SCATTERING BELOW THE GROUND STATE THRESHOLD FOR THE FOCUSING NLS

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Abstract. These notes were originally written to accompany a lecture at Fuzhou University and Fujian Normal University in June, 2018.

We consider the focusing cubic nonlinear Schrödinger equation (NLS) in three space dimensions. We discuss the problem of scattering below the ground state threshold. We first discuss the original proof due to Duyckaerts, Holmer, and Roudenko, which employed the concentration-compactness approach to induction on energy. We then discuss recent work of the author (joint with Dodson) giving simplified proofs of the same result that avoid the use of concentration-compactness. Finally, we discuss the analogous problem for the case of NLS with an inverse-square potential.

1. Introduction

We study the focusing cubic nonlinear Schrödinger equation (NLS) in three dimensions:

$$\begin{cases} i\partial_t u = -\Delta u - |u|^2 u, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^3). \end{cases} \tag{1.1}$$

We will also consider the cubic NLS in the presence of an inverse-square potential, which has the form

$$i\partial_t u = \mathcal{L}_a u - |u|^2 u, \quad \mathcal{L}_a = -\Delta + a|x|^{-2}. \tag{1.2}$$

Using Strichartz estimates, one can show that this equation is locally well-posed in H^1. Furthermore, any solution that remains bounded in H^1 extends to a global solution. For sufficiently small data, solutions scatter; in particular, there exists $u_+ \in H^1$ such that

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta}u_+\|_{H^1} = 0,$$

where $e^{it\Delta} = \mathcal{F}^{-1}e^{-it|\xi|^2}\mathcal{F}$ is the free Schrödinger group. More generally, one can prove that a solution may be extended as long as its $L^5_{t,x}$-norm remains finite, and that a global solution with finite $L^5_{t,x}$-norm on $\mathbb{R}\times\mathbb{R}^3$ scatters.

The equation (1.1) enjoys several symmetries and corresponding conservation laws. Among them are the mass, energy, and momentum, defined via

$$
\begin{aligned}
M(u) &= \int |u|^2\,dx,\\
E(u) &= \int \tfrac{1}{2}|\nabla u|^2 - \tfrac{1}{4}|u|^4\,dx,\\
P(u) &= 2\,\mathrm{Im}\int \bar{u}\nabla u\,dx.
\end{aligned}
$$

The fact that solutions with H^1 data have finite energy is a consequence of the Gagliardo–Nirenberg inequality, which we write in its sharp form as follows:

$$\|f\|_{L^4}^4 \le C_0\|f\|_{L^2}\|\nabla f\|_{L^2}^3 \quad \text{for any} \quad f\in H^1(\mathbb{R}^3). \tag{1.3}$$

The equation (1.1) admits a global but nonscattering solution of the form $u(t,x)=e^{it}Q(x)$, where Q (the ground state) is the unique, positive, radial, decaying solution to

$$-\Delta Q + Q - Q^3 = 0.$$

In fact, Q may be constructed as an optimizer of the Gagliardo–Nirenberg inequality. As we will see, Q can be used to describe a threshold between scattering and blowup behavior for solutions to (1.1). In particular, we will discuss the following theorem. In the following, we define

