0})^{\frac{4-p(d-2)}{dp-4}} E_{a\wedge 0}(Q_{a\wedge 0}), \tag{1.7}$$

where M *and* E_a *are defined in* (1.10) *and* $a \wedge 0 = \min\{a, 0\}$, *then the following dichotomy holds:*

[1]For $a > -(\frac{d-2}{2})^2$, the spaces $\dot{H}^1_a$ and $\dot{H}^1$ are equivalent by the sharp Hardy inequality.

[2]This is due to the lack of compactness coming from space translation. When $a \leq 0$, the compactness can be restored via radial rearrangements and the Riesz rearrangement inequality.

(i) *If*

$$\|u_0\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|u_0\|_{\dot H^1_a} < \|Q_{a\wedge 0}\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|Q_{a\wedge 0}\|_{\dot H^1_{a\wedge 0}}, \tag{1.8}$$

then u is global in time and scatters in both time directions; that is, there exist solutions $v_\pm$ to the linear equation $i\partial_t v_\pm = \mathcal{L}_a v_\pm$ such that

$$\lim_{t\to\pm\infty} \|u(t) - v_\pm(t)\|_{H^1} = 0.$$

(ii) *If*

$$\|u_0\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|u_0\|_{\dot H^1_a} > \|Q_{a\wedge 0}\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|Q_{a\wedge 0}\|_{\dot H^1_{a\wedge 0}},$$

and u_0 is radial or $xu_0 \in L^2$, then u blows up in finite time in both time directions.

Remark 1.2. The precise set of (d,p,a) under consideration is the following:

$$\begin{cases} a > -(\frac{d-2}{2})^2 & \text{for} \quad d=3 \quad \text{and} \quad \frac{4}{3} < p \le 2, \\ a > -(\frac{d-2}{2})^2 + (\frac{d-2}{2} - \frac{1}{p})^2 & \text{for} \quad 3 \le d \le 6 \quad \text{and} \quad \frac{2}{d-2} \vee \frac{4}{d} < p < \frac{4}{d-2}, \end{cases} \tag{1.9}$$

where $x \vee y = \max\{x,y\}$. These constraints come from the H^1 local well-posedness theory and are related to the issue of equivalence of Sobolev spaces defined in terms of $\mathcal{L}_a$ and the standard Sobolev spaces. We discuss the local theory in Section 2.

Remark 1.3. The conserved mass and energy of a solution u to (1.4) are defined by

$$M(u) = \int_{\mathbb{R}^d} |u|^2\,dx, \quad E_a(u) = \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u|^2 + \tfrac{a}{2|x|^2}|u|^2 - \tfrac{1}{p+2}|u|^{p+2}\,dx, \tag{1.10}$$

respectively. Note that $M(u_0), |E_a(u_0)| < \infty$ for $u_0 \in H^1$ (cf. (1.6) above). The power appearing in (1.7) is chosen to make the product invariant under the scaling symmetry (1.3).

Remark 1.4. The analogue of Theorem 1.1 for equation (1.1) was originally established in [**4**, **6**–**8**], and the proof of Theorem 1.1 follows a similar overall strategy. However, because of the broken space-translation symmetry, it is important to understand the sense in which (1.1) may be 'embedded' into (1.4) in certain limiting regimes; in particular, one needs to use the analogous result for (1.1) as a black box in order to treat the equation with broken symmetry. For a further discussion, see Lemma 3.4.

In the rest of this article, we describe the proof of Theorem 1.1. The starting point is the following virial identity, which follows from a direct computation using (1.4): Let u be a solution to (1.4) and

$$V(t) := \int |u(t,x)|^2|x|^2\,dx.$$

Then

$$\begin{aligned} V'(t) &= 4\mathrm{Im}\int \bar u \nabla u \cdot x\,dx, \\ V''(t) &= 8\|u(t)\|_{\dot H^1_a}^2 - \tfrac{4pd}{p+2}\|u(t)\|_{L^{p+2}}^{p+2}. \end{aligned} \tag{1.11}$$

Combining this identity with the sharp Gagliardo–Nirenberg inequality (1.6), one deduces that in scenario (i) of Theorem 1.1 one has $V''(t) > 0$, while in scenario (ii) one has $V''(t) < 0$. In particular, in scenario (i), one may expect the solution to spread out and scatter ($V(t) \to \infty$), while in scenario (ii), one may expect the solution to concentrate and blow up in finite time ($V(t) \to 0$ in finite time).

