LOCAL FACTORS OF L-FUNCTIONS FROM TATE'S THESIS

JON AYCOCK

Abstract. Tate's Thesis expresses various zeta and L-functions as zeta integrals, and uses these to prove various things about them. The computation of the Euler factors is important, and in some cases is well-documented. However, the author could not find a reference (outside of the thesis itself, which is somewhat terse) for the case of a ramified character in Section 3.5, giving the motivation for this note.

Contents

1. Introduction

Tate's Thesis expresses various zeta and L-functions as zeta integrals, and uses these to prove various things about them. A zeta integral looks roughly like this:

$$\zeta_K(\chi, s, \varphi) = \int_{\mathbb{A}_K^\times} \chi(\alpha)|\alpha|^s \varphi(\alpha)\, \mathrm{d}^\times \alpha \tag{1}$$

The computation of the Euler factors is important, and in some cases is well-documented. However, the author could not find a reference (outside of the thesis itself, which is somewhat terse) for the case of a ramified character in Section 3.5, giving the motivation for this note. Other computations are included for context and completeness, though they are readily available from other sources.

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The contents of this paper are not new, and the author invites comments to help make it a better reference. Please contact the author if you notice an issue or a typo.

1.1. **Characters.** Classically, the Riemann zeta function is seen as a holomorphic function of a complex number. For us, the input for an L-function will be a character instead.

Our conventions differ slightly from Tate's thesis.

Definition 1.1.1. A *character* of a group G is a continuous group homomorphism $G \to \mathbb{C}^\times$. It is a *unitary character* if it takes values in the unit circle S^1.

A *Hecke character* is a smooth character $\chi\colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that $K^\times$ is in the kernel. i.e., $\chi(c\alpha) = \chi(\alpha)$ for all $c \in K^\times$, $\alpha \in \mathbb{A}_K^\times$; it descends to a character of the quotient $K^\times \backslash \mathbb{A}_K^\times$.

Remark 1.1.2. In Tate's thesis, a *character* means a *unitary character* in our languange. In contrast, what he calls a *quasi-character* is what we would call a *character*.

For each place v, the absolute value is a character $|\cdot|_v\colon K_v^\times \to \mathbb{C}^\times$. This is important, and we will normalize it at finite places so that the absolute value of a uniformizer is $|\pi|_v = \mathrm{Norm}(\mathfrak{p}_v)^{-1}$. At real places, this will be the usual absolute value, while at complex places it will be the square of the usual absolute value. Note that in every case, the image will land in $\mathbb{R}^\times$. For any complex number s, we get a character $|\cdot|_v^s$ of $K_v^\times$.

Let $U_v = \{\alpha \in K_v \text{ s.t. } |\alpha|_v = 1\}$ denote the kernel of the absolute value on K_v. Note that it is compact. A character is called *unramified* if it is trivial on U_v. Tate classifies all such unramified characters in his Lemma 2.3.1, which we simply record here.

Lemma 1.1.3 (Tate Lemma 2.3.1)**.** *All unramified characters of $K_v^\times$ are of the form $|\cdot|_v^s$ for some complex number s. For Archimedean places v, s is uniquely determined, while it is only determined modulo $\frac{2\pi i}{\log \mathrm{Nm}\,\mathfrak{p}_v}$ if v is the finite place corresponding to the prime $\mathfrak{p}_v$.*

He then states in his Theorem 2.3.1:

Theorem 1.1.4 (Tate Theorem 2.3.1)**.** *Any character χ of $K_v^\times$ is the product of an unramified character $|\cdot|_v^s$ and a unitary character χ_0.*

The unitary character is determined by the restriction of χ to U_v, and the unramified character is simply χ/χ_0.

We briefly turn to global characters. The product of the absolute values determines a character of $\mathbb{A}_K^\times$. Since $x_v \in \mathcal{O}_{K_v}^\times$ for all but finitely many v, we have $|x_v|_v = 1$ for all but finitely many v. Thus we do not have to worry about convergence when defining the character $|\cdot| = \prod_v |\cdot|_v\colon \mathbb{A}_K^\times \to \mathbb{R}^\times$, with $|(x_v)_v| = \prod_v |x_v|_v$. As above, we may write any character χ of $\mathbb{A}_K^\times$ as a product of an unramified character $|\cdot|^s$ and a unitary character χ_0.

One source of unitary characters is the space of *Hecke characters*, which are characters of $\mathbb{A}_K^\times / K^\times U^+(\mathfrak{m}) K_\infty^0$; here K_∞^0 is the subgroup of $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong (\mathbb{R}^r \times \mathbb{C}^s)^\times$ consisting of elements in $\mathbb{R}_{>0}$ in each component, and $U^+(\mathfrak{m}) = U^+(\mathfrak{p}_{v_1}^{e_1} \dots \mathfrak{p}_{v_\ell}^{e_\ell})$ is the neighborhood of 1 consisting of idèles (x_v) with $x_v > 0$ whenever v is a real place and $x_{v_i} - 1 \in \mathfrak{p}_{v_i}^{e_i}$ for all $i = 1, \dots, \ell$. By class field theory, this is a finite group for any choice of $\mathfrak{m}$. The *conductor* of a Hecke character χ is the ideal $\mathfrak{m}$ such that χ is trivial on $U^+(\mathfrak{m})$ but not on $U^+(\mathfrak{m}')$ for any ideal $\mathfrak{m}'$ dividing $\mathfrak{m}$.

Remark 1.1.5. The only global characters we will care about are products of $|\cdot|^s$ with Hecke characters. They decompose as a product $\chi|\cdot|^s = \prod_v \chi_v |\cdot|_v^s$, where χ_v is the restriction $\chi|_{K_v^\times}$. Classically, Hecke L-functions fix a χ and allow s to vary.