$$K(u) = \|u\|_{L^2}\|\nabla u\|_{L^2}.$$

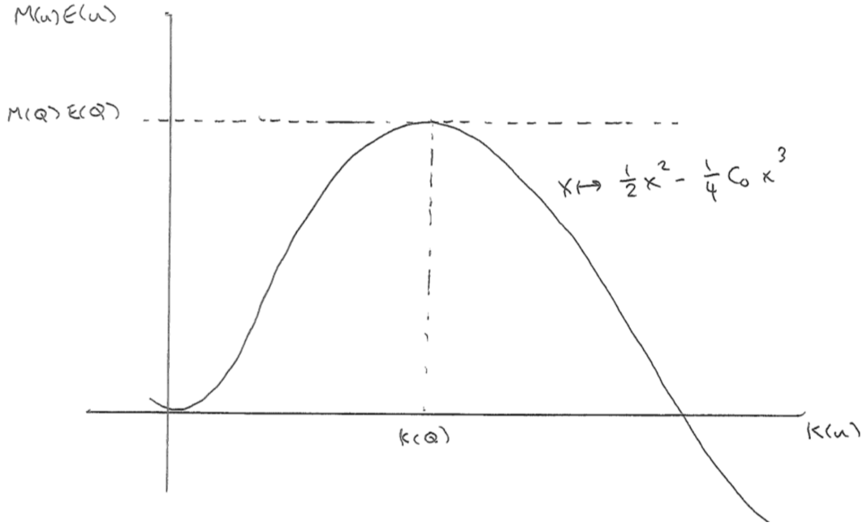
Theorem 1.1. *Let $M(u_0)E(u_0) < M(Q)E(Q)$. Let u be the corresponding solution to* (1.1)*.*

- *If $K(u_0) < K(Q)$, then u scatters.*
- *If $K(u_0) > K(Q)$, then u blows up in finite time.*

We will focus primarily on the scattering result.

This theorem is due originally to Holmer and Roudenko in the radial case and Duyckaerts, Holmer, and Roudenko in the non-radial case. We will first discuss the proofs given by these authors, which involve the concentration-compactness approach to induction on energy. We will then discuss new, simplified proofs of these results, which is joint work with B. Dodson. In particular, these proofs avoid the use of concentration-compactness entirely. Finally, we will discuss an analogue of Theorem 1.1 for (7.1), which is joint work with R. Killip, M. Visan, and J. Zheng.

The conditions in Theorem 1.1 may be pictured as follows:



From the picture (and, more precisely, a continuity argument) we see that sub-threshold solutions remain uniformly bounded in H^1 and hence are global.

2. Virial identity

The starting point for understanding Theorem 1.1 is the virial identity. We suppose u is a solution to (1.1) and define

$$V(t) = 2\,\mathrm{Im} \int \bar{u}\nabla u \cdot \nabla a(x)\,dx = \tfrac{d}{dt} \int |u|^2 a(x)\,dx$$

for a weight $a(x)$. If we let $a(x) = |x|^2$, then a computation using (1.1) and integration by parts yields

$$V'(t) = 8\|\nabla u(t)\|_{L^2}^2 - 6\|u(t)\|_{L^4}^4.$$

Roughly speaking, one expects that if $V'(t) > 0$ then the solution will scatter. For example, if we could show $V'(t) \gtrsim \|u(t)\|_{L^4}^4$, then upon integrating and using the fundamental theorem of calculus we would arrive at the a priori estimate

$$\iint |u|^4\,dx\,dt \lesssim \sup_t |V(t)|.$$

Then (if $\sup_t |V(t)|$ were finite!) we could deduce global space-time bounds, from which we may expect to prove scattering.

This computation is connected to the ground state Q through the sharp Gagliardo–Nirenberg inequality. In particular, one can show that if

$$M(u)E(u) < M(Q)E(Q) \quad \text{and} \quad K(u) < K(Q),$$

then $V'(t) > 0$.

(Similarly, when $K(u) > K(Q)$, we can prove $V' < 0$, which then implies blowup in finite time. Indeed, the second derivative of the positive quantity $\int |u|^2|x|^2\,dx$ is strictly negative.)

The problem with the computation above is that none of the quantities are necessarily finite for data in H^1; indeed, we do not necessarily have $xu_0 \in L^2$. Even if we take this as an additional assumption, the quantity $V(t)$ will grow in time, and hence we will not arrive at any useful space-time estimates.

The resolution of this issue is to use a truncated version of the $|x|^2$ weight and introduce a weight of the form $a(x) = |x|^2\chi_R$, in such a way that $|x|^2$ becomes constant after $|x| > R$ for some large R.