In fact, using the virial identity to establish blow-up in scenario (ii) is fairly standard, and so for the remainder of the article we will focus on proving scattering in scenario (i).

Because we work with merely H^1 data, one cannot implement the virial identity directly—indeed, none of the quantities appearing in the identity are necessarily finite. We therefore follow the concentration compactness approach to induction on energy: we reduce the problem of proving scattering to the problem of precluding the existence of a special type of non-scattering solution. In particular, we reduce to the problem of precluding solutions satisfying (1.7) and (1.8) that have a pre-compact orbit in H^1; for such solutions, a localized version of the virial identity may be implemented in order to reach a contradiction (see Section 3.1).

The main problem therefore reduces to proving the existence of compact 'minimal blowup solutions' under the assumption that Theorem 1.1(i) fails. We will discuss this problem in Section 3; it is here that the broken translation symmetry present in (1.4) plays its most important role.

2. Linear and local theory

In order to treat the nonlinear problem (1.4), it is important to have (i) good estimates for the underlying linear equation and (ii) harmonic analysis tools adapted to the Schrödinger operator $\mathcal{L}_a$.

Strichartz estimates for the linear propagator $e^{-it\mathcal{L}_a}$ were established in [**2**]:

THEOREM 2.1 (Strichartz estimates, [**2**]). *Let $a > -(\frac{d-2}{2})^2$ and $d \geq 3$. Let (q,r) and $(\tilde{q},\tilde{r})$ be admissible pairs, i.e.*

$$2 \leq q, \tilde{q} \leq \infty \quad \text{and} \quad \tfrac{2}{q} + \tfrac{d}{r} = \tfrac{2}{\tilde{q}} + \tfrac{d}{\tilde{r}} = \tfrac{d}{2},$$

with $(q,\tilde{q}) \neq (2,2)$. Suppose $u : I \times \mathbb{R}^d \to \mathbb{C}$ solves

$$(i\partial_t - \mathcal{L}_a)u = F.$$

Then for any $t_0 \in I$, the following estimate holds:

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|u(t_0)\|_{L_x^2(\mathbb{R}^d)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^d)}.$$

One constructs solutions to (1.4) via a fixed point argument and Strichartz estimates; in particular, solutions satisfy the Duhamel formula

$$u(t) = e^{-it\mathcal{L}_a}u_0 + i\int_0^t e^{-i(t-s)\mathcal{L}_a}(|u|^p u)(s)\,ds.$$

One closes the estimates in spaces of the form $C_t H_a^1 \cap L_t^q H_a^{1,r}$, where (q,r) is an admissible pair and $H_a^{s,r}$ denotes the Sobolev space defined in terms of $\mathcal{L}_a$, i.e.

$$\|f\|_{H_a^{s,r}(\mathbb{R}^d)} = \|(1+\mathcal{L}_a)^{\frac{s}{2}} f\|_{L^r(\mathbb{R}^d)}.$$

In particular, one needs to estimate $\sqrt{\mathcal{L}_a}(|u|^p u)$ in some dual Strichartz space. For this purpose, it is useful to substitute $\sqrt{\mathcal{L}_a}$ with ∇, use the chain rule, and subsequently exchange ∇ with $\sqrt{\mathcal{L}_a}$ again. For this to work, however, one needs to know that Sobolev spaces defined in terms of $\mathcal{L}_a$ are equivalent to those defined in terms of Δ. The sharp range of exponents for which this is the case was worked out in [**11**]:

LEMMA 2.2 (Equivalence of Sobolev spaces, [**11**]). *Let* $d \geq 3$, $a > -(\frac{d-2}{2})^2$, *and*

$$\rho = \tfrac{d-2}{2} - \big[\big(\tfrac{d-2}{2}\big)^2 + a\big]^{\frac{1}{2}}.$$

