Minimal N.B. Matrix Diagonalization

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1 Mass Matrix Diagonalization

1.1 Methods

A helpful starting place for the mass matrix diagonalization is the lagrangian since, say for some proposed theory in qft, we expect the lagrangian to give us the couplings of the theory. You can write $\mathcal{L}$ in terms of the mass mass matrix from which, you may diagonalize.

So, let's start with an example and go no further. Revisiting a minimal Nelson-Barr Model of [1], our lagrangian:

$$-\mathcal{L} \supset M_0 \bar{\psi}_L \psi_R + B_{0i} \bar{\psi}_L u_{Ri} + y^u_{ij} \bar{Q}_{Li} \tilde{H} u_{Ri} + y^d_{ij} \bar{Q}_{Li} H d_{Rj}.$$

We may write the Lagrangian as a product of the basis vectors, on the left and right, and the mass matrix (in the middle). By doing the matrix multiplication, one can see that the below is no different from the lagrangian above.

$$-\mathcal{L} \supset \begin{bmatrix} \bar{u}_L & \bar{\psi}_L \end{bmatrix} \underbrace{\begin{bmatrix} \frac{v}{\sqrt{2}} y^u & 0 \\ B_0 & M_0 \end{bmatrix}}_{\mathcal{M}_0} \begin{bmatrix} u_R \\ \psi_R \end{bmatrix},$$

where

$$\bar{m}_{ij} = y^d_{ij} V_H$$

and

$$B_j = (g_j S + g_i^* S^*)\, V_s.$$

Our goal is to diagonalize this mass matrix; let us consider the traditional method first. Given some n × n matrix A, consider a general $n \times n$ matrix A with elements a_{ij}:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Theorem 1.1 An n × n matrix A can be diagonalized iff it has n linearly independent Eigen-vectors.

Also, an $n \times n$ matrix is orthogonally diagonalizable iff A is a symmetric matrix.

So, say, for some matrix A, the set of orthogonal eigenvectors, say $\{u_1, u_2, u_3, \ldots, u_n\}$ form the matrix P, and the diagonal matrix D has the eigenvalues λ_i on its diagonal entries.

$$A = PDP^T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix}^T$$

Hence, in order to diagonalize A, you must find the eigenvalues and the *orthogonal* eigenvectors of A. Now, this may seem easy at first glance, but when you actually try to do this for the mass matrix above, it can be very hard. Let's try something else.

Method 1

This first method of partial diagonalization is taken from [7] & [8]. The Nelson-Barr Lagrangian looks like

$$-\mathcal{L} \supset M_0\bar{\psi}_L\psi_R + B_{0i}\bar{\psi}_L u_{Ri} + y^u_{ij}\bar{Q}_{Li}\tilde{H}u_{Ri} + y^d_{ij}\bar{Q}_{Li}Hd_{Rj}.$$

A generic Lagrangian with couplings only between Standard Model (SM) quarks is

$$-\mathcal{L} \supset M\psi_L\psi_R + B_i\bar{Q}_{Li}\tilde{H}\psi_R + Y^u_{ij}\bar{Q}_{Li}\tilde{H}u_{Ri}y^u_{ij}\bar{Q}_{Li}\tilde{H}u_{Rj} + Y^d_{ij}\bar{Q}_{Li}Hd_{Rj}.$$

The up quark sector of the Nelson-Barr Lagrangian takes the following matrix form:

$$-\mathcal{L} \supset (\bar{u}_L\bar{\psi}_L)\underbrace{\begin{pmatrix} \frac{v}{\sqrt{2}}y^u & 0 \\ B_0 & M_0 \end{pmatrix}}_{\mathcal{M}_0}\begin{pmatrix} u_R \\ \psi_R \end{pmatrix},$$

while the generic Lagrangian takes the following matrix form:

$$-\mathcal{L} \supset (\bar{u}_L\bar{\psi}_L)\underbrace{\begin{pmatrix} \frac{v}{\sqrt{2}}Y^u & \frac{v}{\sqrt{2}}B \\ 0 & M \end{pmatrix}}_{\mathcal{M}}\begin{pmatrix} u_R \\ \psi_R \end{pmatrix}.$$