However, doing this introduces error terms into the virial computation that must be controlled in order to arrive at a useful estimate. In particular, one must be able to make terms like

$$\int_{|x|>R} |\nabla u(t,x)|^2\,dx$$

small by choosing R sufficiently large; the difficulty is that this must be done uniformly in time. This is where the concentration-compactness approach proves to be very useful. In particular, this strategy reduces the proof of Theorem 1.1 to the preclusion of a special type of (sub-threshold) solution that enjoys exactly the type of compactness properties that are useful in the setting of localized virial computations. This is sometimes called the 'Kenig–Merle' roadmap. Ultimately, the localized virial is used to contradict the existence of such a solution, thus completing the proof of the main result.

3. Concentration compactness

Theorem 3.1 (Existence of minimal blowup solutions)**.** *Suppose Theorem 1.1 fails. Then there exists $E_c \in (0, M(Q)E(Q))$ and a global solution u to* (1.1) *satisfying:*

- *$M(u) = 1$, $\|\nabla u(0)\|_{L^2} < K(Q)$, and $E(u) = E_c$,*
- *u blows up in both time directions:*

$$\|u\|_{L^5_{t,x}((-\infty,0)\times\mathbb{R}^3)} = \|u\|_{L^5_{t,x}((0,\infty)\times\mathbb{R}^3)} = \infty,$$

- *there exists* $x : \mathbb{R} \to \mathbb{R}^3$ *such that*
$$\{u(t, \cdot - x(t)) : t \in \mathbb{R}\}$$
is pre-compact in $H^1(\mathbb{R}^3)$.

Remark 3.2. The quantity E_c in Theorem 3.1 is defined to be minimal in the following sense: if $M(u_0)E(u_0) < E_c$ and $K(u_0) < K(Q)$, then the solution with initial data u_0 is global and scatters.

With Theorem 3.1 in place, one can use a localized virial identity (and the sub-threshold assumption) to reach a contradiction; basically, one shows that the solution tends to zero in some averaged sense as $t \to \infty$, which cannot be the case for the solution at hand.

To carry out this argument, one needs to control terms like
$$\int_{|x|>R} |\nabla u(t,x)|^2 \, dx$$
for R sufficiently large, uniformly in t.

In the radial case, one must have $x(t) \equiv 0$, and hence these terms can be controlled by precompactness in H^1.

In the non-radial case, one needs a further argument. In particular, exploiting the Galilei symmetry, one can define another blowup solution
$$\tilde{u}(t,x) = e^{ix\xi_0 - it|\xi_0|^2} u(t, x - 2\xi_0 t), \quad \xi_0 = -2[M(u)]^{-1}P(u).$$
Computing the mass/energy of $\tilde{u}$, one sees that if $P(u) \neq 0$, then $\tilde{u}$ would be a blowup solution strictly below u, contradicting minimality of E_c. Thus one concludes that $P(u) = 0$. Using this and an argument involving the 'truncated position', one can deduce $x(t) = o(t)$. This control over $x(t)$ is sufficient to run a localized virial argument.

To prove Theorem 3.1 requires concentration-compactness techniques. Assuming Theorem 1.1 fails, we take a sequence of solutions u_n that asymptotically blow up their $L^5_{t,x}$-norm (in both the past and future of some sequence t_n). The key is then to show for such a sequence, we have that $\{u_n(t_n, x - x_n)\}$ converges along a subsequence in H^1; indeed, the solution u with the limit as its initial data satisfies the conclusions of Theorem 3.1. To check the compactness, for example, one applies the same argument to $u(\tau_n)$ for an arbitrary sequence τ_n.

We turn to the question of convergence. Normalizing t_n to 0, we apply a 'linear profile decomposition' (adapted to the $H^1 \to L^5_{t,x}$ Strichartz estimate) to the sequence of initial data $u_n(0)$.

