DAMERELL'S FORMULA

JON AYCOCK

ABSTRACT. We make explicit the relationship between values of Eisenstein series at CM points and values of zeta functions for CM fields, beginning with the special case of quadratic imaginary fields. Once we have given this explicit calculation, we briefly describe how they are applied to "algebraicity" and "interpolation" results to construct p-adic L-functions.

CONTENTS

1. INTRODUCTION

In [Ser72], Serre gave a construction of the Kubota Leopoldt p-adic zeta function using the theory of p-adic of modular forms and the fact that the values of holomorphic Eisentein series at the cusps are related to the values of the zeta function. This construction was generalized in [DR80] to construct the p-adic zeta functions of totally real fields.

These two "interpolation" results were preceded by "algebraicity" results: up to a renormalization, the values of these zeta functions at certain inputs are in fact algebraic numbers,

Date: September 28, 2021.

so that it makes sense to ask about congruences between them. For quadratic imaginary fields, this study was initiated by Damerell, and completed using the Maass–Shimura operators and the theory of nearly holomorphic modular forms. From here, p-adic interpolation of the zeta functions of quadratic imaginary fields, and then for CM fields,[1] were given by Katz in [Kat76] and [Kat78] respectively, though with conditions on p.

Here, we make explicit the relationship between values of Eisenstein series at CM points and values of zeta functions for CM fields. We begin by describing the L-functions to be interpolated in Section 2, introducing Hilbert modular forms in Section 3. Once we have both tools, we relate them in Section 4. Section 5 gives some reasons one might care about the result.

The seminal work on the theory of nearly holomorphic automorphic forms is [Shi00]. Katz's construction of p-adic L-functions has been extended in the case of quadratic imaginary fields by Andreatta and Iovita in [AI21]; the author of the present note is working to extend this to the case of CM fields.

This document is a draft, and may contain mistakes. The author welcomes any questions or comments on its content. See the end for contact information.

1.1. **Notation, Conventions, and Assumed Knowledge.** We assume the reader has taken a first course in algebraic number theory (e.g. out of [Mil08]) and has a basic understanding of complex analysis.

2. L-functions

For this section, we fix a totally real field F of degree d, and a CM extension $K = F(\alpha)$. For simplicity, we assume that the ring of integers is $\mathcal{O}_K = \mathcal{O}_F + \alpha\mathcal{O}_F$.[2] We also fix a CM type of K; for each real embedding $\sigma\colon F \to \mathbb{R}$, we choose a preferred embedding of K into $\mathbb{C}$ which agress with σ when restricted to F, which we also call σ. Thus the set of complex embeddings of K is the set of σ's and all $\overline{\sigma}$'s as σ runs over the real embeddings of F. Write I for the set of real embeddings of F.

2.1. **The 1-variable L-function.** The standard, 1-variable Dedekind zeta function of K can be written, for $\mathrm{Re}(s) > 1$, as the sum over all nonzero ideals of $\mathcal{O}_K$.

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{Nm}(\mathfrak{a})^s}. \tag{1}$$

We may rewrite this in order to make most of the summation happen with elements of the field itself. Pick representatives $\mathfrak{a}_1, \dots, \mathfrak{a}_h$ for the class group of $\mathcal{O}_K$, and note that every integral ideal $\mathfrak{a}$ can be written as $\mathfrak{a} = \mathfrak{a}_i^{-1}(\alpha)$ for some i and some $\alpha \in \mathfrak{a}_i$.

Writing $\mathfrak{a}$ as $\mathfrak{a}_i^{-1}(\alpha)$ for some i and some $\alpha \in \mathfrak{a}_i$ is unique up to rescaling α by a unit in $\mathcal{O}_K^\times$. In the quadratic imaginary case, we can sum over all elements of $\mathfrak{a}_i$, and divide by $\#\mathcal{O}_K^\times$ to account for repetition. When K is not quadratic imaginary, its unit group is infinite, and we can no longer do this. We should instead view $\mathfrak{a}_i$ as a $\mathcal{O}_K^\times$-set, where $u \cdot \alpha = u\alpha$ for any unit u and $\alpha \in \mathfrak{a}_i$. This action is useful because each orbit $\alpha\mathcal{O}_K^\times \in \mathfrak{a}_i/\mathcal{O}_K^\times$ corresponds to a unique integral ideal $\mathfrak{a}_i^{-1}(\alpha)$, and each integral ideal $\mathfrak{a}_i^{-1}(\alpha)$ corresponds to a unique orbit $\alpha\mathcal{O}_K^\times$.

[1] Recall that a CM field is a totally imaginary field K which is a degree 2 extension of a totally real field F.

[2] If F is a PID, this is possible because K/F is quadratic. If not, this decomposition should be $\mathcal{O}_K = \mathcal{O}_F + \alpha\mathfrak{a}$ for a fractional ideal $\mathfrak{a}$ of F.

In fact, for reasons that will arise later, we will look at orbits for the restricted action of $\mathcal{O}_F^{\times,+} \subset \mathcal{O}_K^\times$ consisting of *totally positive units* u with $\sigma(u) > 0$ for every real embedding σ of F. The association $\alpha\mathcal{O}_F^{\times,+} \mapsto \mathfrak{a}_i^{-1}(\alpha)$ is then $[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]$-to-one, where $[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]$ is the index of the subgroup $\mathcal{O}_F^{\times,+}$ in $\mathcal{O}_K^\times$.[3] Thus we write the following, where the innermost sum is over cosets $0 \neq \alpha\mathcal{O}_F^{\times,+} \in \mathfrak{a}_i/\mathcal{O}_F^{\times,+}$.

$$\zeta_K(s) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]} \sum_{i=1}^{h} \sum_{\alpha\mathcal{O}_F^{\times,+}} \frac{\mathrm{Nm}(\mathfrak{a}_i)^s}{\mathrm{Nm}(\alpha)^s}. \tag{2}$$

2.2. **Hecke Characters.**

We will give a definition in terms of Hecke characters in order to write down the $(d+1)$-variable zeta function in all cases. However, in order to avoid delving into the idèlic theory, we give the following ideal-theoretic definition of a Hecke character, instead of the standard one.