We seek a matrix W_R which rotates the first form of the mass matrix $\mathcal{M}_0$ into the second $\mathcal{M}$ such that

$$\mathcal{M}_0 W_R = \mathcal{M}$$

We can write the matrix W_R as four blocks such that

$$W_R = \begin{pmatrix} K & R \\ S & T \end{pmatrix}.$$

Now, we have that

$$\begin{pmatrix} \frac{v}{\sqrt{2}}y^u & 0 \\ B_0 & M_0 \end{pmatrix}\begin{pmatrix} K & R \\ S & T \end{pmatrix} = \begin{pmatrix} \frac{v}{\sqrt{2}}Y^u & \frac{v}{\sqrt{2}}B \\ 0 & M \end{pmatrix}.$$

Doing some finagling, we find four equations

$$\begin{aligned} Y^u &= y^u K, \\ B &= y^u R, \\ S &= -\frac{B_0 K}{M_0}, \\ M &= B_0 R + M_0 T. \end{aligned}$$

From the unitarity of W_R, from which follows the relation, $\mathcal{M}_0\mathcal{M}_0^\dagger = \mathcal{M}\mathcal{M}^\dagger$, we obtain the following sum rule:

$$Y^u Y^{u\dagger} + BB^\dagger = y^u y^{u\dagger}.$$

Unitarity of W_R also means that

$$M^2 = H_B \equiv B_0 B_0^\dagger + M_0^2$$

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and

$$KK^\dagger = \mathbb{1} - \frac{B_0^\dagger B_0}{M^2}.$$

Using Eq. 1.1.1 and its square, we see that the simplest consistent choices for R and T are

$$R = \frac{B_0^\dagger}{M} \quad \text{and} \quad T = \frac{M_0}{M}.$$

Finally, using Eq.s 1.1.1 & 1.1.1, we obtain

$$\begin{aligned} Y^u Y^{u\dagger} &= y^u K K^\dagger y^{u\dagger}, \\ &= y^u \left(\mathbb{1} - \frac{B_0^\dagger B_0}{M^2} \right) y^{u\dagger}, \end{aligned}$$

$$\frac{v^2}{2} Y^u Y^{u\dagger} = m^u m^{u\dagger} - \frac{m^u B_0^\dagger B_0 m^{u\dagger}}{M^2}$$

Mass Diagonilization Calculation

Let us now do the mass diagonalization for this Lagrangian.

The lagrangian

$$-\mathcal{L} \supset \mu \bar{q} \bar{q} + \left(g_i \Phi + \tilde{g}_i \Phi^* \right) \bar{u}_i q + y_{ij}^u \tilde{H} Q_i \bar{u}_j + y_{ij}^d H Q_i d_j + h.c$$

In Quark matrix form

$$-\mathcal{L} \supset \begin{bmatrix} q_i & u_{Li} \end{bmatrix} M \begin{bmatrix} \bar{q} \\ u_{Ri} \end{bmatrix},$$

where

$$M = \begin{bmatrix} v y_{ij}^u & 0 \\ B_i & \mu \end{bmatrix} = \begin{bmatrix} \underline{m}_{ij}^u & 0 \\ B_i & \mu \end{bmatrix}, \quad B_i = \left(g_j \Phi + g_i^* \Phi^* \right), \quad S = v e^{i\Theta}, \quad i, j = 1, 2, 3$$

Theorem 1.2 The unitary matrices $U_L^\dagger$, U_R will diagonilize via $U_L^\dagger M U_R = diag\,(M)$

$$U_L^\dagger M U_R = M_d, \quad U_L^\dagger M U_R = \begin{bmatrix} \tilde{m} & 0 \\ 0 & \tilde{M} \end{bmatrix}$$

So starting with 1.2,

$$U_L^\dagger M U_R = M_d$$

$$U_L^\dagger M U_R = \begin{bmatrix} \tilde{m} & 0 \\ 0 & \tilde{M} \end{bmatrix}$$

Now, multiply through by $\left(U_L^\dagger M U_R \right)^\dagger$ on the RHS

$$\left(U_L^\dagger M U_R \right) \left(U_L^\dagger M U_R \right)^\dagger = \begin{bmatrix} \tilde{m} & 0 \\ 0 & \tilde{M} \end{bmatrix} \left(U_L^\dagger M U_R \right)^\dagger$$

$$\left(U_L^\dagger M U_R \right) \left(U_L^\dagger M U_R \right)^\dagger = \begin{bmatrix} \tilde{m} & 0 \\ 0 & \tilde{M} \end{bmatrix} \left(\begin{bmatrix} \tilde{m} & 0 \\ 0 & \tilde{M} \end{bmatrix} \right)^\dagger$$