Proposition 3.3 (Linear profile decomposition)**.** *Passing to a subsequence, there exist $J^* \in \{0,1,2\dots,\infty\}$, non-zero profiles $\{\phi^j\}_{j=1}^{J*}$, and parameters $\{(t_n^j, x_n^j)\}_{j=1}^{J^*}$ satisfying the following.*

For each finite $0 \leq J \leq J^*$, we have

$$f_n = \sum_{j=1}^{J} \phi_n^j + r_n^J, \quad \text{where} \quad \phi_n^j(x) = e^{-it_n^j\Delta}\phi^j(x - x_n^j) \quad \text{and} \quad r_n^J \in H^1.$$

The following decouplings hold for $0 \leq J \leq J^*$:

$$\lim_{n\to\infty} \big[\|f_n\|_{\dot H^s}^2 - \sum_{j=1}^{J} \|\phi_n^j\|_{\dot H^s}^2 - \|r_n^J\|_{\dot H^s}^2\big] = 0, \quad s \in \{0,1\},$$

$$\lim_{n\to\infty} \big[\|f_n\|_{L^4}^4 - \sum_{j=1}^{J} \|\phi_n^j\|_{L^4}^4 - \|r_n^J\|_{L^4}^4\big] = 0.$$

The remainder satisfies $(e^{it_n^j\Delta} r_n^J)(x + x_n^J) \rightharpoonup 0$ weakly in H^1 and

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{it\Delta} r_n^J\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} = 0.$$

The parameters (t_n^j, x_n^j) are orthogonal in the sense that for $j \neq k$ we have

$$\lim_{n\to\infty} \big(|t_n^j - t_n^k| + |x_n^j - x_n^k|\big) = \infty. \tag{3.1}$$

Furthermore, for each j we may assume that either $t_n^j \to \pm\infty$ or $t_n^j \equiv 0$, and either $|x_n^j| \to \infty$ or $x_n^j \equiv 0$. If the f_n are radial, then we have $x_n \equiv 0$.

Briefly, the linear profile decomposition is proved by 'removing one bubble at a time' until the Strichartz norm is depleted. The bubbles are found firstly with a refined Strichartz estimate, which identifies a scale for concentration, and then essentially Hölder's inequality, which identifies a point in space-time for concentration.

The decoupling statements and the sharp Gagliardo–Nirenberg inequality imply that each profile must carry positive energy. To prove convergence, one needs to show that in fact $J^* = 1$, $t_n^1 \equiv 0$, and $r_n^1 \to 0$ in H^1.

We will assume towards a contradiction that $J^* > 1$. In particular, each profile carries strictly less than the critical energy, and hence we can construct a scattering nonlinear solution associated to each profile:

Lemma 3.4. *Let ϕ_n^j be as in Proposition 3.3. Suppose that $M(\phi^j)E(\phi^j) < E_c$ and $K(\phi) < K(Q)$ in the case $t_n^j \equiv 0$, and $\frac{1}{2}K(\phi)^2 < M(Q)E(Q)$ in the case $t_n^j \to \pm\infty$. Then there exists a global solution v_n^j to (1.1) with $v_n^j(0) = \phi_n^j$ satisfying $\|\langle\nabla\rangle v_n^j\|_{S(\mathbb{R})} \lesssim 1$ for any Strichartz norm S.*

One just takes the solution with data ϕ^j and then incorporates the symmetries.

We now define

$$u_n^J(t) = \sum_{j=1}^{J} v_n^j(t) + e^{it\Delta} r_n^J,$$

Using orthogonality of the parameters, one can show that u_n^J is an approximate solution to (1.1). Furthermore, because each v_n^j scatters, one can also show that u_n^J satisfies good space-time bounds. Therefore, since u_n^J agrees with u_n at time zero (by construction), we can apply stability theory for (1.1) to deduce that the u_n obey good space-time bounds. This is a contradiction!