1.2. **Schwartz Functions.** The Schwartz function φ is an analytic object included in the integral to make it converge. A *Schwartz function* on an analytic space X is a function $\varphi\colon \mathbb{A}^\times \to \mathbb{R}$ which is smooth and has rapid decay. We will consider Schwartz functions on $\mathbb{A}^\times$, and we will assume for convenience that φ decomposes as a product $\prod_v \varphi_v$ of Schwartz functions on each local field $K_v^\times$. In general this is not true, as it could be a finite sum of such functions. But in all of our situations, it will be true.

For a nonarchimedean place v, saying that φ_v is smooth is the same as saying that it is locally constant, and saying that it has rapid decay is the same as saying that it is compactly supported.

Remark 1.2.1. The choice of Schwartz functions will seem entirely random. Here we give a few reasons to care about them.

Tate's thesis was dedicated to proving the functional equation for certain L-functions using this framework. As such, it was important for him to be able to relate the Fourier transform of each Schwartz function to itself, which he did in each case. We will not use this, since this is not our goal.

On the other hand, it may be more natural to think of a space of zeta integrals, which consists of all functions $\zeta(\chi, s, \varphi)$ for fixed χ and all Schwartz functions φ. *The* zeta integral in this case is more like a "GCD" of all of these zeta integrals, and the choice of a single φ just gives a specific GCD.

1.3. **Measures.** Because $\mathbb{A}_K^\times$ is a *locally compact Abelian group*, it has a Haar measure, unique up to scale. The same is true for each local piece $K_v^\times$. We pick a specific normalization for each one.

- At real places if $\mathrm{d}x$ denotes the usual Lebesgue measure, and $|x|_v$ is the usual absolute value, we use $\mathrm{d}^\times x = \frac{\mathrm{d}x}{|x|_v}$.
- At complex places, if $\mathrm{d}z$ denotes the usual Lebesgue measure, and $|z|_v = z\overline{z}$ is the square of the usual absolute value, we use $\mathrm{d}^\times z = \frac{2\mathrm{d}z}{|z|_v}$.
- At finite places v, we normalize the Haar measure so that the unit group $\mathcal{O}_{K_v}^\times$ is assigned the size $\sqrt{|\mathrm{Nm}\,\mathfrak{d}_v^{-1}|}$, where $\mathfrak{d}_v^{-1}$ is the *local inverse different*, which is the kernel of the trace map $S = S_{K_v/\mathbb{Q}_p}$. This is 1 at all unramified primes.

There is a Haar measure on the idèles such that, for functions $f = \prod_v f_v$, we have

$$\int_{\mathbb{A}_K^\times} f(\alpha)\mathrm{d}^\times\alpha = \prod_v \int_{K_v^\times} f_v(\alpha)\mathrm{d}^\times\alpha. \tag{2}$$

In the context of zeta integrals, we have

$$\zeta_K(\chi, s, \varphi) = \int_{\mathbb{A}_K^\times} \chi(\alpha)|\alpha|^s\varphi(\alpha)\,\mathrm{d}^\times\alpha = \prod_v \int_{K_v^\times} \chi_v(\alpha)|\alpha|_v^s\varphi_v(\alpha)\,\mathrm{d}^\times\alpha. \tag{3}$$

Each local integral is called an Euler factor,

$$Z_v(\chi, s, \varphi) = \int_{K_v^\times} \chi_v(\alpha)|\alpha|_v^s\varphi_v(\alpha)\,\mathrm{d}^\times\alpha. \tag{4}$$

2. Archimedean Places

Recall that, for a choice of Schwartz function $\varphi = \prod_v \varphi_v$ and a character $\chi|\cdot|^s$, we get a *zeta integral*

$$\zeta(\chi, s, \varphi) = \int_{\mathbb{A}_K^\times} \chi(\alpha)|\alpha|^s\varphi(\alpha)\,\mathrm{d}^\times. \tag{5}$$

The *Euler factor at* v is the local integral

$$Z_v(\chi, s, \varphi) = \int_{K_v^\times} \chi_v(\alpha_v)\varphi_v(\alpha)\mathrm{d}^\times\alpha. \tag{6}$$

Here we compute the Euler factors for Archimedean places v, depending on if v is *real* or *complex*.

By writing $\chi|\cdot|^s$, we are assuming that χ is a Hecke character, so that the local character χ_v is either unramified (trivial) or ramified (the sign function $\mathbb{R}^\times \to \{\pm 1\}$) on any real place, while its restriction to S^1 at any complex place is a map $S^1 \to S^1$, given by $z \mapsto z^n$ for some n.

2.1. **Schwartz Functions.** To follow the cadence of the next section, we introduce the Schwartz functions for Archimedean places here. They will depend on the local character χ_v.

If v is a real place and $\chi_v|_{\{\pm 1\}}$ is trivial, we use the standard Gaussian.

$$\varphi_v(t) = e^{-2\pi t^2}. \tag{7}$$

If v is real and $\chi_v|_{\{\pm 1\}}$ is nontrivial ($-1 \mapsto -1$), using the standard Gaussian would produce 0 as the result. Thus we modify it.

$$\varphi_v(t) = te^{-2\pi t^2}. \tag{8}$$

Finally, for v a complex place, we whave $\chi_v|_{S^1} \colon S^1 \to S^1$ is $z \mapsto z^n$ for some n. We will use the Schwartz function

$$\varphi_v(z) = \begin{cases} \overline{z}^{|n|}e^{-2\pi|z|_v} & , \quad n \geq 0 \\ z^{|n|}e^{-2\pi|z|_v} & , \quad n \leq 0 \end{cases} \tag{9}$$

Writing each $z = re^{i\theta}$ for $0 < r < \infty$ and $0 \leq \theta < 2\pi$, and noting that $|z|_v = r^2$ is the square of the usual absolute value on $\mathbb{C}$, we get

$$\varphi_v(z) = r^{|n|}e^{-in\theta}e^{-2\pi r^2}. \tag{10}$$

2.2. **Real Places.** If v is a real place, the local integral is

$$Z_v(\chi, s, \varphi) \quad = \quad \int_{\mathbb{R}^\times} \chi_v(t)|t|^s\varphi_v(t)\frac{\mathrm{d}t}{|t|}. \tag{11}$$

We split into the ramified and unramified cases.