Definition 2.2.1. An *unramified Hecke character* is a homomorphism $\chi: \mathcal{I}_K \to \mathbb{C}^\times$ from the group of fractional ideals of K to the complex numbers. An unramified Hecke character χ is *unitary* if $|\chi(\mathfrak{a})| = 1$ for all ideals $\mathfrak{a} \in \mathcal{I}_K$. We say that χ has *infinity type* $(k_{1,\sigma}, k_{2,\sigma})_\sigma$ if χ can be written as a product of characters $\chi = \chi_u\chi_\infty$, where χ_u is unitary and $\chi_\infty((\alpha)) = \prod_\sigma \sigma(\alpha)^{k_{1,\sigma}}\overline{\sigma}(\alpha)^{k_{2,\sigma}}$ for all principal ideals (α).

Remark 2.2.2. Unramified, unitary Hecke characters χ have the property that $\chi((\alpha)) = 1$ for all principal ideals (α).

We note that ramified Hecke characters exist. For simplicity, we only work with unramified Hecke characters, but some of what we say will be true for all Hecke characters. We distinguish these statements and constructions by writing "(unramified) Hecke character" when the unramified hypothesis is not needed.

In particular, we note that the map $\mathfrak{a} \mapsto \mathrm{Nm}(\mathfrak{a})^s$ is an unramified Hecke character of infinity type $(s,s)_\sigma$. For any (unramified) Hecke character χ, write:

$$L(\chi, s) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathrm{Nm}(\mathfrak{a})^s}, \qquad L(\chi^{-1}, 0) = \sum_{\mathfrak{a}} \frac{1}{\chi(\mathfrak{a})}. \tag{3}$$

When $\chi = \mathrm{Nm}^s$, we see that $L(\chi^{-1}, 0) = \zeta_K(s)$. More generally, $L(\chi\,\mathrm{Nm}^k, s) = L(\chi, s-k)$ for any (unramified) Hecke character χ and any two complex numbers k and s.

2.3. **The $(d+1)$-variable L-function.**

We build the $(d+1)$-variable L-function following Equation (2). Again, the innermost sum is over cosets $0 \neq \alpha\mathcal{O}_F^\times \in \mathfrak{a}_i/\mathcal{O}_F^\times$.

$$L(\chi, (s_\sigma, t_\sigma)_\sigma) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{i=1}^{h} \sum_{\alpha\mathcal{O}_F^\times} \frac{\chi(\mathfrak{a}_i)}{\chi(\alpha)\prod_\sigma \sigma(\alpha)^{s_\sigma}\overline{\sigma}(\alpha)^{t_\sigma}}. \tag{4}$$

Note that $L(\chi, s) = L(\chi, (s,s)_\sigma)$, and changing the CM type at σ (i.e., replacing the preferred choice of complex embedding σ by $\overline{\sigma}$) interchanges the variables s_σ and t_σ.

[3] This index is finite. In particular, $[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}] \leq 2^d$, so that $\mathcal{O}_F^{\times,+}$ is a finitely generated Abelian group of the same rank as $\mathcal{O}_F^\times$. Since $\mathcal{O}_K^\times$ also has the same rank as $\mathcal{O}_F^\times$, so $\mathcal{O}_F^{\times,+} \subset \mathcal{O}_K^\times$ is an inclusion of finitely generated Abelian groups of the same rank. The quotient is finitely generated of rank 0, hence finite; its size is the index we're looking for.

Remark 2.3.1. For this to be well-defined, we need the sum $s_\sigma + t_\sigma$ to be independent of σ, since α is only defined up to a totally positive unit in F. Write S_0 for this common value. If we replace α by $u\alpha$ for some $u \in \mathcal{O}_F^{\times,+}$, $\chi(u\alpha) = \chi(u)\chi(\alpha)$ does not change; since χ is unramified and unitary, we have $\chi(u) = 1$. However, we must also check that

$$\prod_\sigma \sigma(u\alpha)^{s_\sigma}\overline{\sigma}(u\alpha)^{t_\sigma} = \prod_\sigma \sigma(\alpha)^{s_\sigma}\overline{\sigma}(\alpha)^{t_\sigma}. \tag{5}$$

Since $\sigma(u) = \overline{\sigma}(u)$, the left hand side differs from the right hand side by a factor of

$$\prod_\sigma \sigma(u)^{s_\sigma+t_\sigma} = \prod_\sigma \sigma(u)^{S_0} = \left(\prod_\sigma \sigma(u)\right)^{S_0} = \mathrm{Nm}(u)^{S_0} = 1. \tag{6}$$

Thus we have verified the equality in Equation (5), and so we see that the function defined in Equation (4) is independent of the choices of the representatives $\alpha \in \alpha\mathcal{O}_F^{\times,+}$.

This is why we describe it as a $(d+1)$-variable L-function when it looks like there are $2d$ variables - once the common sum $S_0 = s_\sigma + t_\sigma$ is chosen, the choice of s_σ forces the choice of t_σ, and vice-versa. Thus we are left with $d+1$ variables; one way to choose these variables is to vary S_0 and the s_σ's, though there is no preferred way to choose.

3. Hilbert Modular Forms

For now, we focus on the totally real field F of degree $[F:\mathbb{Q}] = d$. There are many ways of viewing Hilbert modular forms over F. In this section we describe three.

(a) Hilbert modular forms as functions on the space of lattices,
(b) Hilbert modular forms as holomorphic functions on a symmetric space, and
(c) Hilbert modular forms as sections of a line bundle on a moduli space of Abelian varieties.

The first description will be important for relating values of certain Hilbert modular forms to the values of the $(d+1)$-variable L-function. The second is where the Maass–Shimura operators will be described. The third will be important for applications to algebraicity.

3.1. Setup.

Fix a set of representatives $\mathfrak{a}_1, \dots, \mathfrak{a}_h$ for the class group of F. Note that this also serves as a complete set of isomorphism classes of locally free $\mathcal{O}_F$-modules of rank 1.

Each fractional ideal $\mathfrak{a}_i$ lives naturally in F. Since $\mathbb{R}$ is a flat Abelian group, we may also view $\mathfrak{a}_i$ as a subset of $F \otimes_\mathbb{Z} \mathbb{R} \cong \prod_\sigma \mathbb{R}$, where the isomorphism sends a pure tensor $n \otimes t \in F \otimes \mathbb{R}$ to the tuple $(\sigma(n)t)_\sigma \in \prod_\sigma \mathbb{R}$. Each $\mathfrak{a}_i \subset F \otimes \mathbb{R}$ is a discrete subgroup; we assign a volume to the quotient $(F \otimes \mathbb{R})/\mathfrak{a}_i$ by choosing a fundamental domain D[4], and defining the volume of the quotient to be the volume of this subset of $\mathbb{F} \otimes_\mathbb{Z} \mathbb{R}$. We also refer to this as the *covolume* of the lattice $\mathfrak{a}_i$.