We conclude that $J^* = 1$, and another contradiction argument relying on stability implies $t_n^1 \equiv 0$. Finally, energy decoupling already implies $r_n \to 0$ in $\dot{H}^1$, while the minimality of E_c (and another contradiction argument using stability) guarantees $r_n \to 0$ in L^2 as well.

Thus we conclude the sketch of the proof of Theorem 3.1.

4. New approach

Next, we discuss a new and simpler approach for proving Theorem 1.1. In particular, this approach avoids the use of concentration-compactness completely. This is joint work with B. Dodson.

Generally speaking, the strategy is as follows:

- Find a weaker scattering criterion.
- Prove refined versions of localized virial identities.
- Use localized virial identities to prove directly that the appropriate scattering criterion holds.

Analogous to the original proof of Theorem 1.1, we first considered the radial case and then extended the result to the non-radial case.

In the following, we will always assume that we have a global H^1-bounded subthreshold solution u to (1.1).

5. The radial case

The radial case is particularly simple due to the compactness afforded by the radial Sobolev embedding.

We begin with a scattering criterion due to Tao.

Proposition 5.1 (Scattering criterion, radial case)**.** *Suppose u is a global radial solution satisfying*

$$\|u\|_{L_t^\infty H_x^1} \leq E.$$

There exist $\epsilon(E) > 0$ and $R(E) > 0$ so that if

$$\liminf_{t\to\infty} \int_{|x|\leq R} |u(t,x)|^2 \, dx \leq \epsilon,$$

then u scatters forward in time.

The proof of Proposition 5.1 is relatively simple. It is a perturbative argument relying only on Strichartz estimates, the dispersive estimate, the local form of mass conservation, and the radial Sobolev inequality.

We turn to a refined localized virial identity. Here we introduce

$$V(t) = 2\,\mathrm{Im} \int \bar{u} \nabla u \cdot \nabla a \, dx$$

as above, but instead of using $a(x) = |x|^2$, we use a modified weight that was originally introduced by Ogawa and Tsutsumi. It smoothly transitions between $|x|^2$ and the Morawetz weight $|x|$. In particular, we let

$$a(x) = \begin{cases} |x|^2 & |x| \leq R/2 \\ 2R|x| & |x| > R \end{cases}$$

for some large R to be determined, and we will in a in the remaining regions so that it satisfies $\partial_r a, \partial_r^2 a \geq 0$ and $|\partial^\alpha a| \lesssim_\alpha R|x|^{-|\alpha|+1}$ for $|\alpha| \geq 1$.

With this choice of a, we have $\sup |V(t)| \lesssim R$.

Computing the time derivative, we find that the main contribution to V' is given by

$$\int_{|x|\leq R/2} 8|\nabla u|^2 - 6|u|^4 \, dx \gtrsim \int_{|x|\leq R/2} |u|^4 \, dx,$$

provided we choose $R = R(M(u))$ sufficiently large.

This relies on the sharp Gagliardo–Nirenberg inequality and the sub-threshold assumption, and integration by parts (one needs to show that $K(\chi_R u(t)) < K(Q)$ for all t).

As for the error terms, either they have a good sign (because of the presence of the Morawetz weight) or they are controlled by

$$\int_{|x|>R/2} \tfrac{R}{|x|}|u|^4 + \tfrac{R}{|x|^3}|u|^2 \, dx \lesssim R^{-2} M(u)[1 + \|u\|_{L_t^\infty H_x^1}^2],$$

where we use the radial Sobolev embedding

$$\|xf\|_{L^\infty(\mathbb{R}^3)} \lesssim \|f\|_{H^1(\mathbb{R}^3)}.$$

In particular, these are uniformly small provided R is chosen large enough.