Let $0 < s < 2$. *If* $1 < p < \infty$ *satisfies* $\frac{s+\rho}{d} < \frac{1}{p} < \min\{1, \frac{d-\rho}{d}\}$, *then*

$$\||\nabla|^s f\|_{L_x^p} \lesssim_{d,p,s} \|(\mathcal{L}_a)^{\frac{s}{2}} f\|_{L_x^p} \quad \textit{for all} \quad f \in C_c^\infty(\mathbb{R}^d \backslash \{0\}).$$

If $\max\{\frac{s}{d}, \frac{\rho}{d}\} < \frac{1}{p} < \min\{1, \frac{d-\rho}{d}\}$, *then*

$$\|(\mathcal{L}_a)^{\frac{s}{2}} f\|_{L_x^p} \lesssim_{d,p,s} \||\nabla|^s f\|_{L_x^p} \quad \textit{for all} \quad f \in C_c^\infty(\mathbb{R}^d \backslash \{0\}).$$

Using these ideas, one can establish an H^1 well-posedness theory for (1.4), at least for (d, p, a) obeying (1.9). With

$$q_0 = \tfrac{p(d+2)}{2},$$

we have the following result.

THEOREM 2.3 (Local well-posedness). *Let* $t_0 \in \mathbb{R}$, $u_0 \in H^1$, *and suppose* (d, p, a) *satisfy* (1.9).

- *There exist* $T = T(\|u_0\|_{H^1}) > 0$ *and a unique solution* u *to* (1.1) *on* $(t_0 - T, t_0 + T)$ *with* $u(t_0) = u_0$. *In particular, if* u *remains uniformly bounded in* H^1 *throughout its lifespan, then* u *extends to a global solution.*
- *Furthermore, there exists* $\eta_0 > 0$ *so that if*

  $$\|e^{-i(t-t_0)\mathcal{L}} u_0\|_{L_{t,x}^{q_0}((t_0,\infty)\times\mathbb{R}^d)} < \eta \quad \textit{for some} \quad 0 < \eta < \eta_0,$$

  then u *is forward global and obeys*

  $$\|u\|_{L_{t,x}^{q_0}((t_0,\infty)\times\mathbb{R}^d)} \lesssim \eta.$$

  The analogous statement holds backward in time and on all of $\mathbb{R}$.
- *Finally, for any* $\psi \in H^1$ *there exists a solution to* (1.1) *that scatters to* ψ *as* $t \to \infty$, *and the analogous statement holds backward in time.*

Using persistence of regularity arguments, one can also show that if $u(t_0) \in H^1$ and u obeys $L_{t,x}^{q_0}$-bounds on some interval $I \ni t_0$, then in fact $(1+\mathcal{L}_a)^{\frac{1}{2}} u$ is bounded in every Strichartz norm on I. Consequently, if u remains uniformly bounded in $L_{t,x}^{q_0}$ throughout its lifespan, then u is global and scatters.

Related to the local theory is the stability theory for (1.4), which concerns approximate solutions $\tilde{u}$ to (1.4) and will be important in Section 3.

THEOREM 2.4 (Stability). *Let* (d, p, a) *satisfy* (1.9). *Let* $\tilde{u}$ *solve*

$$i\partial_t \tilde{u} = \mathcal{L}_a \tilde{u} - |\tilde{u}|^p \tilde{u} + e$$

on an interval I *for some function* e. *Suppose*

$$\|u_0\|_{H^1} + \|\tilde{u}(t_0)\|_{H^1} \leq E, \quad \|\tilde{u}\|_{L_{t,x}^{q_0}(I\times\mathbb{R}^d)} \leq L.$$

There exists $\varepsilon_0(E, L) > 0$ *so that if* $0 < \varepsilon < \varepsilon_0$ *and*

$$\|u_0 - \tilde{u}(t_0)\|_{H^1} + \||\nabla|^{s_c} e\|_{N(I)} < \varepsilon,$$

where $s_c = \frac{d}{2} - \frac{2}{p}$ *and* N *is a sum of dual Strichartz spaces, then there exists a solution* u *to* (1.4) *with* $u(t_0) = u_0$ *satsifying*

$$\|(\mathcal{L}_a)^{\frac{s_c}{2}} [u - \tilde{u}]\|_{S(I)} \lesssim \varepsilon, \quad \|(1 + \mathcal{L}_a)^{\frac{1}{2}} u\|_{S(I)} \lesssim_{E,L} 1$$

for any Strichartz space S.