We scale the representatives $\mathfrak{a}_i$ for the class group. Let V_i denote the covolume of $\mathfrak{a}_i$ in $F \otimes \mathbb{R}$, and Δ the discriminant of F. Replace $\mathfrak{a}_i$ by the lattice $\frac{\sqrt[2d]{|\Delta|}}{\sqrt[d]{V_i}}\mathfrak{a}_i \subset F \otimes \mathbb{R}$; this need not be a fractional ideal of F, but it is a lattice in $F \otimes \mathbb{R}$ with covolume $\sqrt{|\Delta|}$. In particular, the choice of $\mathfrak{a}_i$ is unique, and the representative of the class consisting of principal ideals is $\mathcal{O}_F$.

[4] D is a connected subset of $F \otimes \mathbb{R}$ such that, for all $x \in F \otimes \mathbb{R}$, exactly one element $d \in x + \mathfrak{a}_i$ is in D.

3.2. **Lattices.** We describe Hilbert modular forms as functions on a space of lattices. First, we should define what a lattice is.

Definition 3.2.1. Consider the vector space $F \otimes_{\mathbb{Z}} \mathbb{C}$. It has the structure of a $\mathcal{O}_F$-module by acting on the first component, and a real vector space by acting on the second component. A *lattice*, or more precisely an $\mathcal{O}_F$-lattice in $F \otimes \mathbb{C}$, is a discrete $\mathcal{O}_F$-submodule L of $F \otimes \mathbb{C}$ which is locally free of rank 2 as an $\mathcal{O}_F$-module, and which spans $F \otimes \mathbb{C}$ over $\mathbb{R}$.

Since a lattice L is locally free of rank 2 as an $\mathcal{O}_F$-module, it is isomorphic to $\mathfrak{a}_{i_1} \oplus \mathfrak{a}_{i_2}$ for two indices i_1 and i_2, where $\mathfrak{a}_i$ refers to one of the representatives of the class group of F chosen above. Picking an isomorphism, we may write $L = \omega_1 \mathfrak{a}_{i_1} \oplus \omega_2 \mathfrak{a}_{i_2}$ for some $\omega_1, \omega_2 \in (F \otimes \mathbb{C})^\times$.

We say two lattices L and L' are *homothetic* if there is some $\lambda \in (F \otimes \mathbb{C})^\times$ for which $\lambda L = \{\lambda \ell \in F \otimes \mathbb{C} \mid \ell \in L\} = L^\times$. By scaling $L = \omega_1 \mathfrak{a}_{i_1} + \omega_2 \mathfrak{a}_{i_2}$, we see that any lattice is homothetic to a lattice of the form $\mathfrak{a}_{i_1} + \tau \mathfrak{a}_{i_2}$ for some $\tau \in (F \otimes \mathbb{C})^\times$; e.g., for $\tau = \frac{\omega_2}{\omega_1}$.

We now define Hilbert modular forms in terms of functions on lattices.

Definition 3.2.2. Let $\mathcal{L}$ be the set of $\mathcal{O}_F$-lattices in $F \otimes \mathbb{C}$, and let $\underline{k} = (k_\sigma)_\sigma$ be a tuple of integers indexed by the real embeddings of F. A *Hilbert modular form* of weight $\underline{k}$ is a function $f \colon \mathcal{L} \to \mathbb{C}$ satisfying some analytic conditions (to be specified in the next section) and the homogeneity property

$$f(\lambda L) = \left(\prod_\sigma \sigma(\lambda)^{-k_\sigma} \right) f(L) \qquad \text{for all } \lambda \in (F \otimes \mathbb{C})^\times. \tag{7}$$

Here, if $\lambda = n \otimes z \in F \otimes \mathbb{C}$ is a pure tensor, we write $\sigma(\lambda) = \sigma(n) z$. It is nonzero for all σ if $\lambda \in (F \otimes \mathbb{C})^\times$.

Notice that f is determined by its values on lattices of the form $\mathfrak{a}_{i_1} \oplus \tau \mathfrak{a}_{i_2}$.

Remark 3.2.3. In Section 4.1, we will give our first example of a Hilbert modular form using this viewpoint. This will be the easiest way for us to relate its values to the values of the L-functions.

3.3. **Holomorphic Functions.** Hilbert modular forms can also be viewed as holomorphic functions on a symmetric space. Let $\mathfrak{h} \subset \mathbb{C}$ be the upper half-plane consisting of complex numbers with positive imaginary part. Then define $\mathfrak{h}_F$ to be the product of d copies of $\mathfrak{h}$, indexed by the real embeddings of F. We write $\underline{z} = (z_\sigma)_\sigma$ for elements of $\mathfrak{h}_F$.

We have an action of the group $\mathrm{SL}_2 F$ on $\mathfrak{h}_F$ given by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_\sigma)_\sigma = \left(\frac{\sigma(a) z_\sigma + \sigma(b)}{\sigma(c) z_\sigma + \sigma(d)} \right)_\sigma. \tag{8}$$

This action gives us the following definition.

Definition 3.3.1. A *holomorphic Hilbert modular form* of weight $\underline{k} = (k_\sigma)_\sigma$ is a holomorphic function $f \colon \mathfrak{h}_F \to \mathbb{C}$ such that $|f(\underline{z})|$ is bounded as every $\mathrm{Im}(z_\sigma)$ goes to ∞ at once[5] and which satisfies the transformation property

$$f(\gamma \cdot \underline{z}) = \left(\prod_\sigma (\sigma(c) z_\sigma + \sigma(d))^{k_\sigma} \right) f(\underline{z}) \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mathcal{O}_F. \tag{9}$$

[5]This condition is important for $F = \mathbb{Q}$ and automatic for $d > 1$.

To connect this with the previous definition, we have to associate a lattice to each $\underline{z} \in \mathfrak{h}_F$. In fact, the space of lattices is disconnected, while $\mathfrak{h}_F$ is connected – a modular form in the sense of Definition 3.2.2 is actually a collection of Hilbert modular forms in the sense of Definition 3.3.1, one for each connected component. Thus we should actually associate multiple lattices to each element of $\mathfrak{h}_F$.