Applying the fundamental theorem of calculus, we arrive at the Morawetz estimate

$$\tfrac{1}{T}\int_0^T \int_{|x|\le R} |u(t,x)|^4\,dx\,dt \lesssim \tfrac{R}{T} + \tfrac{1}{R^2},$$

which implies that there exist $R_n, t_n \to \infty$ so that

$$\lim_{n\to\infty} \int_{|x|\le R_n} |u(t_n,x)|^4\,dx = 0.$$

By Hölder's inequality, this implies mass evacuation and completes the proof.

6. The non-radial case

In the non-radial case, we will use the following scattering criterion.

Proposition 6.1 (Scattering criterion, non-radial case)**.** *Suppose u is a global solution satisfying*

$$\|u\|_{L_t^\infty H_x^1} \le E.$$

There exists $\epsilon(E)$ and $T_0(E)$ such that the following holds: If

$$\forall a \in \mathbb{R} \quad \exists t_0 \in (a, a+T_0) : \|u\|_{L^5_{t,x}([t_0 - T_0^{\frac{1}{3}}, t_0]\times\mathbb{R}^3)} \le \epsilon,$$

then u scatters forward in time.

In other words, if "on any sufficiently large window, there exists a sufficiently large interval on which the norm is sufficiently small", then the solution scatters. It is not quite as catchy as "mass evacuation", but it works well for the non-radial case.

This result is also perturbative, relying only on dispersive and Strichartz estimates.

To prove that the scattering criterion holds, we use an "interaction" version of the virial/Morawetz hybrid used above. Essentially, we now use

$$V(t) = \iint |u(y)|^2 \nabla a(x-y) 2\,\mathrm{Im}\,\bar{u}\nabla u(x)\,dx\,dy,$$

that is, we center the old quantity at each $y \in \mathbb{R}^3$ and average against the mass density. More precisely, we let $\chi = 1$ for $|x| < 1-\varepsilon$ and $\chi = 0$ for $|x| > 1$ (for some small $\varepsilon > 0$, define

$$\phi(x) = \tfrac{1}{R^3}\int_{\mathbb{R}^3} \chi^2(\tfrac{x-s}{R})\chi^2(\tfrac{s}{R})\,ds, \quad \psi(x) = \tfrac{1}{|x|}\int_0^{|x|} \phi(r)\,dr,$$

and let $\nabla a(x-y) = \psi(x-y)(x-y)$. Then ψ is roughly constant for $|x| \le R$ and equals $R|x|^{-1}$ for $|x| > R$, so that ∇a transitions from x to $R\frac{x}{|x|}$. In particular, the weight is similar to the one used in the radial case.

This time, the main contribution is given by

$$\int |\chi(\tfrac{y-s}{R})u(y)|^2\{4|\chi(\tfrac{x-s}{R})\nabla u^\xi(x)|^2 - 3|\chi(\tfrac{x-s}{R})u(x)|^4\}\,dx\,dy\,ds,$$

where $u^\xi(x) = e^{ix\xi}u(x)$ is a boost of u (see below). This term can also be shown to be coercive for R large enough (using the sharp Gagliardo–Nirenberg inequality), with a lower bound of

$$c\int |\chi(\tfrac{y-s}{R})u(y)|^2|\chi(\tfrac{x-s}{R})\nabla u^\xi(x)|^2\,dx\,dy\,ds.$$

In this case, computing the time derivative leads to terms at the $\dot H^1$ level, which is potentially problematic if they need to regarded as error terms. They combine to have the form

$$\int \chi^2(\tfrac{x-s}{R})\chi^2(\tfrac{y-s}{R})\{|u(y)|^2|\nabla u(x)|^2 - \operatorname{Im}[\bar u\nabla u](x)\cdot\operatorname{Im}[\bar u\nabla u](y)\}\,dx\,dy,$$

which turns out to be Galilean invariant, i.e. invariant under $u\mapsto u^\xi$ for any ξ. We choose

$$\xi = \xi(t,s,R) = -\frac{\int\chi^2(\frac{x-s}{R})\operatorname{Im}[\bar u\nabla u](x)\,dx}{\int\chi^2(\frac{x-s}{R})|u(x)|^2\,dx},$$

which removes the term

$$\int \chi^2(\tfrac{x-s}{R})\operatorname{Im}[\bar u\nabla u](x)\,dx.$$

The other term appears as part of the main contribution!