Remark 2.5. The parameter s_c is the critical regularity associated to (1.4). For $\frac{4}{d} < p < \frac{4}{d-2}$, we have $0 < s_c < 1$.

2.1. Harmonic analysis adapted to $\mathcal{L}_a$. In the following, it will be important to have a set of harmonic analysis tools adapted to $\mathcal{L}_a$. Such a toolkit was developed in [**11**], where the central ingredient was a Mihklin-type multiplier theorem for functions of $\mathcal{L}_a$.

We make use of Littlewood–Paley projections defined via the heat kernel:

$$P_N^a := e^{-\mathcal{L}_a/N^2} - e^{-4\mathcal{L}_a/N^2} \quad \text{for} \quad N \in 2^{\mathbb{Z}}.$$

In order to state results, it is convenient to define

$$\tilde{q} := \begin{cases} \infty & \text{if } a \geq 0, \\ \frac{d}{\rho} & \text{if } -(\frac{d-2}{2})^2 < a < 0, \end{cases} \qquad \text{where} \quad \rho = \tfrac{d-2}{2} - \big[\big(\tfrac{d-2}{2}\big)^2 + a\big]^{\frac{1}{2}}.$$

We write $\tilde{q}'$ for the dual exponent to $\tilde{q}$. We summarize the tools we need in the following:

Lemma 2.6 (Harmonic analysis tools, [**11**]). *For $\tilde{q}' < q \leq r < \tilde{q}$,*

$$f = \sum_{N \in 2^{\mathbb{Z}}} P_N^a f \quad \textit{as elements of} \quad L_x^r.$$

Furthermore, we have the following Bernstein estimates:

(i) *The operators P_N^a are bounded on L_x^r.*
(ii) *The operators P_N^a map L_x^q to L_x^r, with norm $\mathcal{O}(N^{\frac{d}{q}-\frac{d}{r}})$.*
(iii) *For any $s \in \mathbb{R}$,*

$$N^s \|P_N^a f\|_{L_x^r} \sim \big\|(\mathcal{L}_a)^{\frac{s}{2}} P_N^a f\big\|_{L_x^r}.$$

Finally, for $0 \leq s < 2$, we have the square function estimate:

$$\bigg\|\bigg(\sum_{N \in 2^{\mathbb{Z}}} N^{2s} |P_N^a f|^2\bigg)^{\frac{1}{2}}\bigg\|_{L_x^r} \sim \|(\mathcal{L}_a)^{\frac{s}{2}} f\|_{L_x^r}.$$

3. Construction and exclusion of minimal blowup solutions

As described in the introduction, the most important step in the proof of Theorem 1.1(i) is to show that if the result fails, there exists a minimal blowup solution with certain compactness properties. In this section, we sketch the proof of this fact. At the end of the section, we also sketch the proof that such solutions cannot exist, completing the sketch of the proof of Theorem 1.1(i).

We first define

$$L(\mathcal{E}) := \sup\big\{\|u\|_{L_{t,x}^{q_0}(I \times \mathbb{R}^d)}\big\}, \quad q_0 = \tfrac{p(d+2)}{2},$$

where the supremum is taken over all maximal-lifespan solutions $u : I \times \mathbb{R}^d$ such that

$$M(u)^{\frac{4-p(d-2)}{dp-4}} E_a(u) \leq \mathcal{E} \quad \text{and} \quad \|u(t)\|_{L_x^2}^{\frac{4-p(d-2)}{dp-4}} \|u(t)\|_{\dot{H}_a^1} < \mathcal{K}_a$$

for some $t \in I$. Recall from Theorem 2.3 that uniform boundedness of the $L_{t,x}^{q_0}$-norm throughout the maximal lifespan of a solution implies scattering. Recall also that $L(\mathcal{E})$ is finite for all $\mathcal{E}$ sufficiently small. Thus if Theorem 1.1(i) fails, there exists some critical $\mathcal{E}_c$ such that

$$0<\mathcal{E}_c<M(Q_{a\wedge 0})^{\frac{4-p(d-2)}{dp-4}}E_{a\wedge 0}(Q_{a\wedge 0})$$

and

$$L(\mathcal{E})<\infty \quad \text{for} \quad \mathcal{E}<\mathcal{E}_c, \quad L(\mathcal{E})=\infty \quad \text{for} \quad \mathcal{E}>\mathcal{E}_c.$$