View $\underline{z} \in \mathfrak{h}_F$ as an element of $F \otimes \mathbb{C} \supset \mathfrak{h}_F$. To every triple $(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \underline{z})$, we associate a lattice $\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2} = \{a_1 + a_2\underline{z} \in F \otimes \mathbb{C} \mid a_j \in \mathfrak{a}_{i_j}\}$. This allows us to translate between Definitions 3.2.2 and 3.3.1. Note that the same lattice may be written in different ways.

For a Hilbert modular form f_L viewed as a function on the space of lattices, we build a function $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, -) : \mathfrak{h}_F \to \mathbb{C}$ for each orered pair $(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2})$ by the formula $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \underline{z}) = f_L(\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2})$. One may check that this function satisfies the requisite transformation property. The analytic conditions mentioned in Definition 3.2.2 correspond to the function $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, -)$ being holomorphic and bounded at the cusps.

Now, for each ordered pair $(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2})$, fix a holomorphic modular form $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, -) : \mathfrak{h}_F \to \mathbb{C}$. Every lattice $\omega_1\mathfrak{a}_{i_1} + \omega_2\mathfrak{a}_{i_2}$ is homothetic to $\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2}$ for some $\underline{z} \in \mathfrak{h}_F$. Specifying that $f_L(\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2}) = f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \underline{z})$, the homogeneity property then determines f_L on all lattices.

Remark 3.3.2. Viewing Hilbert modular forms as functions on the space of $\mathcal{O}_F$-lattices in $F \otimes \mathbb{C}$ will give a tight connection to the relevant values of L-functions. However, one of the important pieces of the theory, the Maass–Shimura operator, is best understood in the realm of holomorphic functions. We give both viewpoints here, and a rough translation between them, in order to take advantage of both sides of the theory. The third viewpoint, in terms of algebraic geometry, is given for the purpose of one of our applications. The next section introduces this, and ties it back to the viewpoint involving lattices.

3.4. **Geometry.** HBAVs. What's the cotangent bundle? HMF: Algebraic function w/ homogeneity property $\{(A, \iota, \psi, \lambda, \omega)\} \to R$. Every complex HBAV is $(F \otimes \mathbb{C})/L$ for some L, which is unique up to homothety. Build it by picking a differential and integrating it against all paths in $H_1(A, \mathbb{Z})$; scaling the differential scales L, homogeneity properties match up.

In order to motivate the following discussion, we give a preliminary definition of classical modular forms; they may be viewed as a special case of Hilbert modular forms over the totally real field $F = \mathbb{Q}$.

Definition 3.4.1. Fix a base ring R. A *modular form* defined over R is an algebraic function f that assigns to every pair (E, ω) of an elliptic curve E defined over some R-algebra S and a basis ω for $\Omega^1_{E/S}$ as an S-module, an element of S. A modular form of weight k is such a function which satisfies the homogeneity property that for any $c \in S^\times$, $f(E, c\omega) = c^{-k} f(E, \omega)$.

Remark 3.4.2. Since $\Omega^1_{E/S}$ is a free S-module of rank 1, any two bases ω_1, ω_2 for $\Omega^1_{E/S}$ are related by $\omega_1 = c\omega_2$ for a unique $c \in S^\times$. Thus the value of $f(E, \omega)$ for any ω is determined by the value of $f(E, \omega_0)$ for some fixed ω_0.

Elliptic curves are one-dimensional examples of Abelian varieties. In order to generalize the definition above, one may look at more general classes of Abelian varieties of higher dimension, or endowed with extra structure. For Hilbert modular forms, we consider Hilbert–Blumenthal Abelian varieties.

Definition 3.4.3. Fix a ring R, and let F be a totally real field of dimension d. A *Hilbert–Blumnenthal Abelian variety*, or HBAV, is a tuple $(A, \iota, \psi, \lambda)$ consisting of

- an Abelian variety A of relative dimension d over R,
- a *real multiplication* $\iota\colon \mathcal{O}_F \to \mathrm{End}(A)$,
- a *level structure* ψ, and
- a *polarization* λ.

For our purposes, the level structure ψ is trivial, and we suppress it from the notation.

Remark 3.4.4. In general, the moduli space of HBAVs is represented by a stack, rather than a scheme. However, with enough level structure, the moduli space is "rigidified" to be represented by a scheme. When we do not fix a polarization as part of the data, this never happens. For our purposes, we will have no level structure, which will mean that we have to deal with stacks; we will ignore the issues that come up.

The real multiplication ι gives $\Omega^1_{A/R}$ the structure of an $\mathcal{O}_F \otimes R$-module. For a dense open subspace of the moduli space of HBAVs, this is a free module of rank 1. In this case, we can form its frame bundle consisting of bases ω for $\Omega^1_{A/R}$ which generate it as an $\mathcal{O}_F \otimes R$-module. Any two such bases ω_1, ω_2 are related by $\omega_1 = c\omega_2$ for a unique $c \in (\mathcal{O}_F \otimes R)^\times$.

Definition 3.4.5. Fix a base ring R. A *Hilbert modular form* defined over R is an algebraic function f that assigns to every tuple $(A, \iota, \lambda, \omega)$ of an HBAV (A, ι, λ) defined over some R-algebra S and a basis ω for $\Omega^1_{E/S}$ as an $\mathcal{O}_F \otimes S$-module, an element of S. A Hilbert modular form of weight $(k_\sigma)_\sigma$ is such a function which satisfies the homogeneity property that for any $c \in (\mathcal{O}_F \otimes S)^\times$, $f(A, \iota, \lambda, c\omega) = \left(\prod_\sigma \sigma(c)^{-k_\sigma}\right) f(A, \iota, \lambda, \omega)$. Here σ acts on pure tensors $n \otimes s \in (\mathcal{O}_F \otimes S)^\times$ by $\sigma(n \otimes s) = \sigma(n) \otimes s$.

The algebraic geometry involved gives a way to investigate algebraicity, and in fact the integrality, of the values of Hilbert modular forms. The modular forms we consider will be defined over $\mathbb{Z}$. Thus the algebraicity of their values will depend on the algebraicity of the inputs at which we evaluate them; the integrality will depend on those inputs and on the structure of the moduli space of HBAVs viewed as a stack, which will contribute some predictable denoninators.