2.2.1. *Unramified Character.* If χ is unramified, we use $\varphi_v(t) = e^{-\pi t^2}$. Then $\chi_v(t) = 1$ and $|t|^s\varphi_v(t) = |-t|^s\varphi_v(-t)$, so we can write

$$Z_v(\chi, s, \varphi) = \int_{\mathbb{R}^\times} \chi_v(t)|t|^s e^{-\pi t^2}\frac{\mathrm{d}t}{|t|} = 2\int_0^\infty t^{s-1}e^{-\pi t^2}\mathrm{d}t. \tag{12}$$

A change of variables $u = \pi t^2$, $\mathrm{d}u = 2\pi t\mathrm{d}t$ gives us

$$2\int_0^\infty t^{s-1}e^{-\pi t^2}\mathrm{d}t \quad = \quad \frac{1}{\pi^{\frac{s}{2}}}\int_0^\infty \pi^{\frac{s}{2}-1}t^{s-2}e^{-\pi t^2}2\pi t\mathrm{d}t \quad = \quad \frac{1}{\pi^{\frac{s}{2}}}\int_0^\infty u^{\frac{s}{2}-1}e^{-u}\mathrm{d}u. \tag{13}$$

Comparing this with the definition of the gamma function, we see

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x, \qquad Z_v(\chi, s, \varphi) = \frac{1}{\pi^{\frac{s}{2}}}\int_0^\infty u^{\frac{s}{2}-1}e^{-u}\mathrm{d}u = \frac{\Gamma(\frac{s}{2})}{\pi^{\frac{s}{2}}}. \tag{14}$$

2.2.2. *Ramified Character.* If χ is ramified at v, we use $\varphi_v(t) = te^{-\pi t^2}$. Then $\chi_v(t) = \text{sign}(t)$, so we get $\chi_v(-t) = -\chi_v(t)$, $|-t|^s = |t|^s$, and $\varphi_v(-t) = -te^{-\pi(-t)^2} = -\varphi_v(t)$. Thus

$$Z_v(\chi, s, \varphi) = \int_{\mathbb{R}^\times} \chi_v(t)|t|^s te^{-\pi t^2} \frac{\mathrm{d}t}{|t|} = 2\int_0^\infty t^s e^{-\pi t^2} \mathrm{d}t. \tag{15}$$

We use the same change of variables $u = \pi t^2$, $\mathrm{d}u = 2\pi t \mathrm{d}t$, giving us

$$2\int_0^\infty t^s e^{-\pi t^2}\mathrm{d}t \quad = \quad \frac{1}{\pi}\cdot\frac{1}{\pi^{\frac{s-1}{2}}}\int_0^\infty \pi^{\frac{s-1}{2}} t^{s-1} e^{-\pi t^2} 2\pi t \mathrm{d}t \quad = \quad \frac{1}{\pi^{\frac{s+1}{2}}}\int_0^\infty u^{\frac{s+1}{2}-1}e^{-u}\mathrm{d}u. \tag{16}$$

Comparing this with the definition of the gamma function, we see

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x, \qquad Z_v(\chi, s, \varphi) = \frac{1}{\pi^{\frac{s+1}{2}}}\int_0^\infty u^{\frac{s+1}{2}-1}e^{-u}\mathrm{d}u = \frac{\Gamma(\frac{s+1}{2})}{\pi^{\frac{s+1}{2}}}. \tag{17}$$

2.3. **Complex Places.** Let v be a complex place, so that $\chi_v|_{S^1}$ is map $z \mapsto z^n$ for some n. Write each $z \in \mathbb{C}^\times$ as $z = re^{i\theta}$ for $r^2 = |z|_v$[1] and some $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Then we use the Schwartz function

$$\varphi_v(re^{i\theta}) = r^{|n|}e^{-in\theta}\cdot e^{-2\pi r^2} \tag{18}$$

Parametrizing the compex plane by all $r \in \mathbb{R}_{>0}$ and $0 \le \theta < 2\pi$, we have $\mathrm{d}z = \frac{2r}{r^2}\mathrm{d}r\mathrm{d}\theta$. Further, $\chi(re^{i\theta}) = e^{in\theta}$, and $|re^{i\theta}|_v^s = r^{2s}$. So we compute the integral.

$$Z_v(\chi, s, \varphi) = \int_{\mathbb{C}^\times} \chi_v(z)|z|_v^s \varphi_v(z)\mathrm{d}^\times z = \int_0^\infty\int_0^{2\pi} e^{in\theta}r^{2s}r^{|n|}e^{-in\theta}\cdot e^{-2\pi r^2}\frac{2r}{r^2}\,\mathrm{d}\theta\,\mathrm{d}r. \tag{19}$$

We simplify some terms, collecting our powers of r and noticing that the $e^{in\theta}$ coming from χ is exactly canceled by the $e^{-in\theta}$ coming from φ_v. Then we can integrate out θ, since the integrand does not depend on it.

$$Z_v(\chi, s, \varphi) = \int_0^\infty\int_0^{2\pi} r^{2s+|n|}e^{-2\pi r^2}\frac{2r}{r^2}\,\mathrm{d}\theta\,\mathrm{d}r = 2\pi\int_0^\infty r^{2s+|n|-2}e^{-2\pi r^2}2r\mathrm{d}r. \tag{20}$$