$$\left(U_L^\dagger M U_R \right) \left(U_L^\dagger M U_R \right)^\dagger = \begin{bmatrix} \tilde{m} & 0 \\ 0 & \tilde{M} \end{bmatrix} \begin{bmatrix} \tilde{m} & 0 \\ 0 & \tilde{M} \end{bmatrix}$$

$$\left(U_L^\dagger M U_R\right)\left(U_L^\dagger M U_R\right)^\dagger = \begin{bmatrix} \tilde{m^2} & 0 \\ 0 & \tilde{M}^2 \end{bmatrix}$$

$$U_L^\dagger M U_R U_R^\dagger M^\dagger U_L = \begin{bmatrix} \tilde{m^2} & 0 \\ 0 & \tilde{M}^2 \end{bmatrix}$$

$$U_L^\dagger M M^\dagger U_L = M_d^2$$

Multiply both sides by U_L on LHS

$$U_L\left(U_L^\dagger M M^\dagger U_L\right) = U_L\left(M_d^2\right)$$

$$U_L U_L^\dagger M M^\dagger U_L = U_L\left(M_d^2\right)$$

$$M M^\dagger U_L = U_L\left(M_d^2\right)$$

This is great! Now, we have rewritten 1.2 with like terms ($M, M^\dagger$, and U_L).

We know from theorem 1.1, and also from the corresponding matrices in which it will multiply through, that the size of U_L must be nxn . Hence, let us now prescribe unknown variables to each element of the matrix for which we will solve.

$$U_L = \begin{bmatrix} K & R \\ S & T \end{bmatrix}$$

Let us put the this and the previous result together.

$$M M^\dagger U_L = U_L\left(M_d^2\right)$$

$$\begin{bmatrix} \underline{m}_{ij}^u & 0 \\ B_i & \mu \end{bmatrix} \begin{bmatrix} \underline{m}_{ij}^u & 0 \\ B_i & \mu \end{bmatrix}^\dagger \begin{bmatrix} K & R \\ S & T \end{bmatrix} = \begin{bmatrix} K & R \\ S & T \end{bmatrix} \begin{bmatrix} \tilde{m^2} & 0 \\ 0 & \tilde{M}^2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{m}_{ij}^u & 0 \\ B_i & \mu \end{bmatrix} \begin{bmatrix} \underline{m}_{ij}^{u\dagger} & B_i^\dagger \\ 0 & \mu \end{bmatrix} \begin{bmatrix} K & R \\ S & T \end{bmatrix} = \begin{bmatrix} K & R \\ S & T \end{bmatrix} \begin{bmatrix} \tilde{m^2} & 0 \\ 0 & \tilde{M}^2 \end{bmatrix}$$

Plow the left matrix into the right.

$$\begin{bmatrix} \underline{m}_{ij}^u \underline{m}_{ij}^{u\dagger} & \underline{m}_{ij}^u B_i^\dagger \\ \underline{m}_{ij}^{u\dagger} B_i & B_i B_i^\dagger + \mu^2 \end{bmatrix} \begin{bmatrix} K & R \\ S & T \end{bmatrix} = \begin{bmatrix} K & R \\ S & T \end{bmatrix} \begin{bmatrix} \tilde{m^2} & 0 \\ 0 & \tilde{M}^2 \end{bmatrix}$$

Likewise, matrix multiply the rest of the matrices.

$$\begin{bmatrix} \underline{m}_{ij}^u \underline{m}_{ij}^{u\dagger} & \underline{m}_{ij}^u B_i^\dagger \\ \underline{m}_{ij}^{u\dagger} B_i & B_i B_i^\dagger + \mu^2 \end{bmatrix} \begin{bmatrix} K & R \\ S & T \end{bmatrix} = \begin{bmatrix} \tilde{m}^2 K & \tilde{M}^2 R \\ \tilde{m}^2 S & \tilde{M}^2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{m}_{ij}^u \underline{m}_{ij}^{u\dagger} K + \underline{m}_{ij}^u B_i^\dagger S & \underline{m}_{ij}^u \underline{m}_{ij}^{u\dagger} R + \underline{m}_{ij}^u B_i^\dagger T \\ \underline{m}_{ij}^{u\dagger} B_i K + \left(B_i B_i^\dagger + \mu^2\right) S & \underline{m}_{ij}^{u\dagger} B_i R + \left(B_i B_i^\dagger + \mu^2\right) T \end{bmatrix} = \begin{bmatrix} \tilde{m}^2 K & \tilde{M}^2 R \\ \tilde{m}^2 S & \tilde{M}^2 T \end{bmatrix}$$

Now, we've been left to solve a system of linear equations (of sorts). We understand that each respective element for the left and right matrices must be equal. Hence, we can set them equal and solve.