We connect this section back to the theory of lattices. Let (A, ι, λ) be an HBAV defined $\mathbb{C}$, and let $\omega \in \Omega^1_{A/R}$ be a basis. The choice of ω gives an isomorphism between Lie algebra $\mathrm{Lie}(A) = H_1(A, \mathbb{R})$ and $F \otimes \mathbb{C}$, by sending a path $\gamma \in H_1(A, \mathbb{R})$ to $\int_\gamma \omega$. To $(A, \iota, \lambda, \omega)$, we associate the lattice $\{\int_\gamma \omega \mid \gamma \in H_1(A, \mathbb{Z}) \subset H_1(A, \mathbb{R})\}$.

On the other hand, to any lattice L, we associate the complex torus $A = (F \otimes \mathbb{C})/L$. Since $\mathcal{O}_F$ acts on $(F \otimes \mathbb{C})$ through the first component, and L is stable under this action, we may associate an endomorphism structure ι. Further, there is a natural alternating pairing on $F \otimes \mathbb{C}$ which induces a polarization on A, assuring us that it is the set of complex points of an Abelian variety, given by the formula

$$((z_\sigma)_\sigma, (w_\sigma)_\sigma) \mapsto \sum_\sigma \mathrm{Im}(z_\sigma \overline{w}_\sigma). \tag{10}$$

Write $\mathrm{d}\tau$ for the natural differential on $\mathbb{C}$ with coordinate τ, viewed as an $F\otimes\mathbb{C}$-basis for $F \otimes \mathbb{C}$. The lattice L can be recovered from A by considering the integrals $\left\{\int_\gamma \frac{\mathrm{d}\tau}{\pi} \mid \gamma \in H_1(A, \mathbb{Z})\right\}$.

We now have three separate ways to view Hilbert modular forms. Our next step is to give some examples of Hilbert modular forms, and use all three viewpoints to use them to learn about L-values.

4. Damerell's Formula

Damerell's formula relates the values of the zeta functions described above to the values of certain Hilbert modular forms, known as Eisenstein series. We give a description of the pieces that go into the construction.

4.1. **Holomorphic Eisenstein Series.** We give examples of Hilbert modular forms, in terms of Definition 3.2.2. First, note that we have an action of $\mathcal{O}_F^{\times,+}$ on any lattice L, induced by the action of $\mathcal{O}_F$. Denote the set of orbits of this action by $L/\mathcal{O}_F^{\times}$, and write the orbit of $\alpha \in L$ by $\alpha\mathcal{O}_F^{\times,+} \in L/\mathcal{O}_F^{\times,+}$.

Definition 4.1.1. Let $k > 2$ be an integer. The *holomorphic Eisenstein series of parallel weight* k is the function on the space of lattices

$$G_k(L) = \sum_{0 \neq \alpha\mathcal{O}_F^{\times,+} \in L/\mathcal{O}_F^{\times,+}} \frac{1}{\prod_\sigma \sigma(\alpha)^k}. \tag{11}$$

Here the sum is over nonzero orbits of the action of $\mathcal{O}_F^{\times,+}$ on L. This is independent of the choice of representatives α since $\prod_\sigma \sigma(u)^k = \mathrm{Nm}(u)^k = 1$ for any totally positive unit u and any integer k.

The holomorphic avatar has a simple formula on the connected component corresponding to lattices of the form $\mathcal{O}_F + \underline{z}\mathcal{O}_F$. We write it

$$G_k(\underline{z}) = \sum_{(c,d)} \frac{1}{\prod_\sigma (\sigma(c) z_\sigma + \sigma(d))^k}. \tag{12}$$

The sum is over nonzero representatives (c,d) of the diagonal action of $\mathcal{O}_F^{\times,+}$ on $\mathcal{O}_F \oplus \mathcal{O}_F$. i.e., we choose exactly one pair (c,d) from the elements (cu, du) as u runs over the totally positive units $u \in \mathcal{O}_F^{\times,+}$, but we exclude $(0,0)$ from the sum. It converges absolutely using the fact that $\mathrm{Im}(z_\sigma) > 0$ for all σ.

We conclude this section with a proposition relating the values of this Eisenstein series at certain lattices to certain values of the $(d+1)$-variable L-function.

Proposition 4.1.2. *Let K be a CM field with totally real subfield F, and fix $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ a set of representatives for the class group of K. Fix a CM type for K and an unramified, unitary Hecke character χ. Then*

$$L(\chi, (k,0)_\sigma) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]} \sum_{i=1}^r \chi(\mathfrak{a}_i) G_k(\mathfrak{a}_i). \tag{13}$$

Proof. Recall that

$$L(\chi, (s_\sigma, t_\sigma)_\sigma) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]} \sum_{i=1}^r \sum_{\alpha\mathcal{O}_F^\times} \frac{\chi(\mathfrak{a}_i)}{\chi(\alpha) \prod_\sigma \sigma(\alpha)^{s_\sigma} \overline{\sigma}(\alpha)^{t_\sigma}}. \tag{14}$$

The innermost sum is over representatives α of nonzero orbits of the $\mathcal{O}_F^{\times,+}$ action on $\mathfrak{a}_i$. We can pull out a factor of $\chi(\mathfrak{a}_i)$ from the innermost sum, and set $\chi(\alpha) = 1$ since χ is unramified and unitary. We are left to verify that

$$G_k(\mathfrak{a}_i) = \sum_{\alpha\mathcal{O}_F^\times} \frac{1}{\prod_\sigma \sigma(\alpha)^k}. \tag{15}$$

This is exactly the sum from Definition 4.1.1. □

4.2. **Maass–Shimura Operators.** An important player in the story is the Maass–Shimura operator.

Definition 4.2.1. Let f be a modular form of weight $\underline{k}$. Write $\underline{z} = (z_\sigma)_\sigma = (x_\sigma + iy_\sigma)_\sigma$ for the coordinate on $\mathfrak{h}_F$, and $s_\sigma = (z_\sigma - \overline{z}_\sigma)^{-1} = \frac{1}{2iy_\sigma}$. The *weight $\underline{k}$ Maass–Shimura operator at σ $\delta^\sigma_{\underline{k}}$* acts on f by the formula

$$\delta^\sigma_{\underline{k}}(f) = s_\sigma^{k_\sigma}\frac{\partial}{\partial z_\sigma}s_\sigma^{-k_\sigma}f = k_\sigma s_\sigma f + \frac{\partial f}{\partial z_\sigma}. \tag{16}$$

The resulting function is a *nearly holomorphic modular form* of weight $k+2\sigma$ and type σ.[6] We iterate the Maass–Shimura operator by the formula

$$\delta^{j\sigma}_{\underline{k}}(f) = \left(\delta^\sigma_{\underline{k}+2j\sigma-2\sigma}\circ\cdots\circ\delta^\sigma_{\underline{k}+2\sigma}\circ\delta^\sigma_{\underline{k}}\right)(f). \tag{17}$$

The resulting function is a nearly holomorphic modular form of weight $k+2j\sigma$ and type $j\sigma$. Using the fact that the partial derivatives $\frac{\partial}{\partial z_\sigma}$ commute for different σ, one may show that a similar definition gives a well-defined operator $\delta^{\sum j_\sigma\sigma}_{\underline{k}}$, which sends a holomorphic Hilbert modular form of weight $\underline{k}$ to a nearly holomorphic Hilbert modular form of weight $k+\sum 2j_\sigma\sigma$ and type $\sum j_\sigma\sigma$.