We collect terms to make the change of variables $u = 2\pi r^2$, $\mathrm{d}u = 4\pi r\mathrm{d}r$ as painless as possible.

$$2\pi\int_0^\infty r^{2s+|n|-2}e^{-2\pi r^2}2r\mathrm{d}r = \frac{1}{(2\pi)^{s+\frac{|n|}{2}-1}}\int_0^\infty (2\pi)^{s+\frac{|n|}{2}-1}r^{2\left(s+\frac{|n|}{2}-1\right)}e^{-2\pi r^2}4\pi r\,\mathrm{d}r. \tag{21}$$

We make the change of variables.

$$\frac{1}{(2\pi)^{s+\frac{|n|}{2}-1}}\int_0^\infty (2\pi)^{s+\frac{|n|}{2}-1}r^{2\left(s+\frac{|n|}{2}-1\right)}e^{-2\pi r^2}4\pi r\,\mathrm{d}r = \frac{1}{(2\pi)^{s+\frac{|n|}{2}-1}}\int_0^\infty u^{s+\frac{|n|}{2}-1}e^{-u}\,\mathrm{d}u \tag{22}$$

Finally, we compare to the definition of the gamma function.

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x, \quad Z_v(\chi, s, \varphi) = \frac{1}{(2\pi)^{s+\frac{|n|}{2}-1}}\int_0^\infty u^{s+\frac{|n|}{2}-1}e^{-u}\,\mathrm{d}u = \frac{\Gamma\left(s+\frac{|n|}{2}\right)}{(2\pi)^{s+\frac{|n|}{2}-1}}. \tag{23}$$

[1]Again, $|z|_v$ is the square of the usual absolute value on $\mathbb{C}$.

2.4. **Archimedean Answers.** Here we collect the final computations. We write $\chi|\cdot|^s$ for the character, letting χ be a unitary Hecke character. Depending on whether v is real or complex, and what χ looks like, we get the following.

- If v is real, and χ is trivial, the local factor is in Equation 14.

$$Z_v(\chi, s, \varphi) = \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}}. \tag{24}$$

- If v is real, and χ is nontrivial, the local factor is in Equation 17.

$$Z_v(\chi, s, \varphi) = \frac{\Gamma\left(\frac{s+1}{2}\right)}{\pi^{\frac{s+1}{2}}}. \tag{25}$$

- If v is complex, $\chi(z) = z^n$ for all $|z|_v = 1$ and some n. The local factor is in Equation 23.

$$Z_v(\chi, s, \varphi) = \frac{\Gamma\left(s + \frac{|n|}{2}\right)}{(2\pi)^{s+\frac{|n|}{2}-1}}. \tag{26}$$

3. Non-Archimedean Places

Recall that, for a choice of Schwartz function $\varphi = \prod_v \varphi_v$ and a character $\chi|\cdot|^s$, we get a *zeta function*

$$\zeta(\chi, s, \varphi) = \int_{\mathbb{A}_K^\times} \chi(\alpha)|\alpha|^s \varphi(\alpha)\, \mathrm{d}^\times. \tag{27}$$

The *Euler factor at* v is the local integral

$$Z_v(\chi, s, \varphi) = \int_{K_v^\times} \chi_v(\alpha_v)\varphi_v(\alpha)\mathrm{d}^\times\alpha. \tag{28}$$

Here we compute the Euler factors for non-Archimedean places v.

By writing $\chi|\cdot|^s$, we are assuming that χ is a Hecke character, so that the local character χ_v is determined by its restriction to $U_v = \mathcal{O}_{K_v}^\times$ and its value on a uniformizer $\chi_v(\pi_v)$.

3.1. **Schwartz Functions.** We will use the following Schwartz functions in each case. Let $\mathfrak{d}_v^{-1}$ be the inverse different of $K_v/\mathbb{Q}_p$. We fix an additive character $\lambda\colon \mathbb{Q}_p \to \mathbb{R}/\mathbb{Z}$ by first setting $\lambda(\mathbb{Z}_p) = 0$. For non-integral elements $t \in \mathbb{Q}_p$ we describe $\lambda(t)$ by stating that it is a rational number with a power of p in the denominator, and that $\lambda(t) - t \in \mathbb{Z}_p$. One may check that $\lambda(t) = \frac{a}{p^v} + \mathbb{Z}$ if and only if $t \equiv a \pmod{p^v}$, which shows the existence of such a map.

Let S denote the trace function $S_{K_v/\mathbb{Q}_p}\colon K_v \to \mathbb{Q}_p$. This is a continuous, additive function, so that $\lambda \circ S$ is an additive character $K_v \to \mathbb{R}/\mathbb{Z}$. Further, its kernel is the set of elements of K_v with trace in $\mathbb{Z}_p$; i.e., the kernel is the local inverse different $\mathfrak{d}_v^{-1}$. We use the local Schwartz functions

$$\varphi_v(\alpha) = \begin{cases} e^{2\pi i\lambda(S(\alpha))} & , \quad \alpha \in \mathfrak{p}_v^{-n-d} \\ 0 & , \quad \text{else} \end{cases} \tag{29}$$

Here d and n are defined such that the local inverse different is $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{-d}$ and the character χ_v has conductor $\mathfrak{p}_v^n$. Note that $\varphi_v(\alpha) = 1$ for any $\alpha \in \mathfrak{d}^{-1}$, so that φ_v is the indicator function of $\mathfrak{d}^{-1}$ when χ is unramified at v. Further, saying that χ_v has conductor $\mathfrak{p}_v^n$ is the same as saying that it is trivial on $1 + \mathfrak{p}_v^n$, but not on $1 + \mathfrak{p}_v^{n-1}$.