The main result of this section is the following:

Theorem 3.1 (Existence of minimal blowup solutions). *Suppose Theorem 1.1(i) fails. Then there exists a global solution v to* (1.4) *satisfying*

$$M(v)=1, \quad E_a(v)=\mathcal{E}_c, \quad \|v\|_{L^{q_0}_{t,x}((-\infty,0)\times\mathbb{R}^d)}=\|v\|_{L^{q_0}_{t,x}((0,\infty)\times\mathbb{R}^d)}=\infty.$$

Moreover, the orbit $\{v(t)\}_{t\in\mathbb{R}}$ is precompact in H^1.

In fact, Theorem 3.1 can be deduced from the following lemma:

Lemma 3.2. *Let (d,p,a) satisfy* (1.9). *Suppose u_n is a sequence of solutions to* (1.4) *such that*

$$M(u_n)\equiv 1, \quad E_a(u_n)\nearrow \mathcal{E}_c,$$

and suppose t_n are such that

$$\|u_n(t_n)\|_{L^2}^{\frac{4-p(d-2)}{dp-4}}\|u_n(t_n)\|_{\dot H^1_a}<\|Q_{a\wedge 0}\|_{L^2}^{\frac{4-p(d-2)}{dp-4}}\|Q_{a\wedge 0}\|_{\dot H^1_{a\wedge 0}}, \tag{3.1}$$

$$\lim_{n\to\infty}\|u_n\|_{L^{q_0}_{t,x}(\{t>t_n\}\times\mathbb{R}^d)}=\lim_{n\to\infty}\|u_n\|_{L^{q_0}_{t,x}(\{t<t_n\}\times\mathbb{R}^d)}=\infty. \tag{3.2}$$

Then $\{u_n(t_n)\}$ converges along a subsequence in H^1.

Proof of Theorem 3.1. Assume that Theorem 1.1(i) fails. Applying a rescaling, we may find a sequence of solutions u_n and times t_n as in Lemma 3.2. Let v be the solution to (1.1) with initial data given by the subsequential limit of $u_n(t_n)$. Another application of Lemma 3.2 to $v(\tau_n)$ (for an arbitrary sequence τ_n) demonstrates the precompactness. □

We turn to a discussion of the proof of Lemma 3.2. To begin, we remark that the solutions u_n are global. In fact, using

$$M(u_n)^{\frac{4-p(d-2)}{dp-4}}E_a(u_n)<M(Q_{a\wedge 0})^{\frac{4-p(d-2)}{dp-4}}E_{a\wedge 0}(Q_{a\wedge 0})$$

and (3.1), the sharp Gagliardo–Nirenberg inequality implies that (3.1) holds throughout the lifespan of u_n. Thus the u_n remain uniformly bounded in H^1 and hence (by Theorem 2.3) are global.

The proof of Lemma 3.2 proceeds in several steps.

Step 1. After translating so that $t_n\equiv 0$, the first step is to expand the H^1-bounded sequence $\{u_n(0)\}$ in a linear profile decomposition, which provides a way of quantifying any possible lack of compactness in the sequence.

Lemma 3.3 (Linear profile decomposition). *Passing to a subsequence, there exist $J^*\in\{0,1,2,\dots,\infty\}$, $\{\phi^j\}_{j=1}^{J^*}\subset H^1$, and $\{(t_n^j,x_n^j)\}_{j=1}^{J^*}\subset\mathbb{R}\times\mathbb{R}^d$ such that for each J,*

$$u_n(0)=\sum_{j=1}^{J}\phi_n^j+r_n^J, \tag{3.3}$$

where $\phi_n^j = [e^{it_n^j \mathcal{L}_a^{n_j}}\phi^j](\cdot - x_n^j)$, *with* $\mathcal{L}_a^{n_j} := -\Delta + \frac{a}{|x+x_n^j|^2}$. *The remainder* r_n^J *satisfies*

$$\big(e^{-it_n^J\mathcal{L}_a} r_n^J\big)(x+x_n^J) \rightharpoonup 0 \quad \text{weakly in} \quad H^1,$$
$$\lim_{J\to J^*}\limsup_{n\to\infty} \|e^{-it\mathcal{L}_a} r_n^J\|_{L^{q_0}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} = 0.$$