Remark 4.2.2. Note that we do not define what it means for a function $\mathfrak{h}_F\to\mathbb{C}$ to be a nearly holomorphic Hilbert modular form, or what its type is. The reader may take it to be the definition that

(a) a nearly holomorphic modular form of type 0 is simply a holomorphic modular form,
(b) the vector space nearly holomorphic modular forms of type $\sum j_\sigma\sigma$ includes the vector space of nearly holomorphic modular forms of type $\sum_\sigma j'_\sigma\sigma$ where $0\le j'_\sigma\le j_\sigma$ for all σ, and
(c) if f is a nearly holomorphic modular form of weight $\underline{k}$ and type $\sum_\tau j_\tau\tau$, then $\delta^\sigma_{\underline{k}}f$ is a nearly holomorphic modular form of weight $\underline{k}+2\sigma$ and type $\sigma+\sum_\tau j_\tau\tau$.

When $2j_\sigma < k_\sigma$ for all σ, this process produces all nearly holomorphic modular forms of weight $\underline{k}$ and type $\sum j_\sigma\sigma$.

4.3. **Nearly Holomorphic Eisenstein Series.** Write $G_{k,\underline{j}} := \delta_k^{\sum_\sigma j_\sigma\sigma}G_k$ for $\underline{j} = (j_\sigma)_\sigma$. We find a formula for this in the following proposition.

Proposition 4.3.1. *Let $k>2$ be an integer, and $\underline{j} = (j_\sigma)_\sigma$ a tuple of non-negative integers. Then a formula for $G_{k,\underline{j}}$ is*

$$G_{k,\underline{j}}(\underline{z}) = \left(\prod_\sigma k(k+1)\dots(k+j-1)s_\sigma^{j_\sigma}\right)\sum_{(c,d)}\frac{\prod_\sigma(\sigma(c)\overline{z}_\sigma+\sigma(d))^{j_\sigma}}{\prod_\sigma(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}}. \tag{18}$$

Proof. We will prove this by induction, noting that the base case $\underline{j} = (0)_\sigma$ we may compare with Equation (12). For the induction step, we want to verify that $G_{k,\underline{j}+\sigma} = \delta^\sigma_{k+\underline{j}}G_{k,\underline{j}}$. Since $\delta^\sigma_{k+\underline{j}}f = (k+2j_\sigma)s_\sigma f + \frac{\partial f}{\partial z_\sigma}$, we may factor out the terms involving variables that do not

[6] Weight $k+2\sigma$ refers to the weight $\underline{k} = (k_\tau)_\tau$ where $k_\tau = k$ for $\tau\ne\sigma$ and $k_\sigma = k+2$. In general, weight $k+\sum 2j_\sigma\sigma$ refers to the weight $\underline{k} = (k_\tau)_\tau$ where $k_\sigma = k+2j_\sigma$ for all σ.

depend on z_σ (including $\overline{z}_\sigma$, but not including s_σ) and simply verify, using that formula, that

$$\delta^\sigma_{k+2\underline{j}}\left(s_\sigma^{j_\sigma}\sum\frac{1}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}}\right)=(k_\sigma+j_\sigma)s_\sigma^{j_\sigma+1}\sum\frac{(\sigma(c)\overline{z}_\sigma+\sigma(d))}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma+1}}. \tag{19}$$

Write $f=s_\sigma^{j_\sigma}\sum\frac{1}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}}$. For the first term of $\delta^\sigma_{k+2\underline{j}}f$, we have

$$(k+2j_\sigma)s_\sigma\left(s_\sigma^{j_\sigma}\sum\frac{1}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}}\right). \tag{20}$$

The term $\frac{\partial f}{\partial z_\sigma}$ splits into two terms by the product rule.

$$\frac{\partial f}{\partial z_\sigma}=\frac{\partial s_\sigma^{j_\sigma}}{\partial z_\sigma}\sum\frac{1}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}}+s_\sigma^{j_\sigma}\frac{\partial}{\partial z_\sigma}\sum\frac{1}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}} \tag{21}$$

Call these terms 2a and 2b respectively. Term 2a simplifies using the formula $\frac{\partial s_\sigma^{j_\sigma}}{\partial z_\sigma}=-j_\sigma s_\sigma^{j_\sigma+1}$, and combines with the $(k+2j_\sigma)s_\sigma f$ term to produce

$$\left[(k+2j_\sigma)-j_\sigma\right]s_\sigma^{j_\sigma+1}\sum\frac{1}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}}=(k+j_\sigma)s_\sigma^{j_\sigma+1}\sum\frac{(\sigma(c)z_\sigma+\sigma(d))}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma+1}} \tag{22}$$

We calculate term 2b.

$$s_\sigma^{j_\sigma}\frac{\partial}{\partial z_\sigma}\sum\frac{1}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma}}=(k+j_\sigma)s_\sigma^{j_\sigma+1}\sum\frac{-\sigma(c)(z_\sigma-\overline{z}_\sigma)}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma+1}} \tag{23}$$

Note that we have added an extra factor of s_σ before the sum, in exchange for adding a factor of $s_\sigma^{-1}=z_\sigma-\overline{z}_\sigma$ to the numerator of every term inside the sum. We now add the final result from Equation (22) to that from Equation (23) to obtain $\delta^\sigma_{k+2\underline{j}}f$. We have arranged for the factors in front to match, as well as the denominators for each term in the sum, so that we only have to add the numerators.

$$\delta^\sigma_{k+\underline{j}}f=(k+j_\sigma)s^{j_\sigma+1}\sum\frac{\sigma(c)z_\sigma+d_\sigma-\sigma(c)(z_\sigma-\overline{z}_\sigma)}{(\sigma(c)z_\sigma+\sigma(d))^{k+j_\sigma+1}}. \tag{24}$$

Use the fact that $\sigma(c)z_\sigma+d_\sigma-\sigma(c)(z_\sigma-\overline{z}_\sigma)=\sigma(c)\overline{z}+\sigma(d)$ to compare with Equation (19). $\square$

We now have values for the $G_{k,\underline{j}}$ that will be used to relate their values at CM points to the values of the L-functions from the previous section.