3.2. **Riemann Zeta.** We first consider the case $K = \mathbb{Q}$, $v = p$, $\chi = 1$, so that we can show off the general strategy in the nonarchimedean case. Since χ is trivial, every local character is $\chi_p = 1$. We let φ_p be the indicator function of $\mathbb{Z}_p$. The local integral is

$$Z_p(\chi, s, \varphi) = \int_{\mathbb{Q}_p^\times} \chi_p(t)|t|_p^s \varphi_p(t)\,\mathrm{d}^\times t = \int_{\mathbb{Z}_p} |t|_p^s\,\mathrm{d}^\times t. \tag{30}$$

We break up the domain of integration into a disjoint union of "annuli" on which the integrand is constant:

$$\mathbb{Z}_p = \bigsqcup_{r=0}^{\infty} A_r, \qquad A_r = \left\{ t \in \mathbb{Q}_p^\times \text{ s.t. } |t|_p = \frac{1}{p^r} \right\} = p^r \mathbb{Z}_p^\times. \tag{31}$$

Since each annulus A_r is a translate $p^r\mathbb{Z}_p^\times$ of $U_p = \mathbb{Z}_p^\times$, each has measure 1 by our normalization of the Haar measure. Thus

$$Z_p(\chi, s, \varphi) = \int_{\mathbb{Z}_p} |t|_p^s\,\mathrm{d}^\times t = \sum_{r=0}^{\infty} \int_{A_r} p^{-rs}\,\mathrm{d}^\times t = \sum_{r=0}^{\infty} (p^{-s})^r = \frac{1}{1 - p^{-s}}. \tag{32}$$

3.3. **Unramified Case.** Let K be a number field, v a finite place corresponding to a prime $\mathfrak{p}_v$ which is unramified in $K/\mathbb{Q}$, and π_v be a uniformizer at v. We let $\mathrm{Nm} = \mathrm{Nm}_{K/\mathbb{Q}}$ be the norm, so that $|\pi_v|_v = \mathrm{Nm}(\mathfrak{p}_v)$. Then fix a character χ so that χ_v is unramified. Our Schwartz function will be the indicator function of $\mathcal{O}_{K_v}$, since $n = d = 0$ in the notation of Section 3.1. Our local integral is

$$Z_v(\chi, s, \varphi) = \int_{K_v^\times} \chi_v(\alpha)|\alpha|_v^s \varphi_v(\alpha)\,\mathrm{d}^\times \alpha = \int_{\mathcal{O}_{K_v}} \chi_v(\alpha)|\alpha|_v^s\,\mathrm{d}^\times \alpha. \tag{33}$$

We break up $\mathcal{O}_{K_v}$ into pieces where the integrand is constant:

$$\mathcal{O}_{K_v} = \bigsqcup_{r=0}^{\infty} A_r, \qquad A_r = \left\{ t \in K_v^\times \text{ s.t. } |t|_v = \frac{1}{\mathrm{Nm}(\mathfrak{p}_v)^r} \right\} = \pi_v^r \mathcal{O}_{K_v}^\times \tag{34}$$

For any $\alpha \in A_r = \pi_v^r \mathcal{O}_{K_v}^\times$, we have

$$\chi_v(\alpha)|\alpha|_v^s \varphi_v(\alpha) = \chi_v(\pi_v^r)|\pi_v^r|_v^s = \chi_v(\pi_v)^r\,\mathrm{Nm}(\mathfrak{p}_v)^{-rs}. \tag{35}$$

Since each annulus A_r is a translate $\pi_v^r\mathcal{O}_{K_v}^\times$ of $U_v = \mathcal{O}_{K_v}^\times$, each has measure 1 by our normalization of the Haar measure. Thus

$$Z_v(\chi, s, \varphi) = \int_{\mathcal{O}_{K_v}} \chi_v(\alpha)|\alpha|_v^s\,\mathrm{d}^\times \alpha = \sum_{r=0}^{\infty} \chi_v(\pi_v)^r \left(\mathrm{Nm}(\mathfrak{p}_v)^s\right)^r \int_{\pi_v^r \mathcal{O}_{K_v}^\times} \mathrm{d}^\times \alpha. \tag{36}$$

Using the fact that $\mathcal{O}_{K_v}^\times$ is given measure 1, we have

$$Z_v(\chi, s, \varphi) = \sum_{r=0}^{\infty} \chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v^{-s})^r = \frac{1}{1 - \chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v)^{-s}}. \tag{37}$$

For $K = \mathbb{Q}$ and $v = p$ some finite prime, this specializes to

$$Z_p(\chi, s, \varphi) = \frac{1}{1 - \chi_p(p)p^{-s}}. \tag{38}$$

If χ corresponds to some classical Dirichlet character $\chi_D\colon \mathbb{N} \to \mathbb{C}^\times$ which, by assumption, is unramified at p, then $\chi_D(p) = \chi_p(p)$, recovering the classical Euler product.

3.4. **Unramified Character.** Let K be a number field, v a finite place corresponding to a prime $\mathfrak{p}_v$, and π_v be a uniformizer at v. We let $\mathrm{Nm} = \mathrm{Nm}_{K/\mathbb{Q}}$ be the norm, so that $|\pi_v|_v = \mathrm{Nm}(\mathfrak{p}_v)$. Let $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{-d}$ be the local different. Then fix a character χ so that χ_v is unramified, meaning it is trivial on $U_v := \mathcal{O}_{K_v}^\times$. Our Schwartz function will be the indicator function of $\mathfrak{d}_v^{-1}$, since $n = 0$ in the notation of Section 3.1. Our local integral is

$$Z_v(\chi, s, \varphi) = \int_{K_v^\times} \chi_v(\alpha)|\alpha|_v^s \varphi_v(\alpha)\,\mathrm{d}^\times\alpha = \int_{\mathfrak{d}_v^{-1}} \chi_v(\alpha)|\alpha|_v^s\,\mathrm{d}^\times\alpha. \tag{39}$$