The parameters (t_n^j, x_n^j) *are asymptotically orthogonal: for any* $j \neq k$,

$$\lim_{n\to\infty}\big(|t_n^j - t_n^k| + |x_n^j - x_n^k|\big) = \infty. \tag{3.4}$$

Furthermore, for each j, *we may assume that either* $t_n^j \to \pm\infty$ *or* $t_n^j \equiv 0$, *and either* $|x_n^j| \to \infty$ *or* $x_n^j \equiv 0$. *Finally, we have the following decoupling:*

$$\lim_{n\to\infty}\Big\{\|(\mathcal{L}_a)^{\frac{s}{2}} u_n(0)\|_{L_x^2}^2 - \sum_{j=1}^J \|(\mathcal{L}_a)^{\frac{s}{2}}\phi_n^j\|_{L_x^2}^2 - \|(\mathcal{L}_a)^{\frac{s}{2}} r_n^J\|_{L_x^2}^2\Big\} = 0, \ s\in\{0,1\},$$
$$\lim_{n\to\infty}\Big\{\|u_n(0)\|_{L_x^{p+2}}^{p+2} - \sum_{j=1}^J \|\phi_n^j\|_{L_x^{p+2}}^{p+2} - \|r_n^J\|_{L_x^{p+2}}^{p+2}\Big\} = 0.$$

Using the energy decoupling and the sharp Gagliardo–Nirenberg inequality, one can verify that each profile carries positive energy. Note that with this decomposition, the proof of Lemma 3.2 boils down to proving $J = 1$, $t_n^1 \equiv 0$, $x_n^1 \equiv 0$, and $r_n^1 \to 0$ strongly in H^1. Before proceeding to this, we say a few words about the proof of Lemma 3.3.

To prove Lemma 3.3, one successively extracts bubbles of Strichartz-norm concentration from the sequence $\{u_n(0)\}$, each with a well-defined position in space-time and scale. The starting point is a refined Strichartz estimate of the following form:

$$\|e^{-it\mathcal{L}_a}u_n(0)\|_{L^{q_0}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_n(0)\|_{\dot H_a^{s_c}}^{\theta} \sup_{N\in 2^{\mathbb{Z}}} \|e^{-it\mathcal{L}_a}P_N^a u_n(0)\|_{L^{q_0}_{t,x}(\mathbb{R}\times\mathbb{R}^d)}^{1-\theta},$$

which allows one to identify a scale N_n^{-1} responsible for $L^{q_0}_{t,x}$-norm concentration. This estimate is proved by employing harmonic analysis tools adapted to $\mathcal{L}_a$, specifically the square function estimate and Bernstein estimates of Lemma 2.6. After identifying a scale, one can use Hölder's inequality, Strichartz estimates, and Bernstein estimates to identify a location in space-time (τ_n, x_n) where concentration occurs.

Because the sequence is H^1-bounded, one can prove upper and lower bounds for the scale N_n^{-1}; thus without loss of generality $N_n \equiv 1$ (say). One can also arrange that $x_n \to x_\infty \in \mathbb{R}^d\backslash\{0\}$ or $|x_n| \to \infty$, and that $\tau_n \to \tau_\infty \in [-\infty, \infty]$. The first profile ϕ^1 is obtained as the weak limit of the sequence

$$[e^{-it_n\mathcal{L}_a}u_n(0)](x + x_n),$$

where $t_n \equiv 0$ if $\tau_\infty \in \mathbb{R}$ and $\tau_n = t_n$ otherwise. After removing this first bubble from the sequence, one repeats the argument to extract the second bubble, and so on. One can then prove that the sequence of profiles, space-time positions, and remainders obtained in this way satisfy the conclusions of Lemma 3.3. At a technical level, one encounters some issues related to the fact that translation does not commute with $\mathcal{L}_a$, resulting in the need to prove some convergence properties for certain linear operators.