4.4. **Damerell's Formula.** Fix representatives $\mathfrak{a}_1,\ldots,\mathfrak{a}_h$ for the class group of K, and let χ be an unramified, unitary Hecke character. We write the $(d+1)$-variable L-function, recalling that the inner sum is over nonzero cosets $0\neq\alpha\mathcal{O}_F^{\times,+}\in\mathfrak{a}_i/\mathcal{O}_F^{\times,+}$.

$$L(\chi,(s_\sigma,t_\sigma)_\sigma)=\frac{1}{[\mathcal{O}_K^\times:\mathcal{O}_F^{\times,+}]}\sum_{i=1}^h\sum_{\alpha\mathcal{O}_F^{\times,+}}\frac{\chi(\mathfrak{a}_i)}{\prod_\sigma\sigma(\alpha)^{s_\sigma}\overline{\sigma}(\alpha)^{t_\sigma}}. \tag{25}$$

We have left our a factor of $\chi(\alpha)$ from the denominator; these are forced to be 1 by the assumptions that χ be unramified and unitary.

In the previous section, we gave the formula for $G_{k,\underline{j}}$ as a holomorphic function on $\mathfrak{h}_F$. As a function on lattices, we have

$$G_{k,\underline{j}}(L) = \left(\prod_{\sigma} k(k+1)\dots(k+j_\sigma-1)s_\sigma^{j_\sigma}\right) \sum_{\alpha\mathcal{O}_F^{\times,+}} \frac{\prod_\sigma \overline{\sigma}(\alpha)^{j_\sigma}}{\prod_\sigma \sigma(\alpha)^{k+j_\sigma}}. \tag{26}$$

Here the sum is over nonzero orbits $0 \neq \alpha\mathcal{O}_F^{\times,+} \in L/\mathcal{O}_F^{\times,+}$. Compare to Equation (19) to see that this is the correct formula, at least for lattices $L = \mathcal{O}_F + \underline{z}\mathcal{O}_F$. We might describe $y_\sigma = \frac{1}{2is_\sigma}$ as the "covolume of L at σ", noting that $\prod_\sigma \frac{1}{2is_\sigma}$ is the covolume of L.

Theorem 4.4.1 (Damerell's Formula)**.** *Fix an integer $k > 2$, a tuple $\underline{j} = (j_\sigma)_\sigma$ of positive integers indexed by the real embeddings of F, and an unramified unitary Hecke character χ. Then we may relate special values of the $(d+1)$-variable L-function with the values of various Eisenstein series at lattices corresponding to fractional ideals of K as follows:*

$$L(\chi,(k+j_\sigma,-j_\sigma)_\sigma) = \frac{\sum_{i=1}^h \chi(\mathfrak{a}_i)G_{k,\underline{j}}(\mathfrak{a}_i)}{[\mathcal{O}_K^\times,\mathcal{O}_F^{\times,+}]\left(\prod_\sigma k(k+1)\dots(k+j_\sigma-1)s_\sigma^{j_\sigma}\right)}. \tag{27}$$

Proof. Compare Equations (26) and (4). □

5. Takeaways

In this section we describe some applications for Damerell's formula as discussed in the previous two sections. Specifically, we carry over the setup from Section 3: fix a totally real field F of degree d, and a CM extension $K = F(\alpha)$ with $\mathcal{O}_K = \mathcal{O}_F + \alpha\mathcal{O}_F$. We also fix a CM type of K, i.e., a preferred extension of each real embedding $\sigma: F \to \mathbb{R}$ to a complex embedding $K \to \mathbb{C}$ which we also call σ. Thus the set of complex embeddings of K is the set of all σ's and all $\overline{\sigma}$'s as σ runs over the real embeddings of F.

5.1. Algebraicity of L-Values: Damerell's Theorem.

In order to talk about algebraicity, we recall the algebraic definition from Section 3.4. For simplicity, we focus on the quadratic imaginary case.

Definition 5.1.1 (Definition 3.4.1)**.** Fix a base ring R. A *modular form* defined over R is an algebraic function f that assigns to every pair (E,ω) of an elliptic curve E defined over some R-algebra S and a basis ω for $\Omega^1_{E/S}$ as an S-module, an element of S. A modular form of weight k is such a function which satisfies the homogeneity property that for any $c \in S^\times$, $f(E,c\omega) = c^{-k}f(E,\omega)$.

In particular, given a modular form f defined over a number field H, an elliptic curve E defined over H, and a generator $\omega \in \Omega^1_{E/H}$, the value $f(E,\omega) \in H$ is algebraic. A similar definition, and a similar statement about algebraicity, can be made for nearly modular forms.

To prove the algebraicity of certain values of L-functions, we can use our previous results relating these values to the values of Eisenstein series. Eisenstein series are in fact defined over $\mathbb{Z}$, so it will be enough to show that evaluating their complex avatars at the CM points of the modular curve corresponds to evaluating their algebraic avatars at a pair consisting of an elliptic curve defined over a number field H and a generator $\mathrm{d}w \in \Omega^1_{E/\mathbb{C}}$ which is explicitly related to an algebraic generator $\omega \in \Omega^1_{E/H}$ by a constant $\mathrm{d}w = c\cdot\omega$.

In fact, each point of the upper half plane $\tau \in \mathfrak{h}$ corresponds to a specific complex elliptic curve E_τ and a chosen basis $\mathrm{d}w \in \Omega^1_{E/\mathbb{C}}$. Any complex torus is an elliptic curve; we let L_τ denote the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$, and write $E_\tau = \mathbb{C}/L_\tau$. The projection $\mathbb{C} \to E_\tau$ gives an isomorphism between the cotangent bundle of E_τ and the cotangent bundle of $\mathbb{C}$. Writing w for the coordinate on $\mathbb{C}$, we get a generator $\mathrm{d}w$ of $\Omega^1_{E_\tau/\mathbb{C}}$. Evaluating the holomorphic function at $\tau \in \mathfrak{h}$ corresponds to evaluating the algebraic function at the pair $(E_\tau, \mathrm{d}w)$.