We break up $\mathfrak{d}_v^{-1}$ into pieces where the integrand is constant:

$$\mathfrak{d}_v^{-1} = \bigsqcup_{r=-d}^{\infty} A_r, \qquad A_r = \left\{ t \in K_v^\times \text{ s.t. } |t|_v = \frac{1}{\mathrm{Nm}(\mathfrak{p}_v)^r} \right\} = \pi_v^r \mathcal{O}_{K_v}^\times \tag{40}$$

For any $\alpha \in A_r = \pi_v^r \mathcal{O}_{K_v}^\times$, we have

$$\chi_v(\alpha)|\alpha|_v^s \varphi_v(\alpha) = \chi_v(\pi_v^r)|\pi_v^r|_v^s = \chi_v(\pi_v)^r\,\mathrm{Nm}(\mathfrak{p}_v)^{-rs}. \tag{41}$$

This turns the integral into a sum.

$$Z_v(\chi, s, \varphi) = \int_{\mathfrak{d}_v^{-1}} \chi_v(\alpha)|\alpha|_v^s\,\mathrm{d}^\times\alpha = \sum_{r=-d}^{\infty} \chi_v(\pi_v)^r \left(\mathrm{Nm}(\mathfrak{p}_v)^{-s}\right)^r \int_{\pi_v^r \mathcal{O}_{K_v}^\times} \mathrm{d}^\times\alpha. \tag{42}$$

Since each annulus A_r is a translate $\pi_v^r \mathcal{O}_{K_v}^\times$ of $U_v = \mathcal{O}_{K_v}^\times$, each has measure $\left(\mathrm{Nm}(\mathfrak{d}_v^{-1})\right)^{\frac{1}{2}}$ by our normalization of the Haar measure. We also have that

$$\sum_{r=-d}^{\infty} \left(\chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v)^{-s}\right)^r = \chi_v(\pi_v)^{-d} \left(\mathrm{Nm}(\mathfrak{d}_v^{-1})\right)^{-s} \sum_{r=0}^{\infty} \left(\chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v)^{-s}\right)^r. \tag{43}$$

Thus we can calculate

$$Z_v(\chi, s, \varphi) = \left(\mathrm{Nm}(\mathfrak{d}_v^{-1})\right)^{\frac{1}{2}-s} \sum_{r=0}^{\infty} \left(\chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v)^{-s}\right)^r = \frac{\left(\mathrm{Nm}(\mathfrak{d}_v^{-1})\right)^{\frac{1}{2}-s} \chi_v(\pi_v)^{-d}}{1 - \chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v)^{-s}}. \tag{44}$$

Notice that this recovers the result of the previous section, since $\mathrm{Nm}\,\mathfrak{d}_v^{-1} = 1$ and $\chi_v(\pi_v)^{-d} = \chi_v(\pi_v)^0 = 1$.

3.5. **Ramified Character.** Let K be a number field, v a finite place corresponding to a prime $\mathfrak{p}_v$, and π_v be a uniformizer at v. We let $\mathrm{Nm} = \mathrm{Nm}_{K/\mathbb{Q}}$ be the norm, so that $|\pi_v|_v = \mathrm{Nm}(\mathfrak{p}_v)$. Let $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{-d}$ be the local different. Then fix a character χ so that χ_v is ramified, meaning it is nontrivial on $U_v := \mathcal{O}_{K_v}^\times$. The local conductor $\mathfrak{p}_v^n$ is such that χ_v is trivial on $1 + \mathfrak{p}_v^n$ but not on $1 + \mathfrak{p}_v^{n-1}$. In particular, $n \geq 1$. We fix coset representatives

$$U_v = \mathcal{O}_{K_v}^\times = \bigsqcup_{i=1}^{h} \alpha_i + \mathfrak{p}_v^n. \tag{45}$$

We will finally use the whole scope of 3.1 in defining our Schwartz function. Recall that $\lambda\colon \mathbb{Q}_p \to \mathbb{R}/\mathbb{Z}$ is a fixed additive character with $\lambda(a/p^r) = a/p^r$ for all integers a, and $S = S_{K_v/\mathbb{Q}_p}$ is the trace. We use the Schwartz function

$$\varphi_v(\alpha) = \begin{cases} e^{2\pi i \lambda(S(\alpha))} & , \quad \alpha \in \mathfrak{p}_v^{-n-d} \\ 0 & , \quad \text{else} \end{cases} \tag{46}$$

The local integral is

$$Z_v(\chi, s, \varphi) = \int_{K_v^\times} \chi_v(\alpha)|\alpha|_v^s \varphi_p(\alpha)\, \mathrm{d}^\times\alpha = \int_{\mathfrak{p}_v^{-n-d}} \chi_v(\alpha)|\alpha|_v^s \varphi_p(\alpha)\, \mathrm{d}^\times\alpha. \tag{47}$$

We break up the integral into pieces where the integrand is constant. First, we stratify in terms of $|\cdot|_v$.

$$\mathfrak{p}_v^{-n-d} = \bigsqcup_{r=-n-d}^{\infty} A_r, \quad A_r = \left\{\alpha \in K_v^\times \text{ s.t. } |\alpha|_v = \frac{1}{\mathrm{Nm}(\mathfrak{p})^r}\right\} = \pi_v^r U_v. \tag{48}$$

Each of these must break up further.

$$A_r = \bigsqcup_{i=1}^{h} A_r^i, \quad A_r^i = \pi_v^r\left(\alpha_i + \mathfrak{p}_v^n\right), \qquad \mathfrak{p}_v^{-n-d} = \bigsqcup_{r=-n-d}^{\infty} \bigsqcup_{i=1}^{h} A_r^i. \tag{49}$$

Finally, we have that the integrand is constant on each A_r^i. The integral turns into a sum.