Step 2. Having written $u_n(0)$ in the form (3.3), the next step is to construct nonlinear profiles associated to each ϕ^j. If $x_n^j \equiv 0$ and $t_n^j \equiv 0$, we let v^j be the solution to (1.4) with $v^j(0) = \phi^j$; if instead $t_n^j \to \pm\infty$ we use Theorem 2.3 to find a solution that scatters to $e^{-it\mathcal{L}_a}\phi^j$ as $t \to \pm\infty$. In either case, we then set $v_n^j(t,x) = v^j(t + t_n^j, x)$.

If $|x_n^j| \to \infty$, however, we encounter a significant obstacle: because translation symmetry is broken in (1.4), we cannot simply construct the solution v^j with data ϕ^j and then define v_n^j via translation by x_n^j. Indeed, v_n^j would fail to be a solution.

The following result shows that we may construct nonlinear solutions corresponding to profiles with $|x_n^j| \to \infty$, provided they are below the *Euclidean* ground state. Furthermore, these solutions are global and scatter.

Lemma 3.4 (Embedding of nonlinear profiles). *Let (d, p, a) satisfy (1.9). Suppose $t_n \equiv 0$ or $t_n \to \pm\infty$ and assume $|x_n| \to \infty$. Let ϕ in H^1 and define*

$$\phi_n(x) = [e^{-it_n\mathcal{L}_a^n}\phi](x - x_n), \quad \mathcal{L}_a^n := -\Delta + \tfrac{a}{|x+x_n|^2}.$$

Suppose

$$M(\phi)^{\frac{4-p(d-2)}{dp-4}} E_0(\phi) < M(Q_0)^{\frac{4-p(d-2)}{dp-4}} E_0(Q_0),$$
$$\|\phi\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|\phi\|_{\dot H^1} < \|Q_0\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|Q_0\|_{\dot H^1},$$

in the case $t_n \equiv 0$ and

$$\tfrac12 \|\phi\|_{L^2}^{\frac{2[4-p(d-2)]}{dp-4}} \|\phi\|_{\dot H^1}^2 < M(Q_0)^{\frac{4-p(d-2)}{dp-4}} E_0(Q_0)$$

in the case $t_n \to \pm\infty$. Then there exists a global solution v_n to (1.4) with $v_n(0) = \phi_n$ satisfying $\|(1+\mathcal{L}_a)^{\frac12} v_n\|_{S(\mathbb{R})}$ for any Strichartz norm S.

The key to Lemma 3.4 is to observe that as $|x| \to \infty$, the effect of the potential becomes increasingly weak. In particular, we expect that the desired solution v_n should approximately solve the *free* NLS. Indeed, roughly speaking, the idea of the proof of Lemma 3.4 is as follows:

Using the assumption that ϕ is below the Euclidean ground state, one can use the results of [**1**, **4**, **7**] to construct scattering solutions $\tilde v_n$ to the free NLS (1.1) with data ϕ_n. Relying on convergence results for various linear operators (related to the sense in which $\mathcal{L}_a^n$ 'converges' to $-\Delta$ when $|x_n| \to \infty$), one can prove that the $\tilde v_n$ approximately solve (1.4). Thus, using the stability result (Theorem 2.4), one can construct a true solution v_n to (1.4) with data ϕ_n; furthermore, this solution inherits good space-time bounds from $\tilde v_n$.

It is an important but simple observation that the ground state thresholds for (1.4) are smaller than the thresholds for (1.1). In particular, because $\mathcal{E}_c$ is below the threshold appearing in Theorem 1.1, we know that all of the profiles in the decomposition of $u_n(0)$ are in fact below the Euclidean threshold; thus we may use Lemma 3.4 to construct nonlinear solutions associated to profiles with $|x_n^j| \to \infty$.