(i) $\underline{m}_{ij}^u \underline{m}_{ij}^{u\dagger} K + \underline{m}_{ij}^u B_i^\dagger S = \tilde{m}^2 K$
(ii) $\underline{m}_{ij}^u \underline{m}_{ij}^{u\dagger} R + \underline{m}_{ij}^u B_i^\dagger T = \tilde{M}^2 R$
(iii) $\underline{m}_{ij}^{u\dagger} B_i K + \left(B_i B_i^\dagger + \mu^2\right) S = \tilde{m}^2 S$

(iv) $\underline{m}^{u\dagger}_{ij} B_i R + \left(B_i B_i^\dagger + \mu^2\right) T = \tilde{M}^2 T$

Using equation (iii), we find:

$$S = \frac{\underline{m}^{u\dagger}_{ij} B_i K}{B_i B_i^\dagger - \tilde{m}^2 - \mu^2}$$

Plugging this into the original equation:

$$\underline{m}^{u}_{ij} \underline{m}^{u\dagger}_{ij} K - \underline{m}^{u}_{ij} \underline{m}^{u\dagger}_{ij} \left(\frac{\underline{m}^{u\dagger}_{ij} B_i K}{B_i B_i^\dagger - \tilde{m}^2 - \mu^2} \right) = \tilde{m}^2 K$$

Let

$$K^{-1} \tilde{m}^2 K = m^u_{ij} m^{u\dagger}_{ij} + \underline{m}^{u\dagger}_{ij} B_i \left(\frac{m_u^2 B_i}{u^2 - \tilde{m}^2} \right)$$

where

$$\tilde{m}^2 = m^u_{ij} m^{u\dagger}_{ij} + m_u^* B_i m^{u\dagger}_{ij} B_i \left(\frac{1}{u^2} \left(\tilde{m}^2 - 1\right)^{-1} \right)$$

Assuming vector-like masses, we have $m_u\left(\Lambda\right) \gg m_u \varepsilon$.
Then,

$$\tilde{m}^u_{ij} \tilde{m}^{u\dagger}_{ij} = \left(m^u_{ij} m^{u\dagger}_{ij} - m^u_{ij} B_i B_i^\dagger m^{u\dagger}_{ij} \right) \left(\mu^2 + B_i B_i^\dagger \right)^{-1}$$

Riferimenti bibliografici

[1] Luis Bento, Gustavo C. Branco, and Paulo A. Parada, *A minimal model with natural suppression of strong CP violation*, Physics Letters B, vol. 267, no. 1, pp. 95–99, Elsevier, 1991.

[2] Michael Dine, Gilad Perez, Wolfram Ratzinger, and Inbar Savoray, *Nelson-Barr Ultralight Dark Matter*,

[3] Howard E. Haber and Ze'ev Surujon, *Group-theoretic condition for spontaneous CP violation*, Physical Review D, vol. 86, no. 7, Article 075007, American Physical Society (APS), 2012, http://dx.doi.org/10.1103/PhysRevD.86.075007.

[4] Kai Murai and Kazunori Nakayama, *Revisiting the Minimal Nelson-Barr Model*, arXiv:2407.16202, 2024, https://arxiv.org/abs/2407.16202.

[5] Pouya Asadi, Samuel Homiller, Qianshu Lu, and Matthew Reece, *Chiral Nelson-Barr Models: Quality and Cosmology*, Phys. Rev. D, vol. 107, no. 11, Article 115012, American Physical Society, 2023, https://link.aps.org/doi/10.1103/PhysRevD.107.115012.

[6] Alessandro Valenti and Luca Vecchi, *The CKM phase and $\overline{\vartheta}$ in Nelson-Barr Models*, Journal of High Energy Physics, vol. 2021, no. 7, Article 203, Springer, 2021, https://link.springer.com/10.1007/JHEP07(2021)203.

[7] A. L. Cherchiglia, G. De Conto, and C. C. Nishi, *Flavor Constraints for a Vector-like Quark of Nelson-Barr Type*, Journal of High Energy Physics, vol. 2021, no. 11, Article 93, Springer, 2021, https://link.springer.com/10.1007/JHEP11(2021)093.

[8] A. L. Cherchiglia and C. C. Nishi, *Consequences of Vector-like Quarks of Nelson-Barr Type*, Journal of High Energy Physics, vol. 2020, no. 8, Article 104, Springer, 2020, https://link.springer.com/10.1007/JHEP08(2020)104.