$$Z_v(\chi, s, \varphi) = \sum_{r=-n-d}^{\infty} \sum_{i=1}^{h} \int_{A_r^i} \chi(\alpha)|\alpha|_v^s \varphi_p(\alpha)\, \mathrm{d}^\times\alpha. \tag{50}$$

For every $\alpha \in A_r^i$, we get

$$\chi(\alpha)|\alpha|_v^s \varphi_p(\alpha) = \chi(\alpha_i)\chi(\pi_v)^r\, \mathrm{Nm}(\mathfrak{p}_v)^{-rs} \varphi_v(\pi_v^r \alpha_i). \tag{51}$$

This translates to

$$Z_v(\chi, s, \varphi) = \sum_{r=-n-d}^{\infty} \sum_{i=1}^{h} \int_{A_r^i} \chi_v(\alpha_i)\chi_v(\pi_v)^r\, \mathrm{Nm}(\mathfrak{p}_v)^{-rs} \varphi_v(\pi_v^r \alpha_i)\, \mathrm{d}^\times\alpha. \tag{52}$$

Because the integrand is constant on each A_r^i, and A_r^i has measure $\frac{1}{h}|\mathrm{Nm}\,\mathfrak{d}_v^{-1}|^{\frac{1}{2}}$, we can lose any mention to integration.

$$Z_v(\chi, s, \varphi) = \frac{1}{h}|\mathrm{Nm}\,\mathfrak{d}_v^{-1}|^{\frac{1}{2}} \sum_{r=-n-d}^{\infty} \sum_{i=1}^{h} \chi_v(\alpha_i)\chi_v(\pi_v)^r\, \mathrm{Nm}(\mathfrak{p}_v)^{-rs} \varphi_v(\pi_v^r \alpha_i). \tag{53}$$

The crux of the argument is that the sum over i is zero whenever $r > -n - d$. This is most easily seen when $r \geq -d$, in which case $\varphi_p(\alpha) = 1$:

$$\sum_{i=1}^{h} \chi_v(\alpha_i)\chi_v(\pi_v)^r\, \mathrm{Nm}(\mathfrak{p}_v)^{-rs} = \chi_v(\pi_v)^r\, \mathrm{Nm}(\mathfrak{p}_v)^{-rs} \sum_{i=1}^{h} \chi_v(\alpha_i) = 0. \tag{54}$$

This sum collapses because it is the sum of the values of a nontrivial character. The same argument will be used for $-d > r > -n - d$, but it will need to be modified to reflect the fact that φ_v is not constant on A_r.

We already have a decomposition of $A_r = \pi_v^r U_v$ as a disjoint union of the sets A_r^i, but we will need an intermediate decomposion as well. Let $I_r = \{i \mid \alpha_i \in 1 + \mathfrak{p}_v^{-r}\}$, and note that the subgroup $1 + \mathfrak{p}_v^{-r}$ of U_v is comprised of the cosets $\alpha_i + \mathfrak{p}_v^n$ for $i \in I_r$. Further note that χ_v is not trivial on $1 + \mathfrak{p}_v^{-r}$, since $-r < n$, but φ_v is constant on $\pi_v^r + \mathcal{O}_{K_v} = \bigsqcup_{i \in I_j} \pi_v^r\left(\alpha_i + \mathfrak{p}_v^n\right)$.

Pick a maximal subset $J_r \subset \{1, \ldots, h\}$ such that $\alpha_{j_1} \not\equiv \alpha_{j_2} \pmod{\mathfrak{p}_v^{-r}}$ for any pair of distinct indices $j_1 \neq j_2$ from J_r. Thus, for any $j \in J_r$, we have that $\alpha_j + \mathfrak{p}_v^{-r}$ is comprised of the cosets $\alpha_j\alpha_i + \mathfrak{p}^n$ where i ranges over the elements of I_r. We get our decomposition.

$$A_r = \bigsqcup_{i \in I_r} \bigsqcup_{j \in J_r} A_r^{ij}, \qquad A_r^{ij} := \pi_v^r(\alpha_i\alpha_j + \mathfrak{p}_v^n). \tag{55}$$

Notice that this is the same decomposition as before: each set A_r^{ij} for $i \in I_r$ and $j \in J_r$ is equal to a set A_r^i for some $i = 1, \dots, h$. Thus, on A_r^{ij}, the integrand is constant.

$$\chi(\alpha)|\alpha|_v^s \varphi_v(\alpha) = \chi_v(\alpha_j)\chi_v(\alpha_i)\chi_v(\pi_v)^r \operatorname{Nm}(\mathfrak{p}_v)^{-rs}\varphi_v(\pi_v^r\alpha_j). \tag{56}$$

The disjoint union allows us to write

$$Z_v(\chi, s, \varphi) = \sum_{j\in J_r}\sum_{i\in I_r}\int_{A_r^{ij}} \chi(\alpha)|\alpha|_v^s \varphi_p(\alpha)\,\mathrm{d}^\times\alpha \tag{57}$$

Using the fact that A_r^{ij} has measure $\frac{1}{h}|\operatorname{Nm}\mathfrak{d}_v^{-1}|^{\frac{1}{2}}$, we get the sum

$$Z_v(\chi, s, \varphi) = \frac{1}{h}|\operatorname{Nm}\mathfrak{d}_v^{-1}|^{\frac{1}{2}} \sum_{j\in J_r}\sum_{i\in I_r} \chi_v(\alpha_j)\chi_v(\alpha_i)\chi_v(\pi_v)^r \operatorname{Nm}(\mathfrak{p}_v)^{-rs}\varphi_p(\pi^r\alpha_j). \tag{58}$$