When $\tau = \alpha$ is a CM point, the corresponding elliptic curve E_α has complex multiplication by the quadratic imaginary field $K = \mathbb{Q}(\alpha)$. A celebrated result in explicit class field theory gives that this E_α is defined over the Hilbert class field H of K. We also have that $\pi\mathrm{d}w$ is defined over H, so that

$$G_k\left(E_\alpha, \pi\mathrm{d}w\right) = \pi^{-k} G_k(E_\alpha, \mathrm{d}w) = \pi^{-k} G_k(\alpha) \in H. \tag{28}$$

Thus by Damerell's Formula (Theorem 4.4.1), we have

$$\frac{L(\chi, k+j, -j)}{\pi^k} = \frac{\sum_{i=1}^h \chi(\mathfrak{a}_i) G_{k,j}(\mathfrak{a}_i)/s(\mathfrak{a}_i)^j}{k(k+1)\dots(k+j-1)(\#\mathcal{O}_K^\times)\pi^k} \in H(\mu_h). \tag{29}$$

Here we write $s(\mathfrak{a}_i)$ for the covolume of $\mathfrak{a}_i$ in $F \otimes_{\mathbb{Z}} \mathbb{C}$. Notice that since χ is a character of a group of order h, it takes values in the hth roots of unity μ_h. This is a standard algebraicity result showing that, up to a "period" π^k, the value $L(\chi, k+j, -j)$ lives in a particular number field. Above we wrote $H(\mu_h)$, but many authors would bound it more precisely by writing $H(\text{the values of } \chi)$. In addition to algebraicity, there are integrality results, bounding the denominators of $\pi^{-k} L(\chi, k+j, -j) \notin \mathcal{O}_H(\mu_h)$.

A similar construction can be carried out for a general CM field K, using Hilbert modular forms on its maximal totally real subfield and Hilbert Blumenthal Abelian varieties in place of elliptic curves. We omit it here, as it is nearly the same once we have the geometric desctiption of Hilbert modular forms from Section 3.4.

5.2. ***p*-adic Interpolation.** One application that requires the algebraicity and integrality results described in the previous section is the construction of the p-adic L-function for a CM field. These are laid out carefully in [Kat76] and [AI21] for quadratic imaginary K (respectively for p split in K and p nonsplit in K), and in [Kat78] for a general CM field K (under the assumption that all primes of the maximal totally real subfield K^+ of K which divide p are split in K/K^+). The case when primes above p are nonsplit in K/K^+ is not settled. Very roughly, the construction goes like this.

First, we relate the values of the L-function at certain inputs to the values of certain modular forms. We then modify it to write

$$\begin{aligned} L_p(\chi, (k+j_\sigma, -j_\sigma)_\sigma) &= \left(\textstyle\prod_{\mathfrak{p}|p} 1 - \chi(\mathfrak{p})\,\mathrm{Nm}(\mathfrak{p})^k\right) L(\chi, (k+j_\sigma, -j_\sigma)_\sigma) = \\ &= [\text{predictable constants}] \textstyle\sum_{i=1}^h \chi(\mathfrak{a}_i) G^{[p]}_{k,\underline{j}}(\mathfrak{a}_i). \end{aligned} \tag{30}$$

Here L_p is the function to be interpolated, and it is modified from L by removing the Euler factor at p. We also have $G^{[p]}_{k,\underline{j}}$, the p-depletion of the Eisenstein series $G_{k,\underline{j}}$. When K is quadratic imaginary, one might describe $G^{[p]}_{k,\underline{j}}$ in terms of its q-expansion as the form whose nth Fourier coefficient is 0 whenever p divides n, whose nth Fourier coefficient is the same as that for $G_{k,\underline{j}}$ when n is prime to p.

Second, we prove congruences modulo p^n between the values $G_{k,0}(\mathfrak{a}_i)$ and $G_{k',0}(\mathfrak{a}_i)$ whenever $k \equiv k' \pmod{(p-1)p^{n-1}}$. For K quadratic imaginary, this is established using a q-expansion principle in [Ser72]. Serre's intended use is to p-adically interpolate the standard Riemann zeta function, but the result is useful here as well.

Finally, we prove congruences modulo p^n between $\delta_k^{\sum j_\sigma \sigma} f$ and $\delta_k^{\sum j'_\sigma \sigma} f$ whenever f is a p-depleted form and each $j_\sigma \equiv j'_\sigma \pmod{(p-1)p^{n-1}}$. From here we have the neccessary congruences to claim that $L_p(\chi, -)$ is p-adically continuous as a function of the $d+1$ variables. Thus by the p-adic density of the set of characters of infinity type $(k + j_\sigma, -j_\sigma)_\sigma$ in the set of all p-adic characters, we get a unique extension of L_p to a continuous function on the set of all p-adic characters, which is the p-adic L-function.

Remark 5.2.1. Note that this is a very rough outline of the arguments. In particular, we made no reference to formal or rigid geometry which is a key tool in defining Hilbert modular forms in a p-adic setting, or to the Frobenius and U_p operators that give an algebraic way to p-deplete forms. We also made no reference to how the behavior of p takes this result from one proven in the 1970's when p splits to one proven in the 2010's when p is nonsplit; we note here that it has to do with whether or not the HBAVs with CM by K are ordinary at p.

References

[AI21] Fabrizio Andreatta and Adrian Iovita. Katz type p-adic l-functions for primes p non-split in the cm field, 2021.

[DR80] Pierre Deligne and Kenneth A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[Kat76] Nicholas M. Katz. p-adic interpolation of real analytic Eisenstein series. *Ann. of Math. (2)*, 104(3):459–571, 1976.

[Kat78] Nicholas M. Katz. p-adic L-functions for CM fields. *Invent. Math.*, 49(3):199–297, 1978.

[Mil08] James S. Milne. Algebraic number theory (v3.00), 2008. Available at www.jmilne.org/math/.

[Ser72] J.P. Serre. Formes modulaires et fonctions zêta p-adiques. In *Proc. Internat. Summer School, Univ. Antwerp*, pages 101–268, 1972.

[Shi00] Goro Shimura. *Arithmeticity in the theory of automorphic forms*, volume 82 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

Jon Aycock, Department of Mathematics, University of Oregon, Fenton Hall, Eugene, OR 97403-1222, USA

E-mail address: `jaycock@uoregon.edu`