The remaining error terms either have a good sign or can be handled through a logarithmic averaging trick (due originally to the I-team).

As in the radial case, we average in time (over an interval of the form $(a,a+T_0)$), as well as over $R\in[R_0,R_0e^J]$, which yields an estimate of the form

$$\tfrac{1}{R^3}\int |\chi(\tfrac{y-s}{R})u(t_1,y)|^2|\chi(\tfrac{x-s}{R})\nabla u^{\xi_1}(t_1,x)|^2\,dx\,dy\,ds \ll 1$$

for some $t_1\in(a,a+T_0)$ and some R; here $\xi_1=\xi(t_1,s,R)$. Splitting into cubes and changing variables, we can reduce this to

$$\sum_{z\in\mathbb{Z}^3}\int_{|y-Rz|\le 4R}|u(t_1,y)|^2\,dy\int_{|x-Rz|\le 4R}|\nabla u^{\xi_1}(t_1,x)|^2\,dx \ll 1. \quad (*)$$

We wish to estimate u in $L^5_{t,x}$ on an interval of the form $(t_1,t_1+T_0^{\frac13})$. It is enough to estimate

$$u_L(t) = e^{i(t-t_1)\Delta}u(t_1).$$

The strategy is to split $u_L(t,x)=\sum_z v(z,t,x)$, where $v(z,t,x)$ is localized to $|x-Rz|\le 4R$. The estimate (*) is then essentially a 'small data' condition for v (roughly, it should control the $\dot{H}^{\frac{1}{2}}$-norm), which will allow us to prove good bounds for $v(z)$. Recombining (i.e. taking ℓ_z^5-norms and using the essentially non-overlapping decomposition of v) will yield good bounds for u_L.

Each v solves the equation

$$(i\partial_t + \Delta)v(z) = 2\nabla u_L \cdot \nabla[\chi_R] + u_L \Delta[\chi_R].$$

So to estimate, we use the Duhamel formula and local smoothing estimates for the inhomogeneous terms. To estimate the initial data term, we also need to exploit Galilean invariance in order to to utilize $(*)$ appropriately.

A technical point: the small constant in $(*)$ should be thought of as containing T_0^{-1} from the averaging in time. On the other hand, to control the inhomogeneous terms we essentially have to give up the time integral; thus it is important that we only need to estimate on an interval of length T_0^{1-}.

Collecting the above, we can control u in $L_{t,x}^5$ on a sufficiently long interval, and hence the scattering criterion is satisfied and the theorem is complete.

7. NLS with an inverse-square potential

We return now to the NLS with an inverse-square potential, i.e.

$$i\partial_t u = \mathcal{L}_a u - |u|^2 u, \quad \mathcal{L}_a = -\Delta + a|x|^{-2}. \tag{7.1}$$

Here $a > -\frac{1}{4}$, which guarantees positivity of $\mathcal{L}_a$. This equation is also locally well-posed in H^1, with small-data scattering, and $L_{t,x}^5$ scattering criterion, and global existence for solutions obeying uniform H^1 bounds throughout their lifespan. The proof of these facts is analogous to the free case, with the key ingredients being (i) Strichartz estimates adapted to $e^{-it\mathcal{L}_a}$ and (ii) a result concerning equivalence of Sobolev spaces (allowing for interchange of $(-\Delta)^s$ and $\mathcal{L}_a^s$ for suitable s and Lebesgue exponents).