Only one factor depends on α_i, so we rearrange

$$Z_v(\chi, s, \varphi) = \frac{1}{h}|\operatorname{Nm}\mathfrak{d}_v^{-1}|^{\frac{1}{2}} \sum_{j\in J_r} \chi_v(\alpha_j)\chi_v(\pi_v)^r \operatorname{Nm}(\mathfrak{p}_v)^{-rs}\varphi_p(\pi^r\alpha_j) \sum_{i\in I_r}\chi_v(\alpha_i). \tag{59}$$

The last factor (the sum over $i \in I_r$) is the sum of the values of a nontrivial character $\chi_v|_{1+\mathfrak{p}^{-r}}$, and is thus zero. This leaves that the only value of r that contributes to the sum in Equation 52 is $r = -n - d$.

$$Z_v(\chi, s, \varphi) = \sum_{i=1}^{h}\int_{A^i_{-n-d}} \chi_v(\alpha_i)\chi_v(\pi_v)^{-n-d}\operatorname{Nm}(\mathfrak{p}_v)^{(n+d)s}\varphi_v(\pi_v^{-n-d}\alpha_i)\,\mathrm{d}^\times\alpha. \tag{60}$$

Pulling out the factors of $\chi(\pi_v)^{-n-d}\operatorname{Nm}(\mathfrak{p}_v)^{(n+d)s}$, and using again that each A^i_{-n-d} has measure $\frac{1}{h}|\operatorname{Nm}\mathfrak{d}_v^{-1}|^{\frac{1}{2}}$, we see that our local integral is a nonzero multiple of a *Gauss sum*, and is this nonzero.

$$Z_v(\chi, s, \varphi) = \frac{\operatorname{Nm}(\mathfrak{p}_v)^{\frac{d}{2}+(n+d)s}}{h\chi(\pi_v)^{n+d}}\tau(\chi_v), \qquad \tau(\chi_v) = \sum_{i=1}^{h}\chi_v(\alpha_i)\varphi_v(\pi_v^{-n-d}\alpha_i). \tag{61}$$

To see that this final sum is a Gauss sum, we specialize to the case of a Dirichlet character and specific choices of α_i. For $K = \mathbb{Q}$, we have $d = 0$, and we can ignore the trace in the definition of φ_p. Let $\chi_p^D\colon \mathbb{N} \to \mathbb{C}^\times$ be a classical Dirichlet character of conductor p^n satisfying

$$\chi_p^D(N) = \begin{cases} \chi_p(N) & , \quad p \nmid N \\ 0 & , \quad \text{else} \end{cases} \tag{62}$$

We let each coset representative α_i be an integer with $1 \le \alpha_i \le p^n$, which is necessarily prime to p. We have

$$\tau_p(\chi_p) = \sum_{i=1}^{h}\chi_p(\alpha_i)\varphi_p\left(\frac{\alpha_i}{p^n}\right) = \sum_{i=1}^{h}\chi_p(\alpha_i)e^{\frac{2\pi i\alpha_i}{p^n}}. \tag{63}$$

Because each α_i is an integer, we have $\chi_p(\alpha_i) = \chi_p^D(\alpha_i)$; also, since $\chi_p^D(a) = 0$ for any integer $1 \le a \le p^n$ which is *not* equal to one of the α_i's, we can add in more terms to the sum without changing the value:

$$\tau(\chi_p) = \sum_{i=1}^{h}\chi_p(\alpha_i)e^{\frac{2\pi i\alpha_i}{p^n}} = \sum_{a=1}^{p^n}\chi_p^D(a)e^{\frac{2\pi ia}{p^n}} \tag{64}$$

This is the classical Gauss sum attached to the Dirichlet character χ_p^D. It is nonzero, and in fact it has magnitude $|\tau(\chi_p)| = p^{\frac{n}{2}}$.

3.6. **Nonarchimedean Answers.** Here we collect the final computations. We write $\chi|\cdot|^s$ for the character, letting χ be a unitary Hecke character. We did four computations:

- If $K = \mathbb{Q}$ and $\chi = 1$, we computed the Euler factor of the Riemann zeta function in Equation 32:

$$Z_p(\chi, s, \varphi) = \frac{1}{1 - p^{-s}}. \tag{65}$$

- If v is a finite place corresponding to a prime $\mathfrak{p}_v$ which is unramified in $K/\mathbb{Q}$, and χ is a Hecke character unramified at v, we found in Equation 37 that

$$Z_v(\chi, s, \varphi) = \frac{1}{1 - \chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v)^{-s}}. \tag{66}$$

- If v is a finite place corresponding to a prime $\mathfrak{p}_v$ so that the local different is $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{-d}$, and χ is a Hecke character unramified at v, we found in Equation 44 that

$$Z_v(\chi, s, \varphi) = \frac{(\mathrm{Nm}(\mathfrak{d}_v^{-1}))^{\frac{1}{2}-s}\chi_v(\pi_v)^{-d}}{1 - \chi_v(\pi_v)\,\mathrm{Nm}(\mathfrak{p}_v)^{-s}}. \tag{67}$$

- Finally, if v is a finite place corresponding to a prime $\mathfrak{p}_v$ so that the local different is $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{-d}$, and χ is a Hecke character ramified at v of conductor $\mathfrak{p}_v^n$, we found in Equation 61 that

$$Z_v(\chi, s, \varphi) = \frac{\mathrm{Nm}(\mathfrak{p}_v)^{\frac{d}{2}+(n+d)s}}{h\chi(\pi_v)^{n+d}}\tau(\chi_v), \qquad \tau(\chi_v) = \sum_{i=1}^{h} \chi_v(\alpha_i)\varphi_v(\pi_v^{-n-d}\alpha_i). \tag{68}$$

  Classically, we would expect a contribution of 1 at these places - instead we get a factor that is important when defining the functional equation, just like in the Archimedean case.

Jon Aycock, Department of Mathematics, University of Oregon, Fenton Hall, Eugene, OR 97403-1222, USA
E-mail address: jaycock@uoregon.edu