Step 3. The third step is to establish that $J^* = 1$, that is, there can be only one profile. Equivalently, this means that one profile captures all of the critical energy. This is achieved by a contradiction argument. If there were multiple profiles, then each would carry strictly less than the critical energy $\mathcal{E}_c$. In particular, by the definition of $\mathcal{E}_c$, the corresponding nonlinear profiles v_n^j would scatter and obey global space-time bounds. One then considers the following function:

$$u_n^J(t) = \sum_{j=1}^J v_n^j(t) + e^{-it\mathcal{L}_a} r_n^J.$$

This function has the following properties: (i) u_n^J agrees with the true solutions u_n at time $t=0$, (ii) u_n^J obeys global space-time bounds, and (iii) u_n^J approximately solves (1.4). Indeed, (i) follows by construction. Along with the space-time bounds obeyed by the v_n^j, the key ingredient to establishing (ii) and (iii) is the orthogonality (3.4) satisfied by the profiles. Indeed, orthogonality is essential if one hopes to (approximately) solve a nonlinear equation by a linear combination of solutions. Using properties (i)–(iii), the stability result Theorem 2.4 implies uniform space-time bounds for the solutions u_n, yielding a contradiction to (3.2).

Step 4. We have reduced the decomposition (3.3) to the form

$$u_n(0) = [e^{it_n\mathcal{L}_a^n}\phi](\cdot - x_n) + r_n, \quad \mathcal{L}_a^n = -\Delta + \tfrac{a}{|x+x_n|^2}.$$

To complete the proof of Lemma 3.2, we need to show $t_n \equiv 0$, $x_n \equiv 0$, and $r_n \to 0$ in H^1. In fact, by the energy decoupling one can see already that $r_n \to 0$ in $\dot H^1$. To see that $x_n \equiv 0$, we consider the alternative, namely $|x_n| \to \infty$. In this case, an application of the embedding result Lemma 3.4 and the stability result Theorem 2.4 would imply uniform space-time bounds for the u_n, contradicting (3.2). Similarly, if $t_n \to \pm\infty$, then an application of Theorem 2.4 (comparing u_n with the linear solutions $e^{-it\mathcal{L}_a}u_n(0)$) would contradict (3.2). Finally, to establish $r_n \to 0$ in L^2, it suffices to show $\|\phi\|_{L^2} = 1$. In fact, if $\|\phi\|_{L^2} < 1$, then the definition of $\mathcal{E}_c$ would imply that ϕ leads to a scattering solution, and Theorem 2.4 would once again lead to a contradiction to (3.2). Thus, we complete the sketch of the proof of Lemma 3.2, and hence of Theorem 3.1.

3.1. Preclusion of minimal blowup solutions.

To complete our discussion of the proof of Theorem 1.1(i), it remains to describe how one can preclude the existence of solutions as in Theorem 3.1. As mentioned in the introduction, the key is to use a localized version of the virial identity, as follows.

Suppose toward a contradiction that v is a solution as in Theorem 3.1 and let $\varepsilon > 0$. By precompactness, there exists R large enough that

$$\int_{|x|>R} |v(t,x)|^2 + |\nabla v(t,x)|^2 + |v(t,x)|^{p+2}\,dx < \varepsilon$$

uniformly in $t \in \mathbb{R}$. As v is below the ground state threshold, one can use the sharp Gagliardo–Nirenberg inequality and precompactness to deduce

$$8\|u(t)\|_{\dot H_a^1}^2 - \tfrac{4dp}{p+2}\|u(t)\|_{L^{p+2}}^{p+2} \gtrsim \|u(t)\|_{\dot H_a^1}^2 \gtrsim 1$$

uniformly in $t \in \mathbb{R}$. For ε sufficiently small, one can combine these last two bounds with identities related to (1.11) to deduce that

$$\partial_t^2 \int a_R(x)|u(t,x)|^2\,dx \gtrsim 1,$$

$$\left|\partial_t a_R(x)|u(t,x)|^2\,dx\right| \lesssim R,$$

uniformly in $t \in \mathbb{R}$, where $a_R(x)$ is a smooth function equal to $|x|^2$ for $|x| \le R$ and constant for $|x| \ge 2R$. Integrating over any interval of the form $[0,T]$ and using the fundamental theorem of calculus, one can deduce $T \lesssim R$ (uniformly in T). Choosing T sufficiently large now leads to the desired contradiction and completes the proof of Theorem 1.1(i).

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Department of Mathematics, University of California, Berkeley

Department of Mathematics and Statistics, Missouri University of Science and Technology

E-mail address: `jason.murphy@mst.edu`