The energy now takes the form

$$E_a(u) = \int \tfrac{1}{2}|\sqrt{\mathcal{L}_a} u|^2 - \tfrac{1}{4}|u|^4\,dx = \tfrac{1}{2}\|u\|_{\dot{H}_a^1}^2 - \tfrac{1}{4}\|u\|_{L_x^4}^4,$$

and the relevant (sharp) Gagliardo–Nirenberg inequality has the form

$$\|f\|_{L^4}^4 \le C_a \|f\|_{\dot{H}_a^1}^3 \|f\|_{L^2}.$$

In this case, one finds that for $-\frac{1}{4} < a \le 0$, there is an optimizer Q_a satisfying

$$\mathcal{L}_a Q + Q - Q^3 = 0,$$

while for $a > 0$ one has $C_a = C_0$ but the sharp constant is never attained. With this in mind, we define

$$\mathcal{E}_a = \begin{cases} M(Q)E_a(Q_a) & a \leq 0, \\ M(Q)E(Q) & a = 0, \end{cases}$$

and

$$K_a(u) = \|u\|_{L^2}\|u\|_{\dot{H}^1_a}, \quad \mathcal{K}_a = \begin{cases} K_a(Q_a) & a \leq 0, \\ K(Q) & a > 0 \end{cases}.$$

We can then prove the following analogue to Theorem 1.1:

Theorem 7.1. *Let $u_0 \in H^1$ satisfy $M(u_0)E_a(u_0) < \mathcal{E}_a$. Let u be the corresponding solution to* (7.1)*. If $K_a(u_0) < \mathcal{K}_a$, then u is global and scatters.*

The proof follows the concentration compactness approach sketched in Section 3. In particular, one uses a contradiction argument. Let us briefly go through the main steps of the argument, outlining what changes in the case of the inverse-square potential.

First, some additional technical difficulties arise in the linear profile decomposition, related to the failure of broken translation symmetry. In particular, one needs to introduce the operators $\mathcal{L}_a^n$ satisfying

$$\mathcal{L}_a[\phi(x - x_n)] = [\mathcal{L}_a^n \phi](x - x_n)$$

and then understand the sense in which these converge to some 'limiting' operator if $x_n \to x_0$ or $|x_n| \to \infty$.

It is in the construction of the minimal blowup solution that the most significant challenge arises. Specifically, the construction of nonlinear profiles (Lemma 3.4) in the case when $|x_n^j| \to \infty$ poses a new problem, as one cannot simply use the profile ϕ^j and then incorporate the space-translation. Indeed, space-translation symmetry is broken for this equation. Instead, one observes that in the regime $|x_n^j| \to \infty$, the equation is well-approximated by the NLS without potential. Thus, one can use a solution to the free NLS (given by Theorem 1.1) to construct an approximate solution to (7.1) with good space-time bounds; an application of stability theory then yields a true solution (with good space-time bounds). This again requires understanding the convergence of the operators $\mathcal{L}_a^n$; it also requires that the threshold for (7.1) is strictly below that of the threshold for (1.1), which can be verified directly by comparing the sharp constants in the corresponding Gagliardo–Nirenberg inequalities.

This argument also shows that the minimal blowup solution constructed must have $x(t) \equiv 0$ (indeed, in the profile decomposition we cannot have $|x_n^j| \to \infty$ for any profile, for otherwise the corresponding nonlinear profile in the argument above is a scattering solution).

It therefore remains to preclude a compact solution in H^1. For this, we again use the virial identity. As the virial identity is ultimately linked to scaling, which is preserved in (7.1), we find that an analogous virial identity holds for (7.1) without any new error terms arising from the potential. [This is in contrast to other external potentials, for which one typically needs to impose a 'repulsive' condition $x \cdot \nabla V(x) \leq 0$.] In particular, we deduce the same contradiction as before, by employing a localized virial argument and compactness to deduce that the minimal blowup solution must be identically zero, yielding a contradiction.