

# Minimal N.B. one-loop correction

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## 1 One Loop Corrections

### One Loop calculation

Let us do the one-loop correction for the minimal Nelson bar model of [1]

Following [4], we see that in order for one-loop corrections to have an effect on the value of  $\bar{\vartheta}$  they have to either produce two nonzero entries in the off-diagonals of the mass mixing matrix, or simply introduce a single phase dependence in the diagonals of the mass mixing matrix such that there are nonzero complexity and determinant. This means that in the simplest scenario, we have that in the diagonalized mass basis,

$$-\mathcal{L} = \bar{u}'_{L\alpha}(m_\alpha + \delta m_\alpha)u'_{R\alpha} + \text{h.c.},$$

where  $u_{L\alpha} = V_{\alpha\beta}u'_{L\beta}$ ,  $u_{R\alpha} = U_{\alpha\beta}u'_{R\beta}$ ,  $m_\alpha$  are the diagonal entries of the mass matrix,  $\delta m_\alpha$  are the corrections to the diagonal entries of the mass matrix, and  $\alpha = 1, \dots, 4$ . We work in the Weyl basis, such that

$$\gamma_5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

We define the vectors  $u_\alpha = \begin{pmatrix} u_{L\alpha} \\ u_{R\alpha} \end{pmatrix}$  and  $\bar{u}_\alpha = (\bar{u}_{L\alpha} \quad \bar{u}_{R\alpha})$ . These let us write a combined expression:

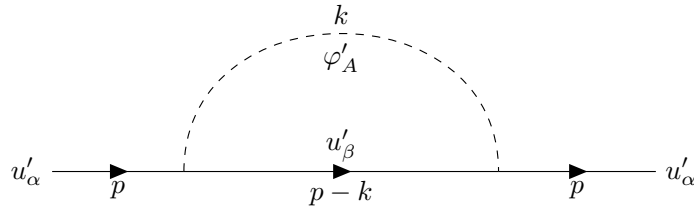
$$-\mathcal{L} = \bar{u}'_\alpha(m_\alpha + \Sigma_\alpha)u'_\alpha,$$

where  $\Sigma_\alpha = \text{Re}(\delta m_\alpha) + i\gamma_5 \text{Im}(\delta m_\alpha)$ . The quantity  $\bar{\vartheta}$  is then given as

$$\bar{\vartheta} = \arg[\det(m_\alpha + \Sigma_\alpha)] \simeq \arg[\text{Tr}(\mathbf{1} + m_\alpha^{-1}\Sigma_\alpha)] = \arg[1 + \text{Tr}(m_\alpha^{-1}\Sigma_\alpha)] \simeq \text{Tr}[m_\alpha^{-1}\text{Im}(\Sigma_\alpha)].$$

The one-loop self-energy is calculated as,

$$\Sigma_\alpha = -i \sum_{A,\beta} R_A^a R_A^b \tilde{\Gamma}'_{\alpha\beta}{}^a I_{A,\alpha,\beta}(\not{p}) \tilde{\Gamma}'_{\beta\alpha}{}^b,$$



**Figure 1.** The diagram for the one-loop correction to the mass matrix.

## 1 One Loop Corrections

where the primes denote the mass basis,  $R_A^a$  is a real orthogonal matrix,

$$\Gamma_{\alpha\beta}^h = \frac{1}{\sqrt{2}} \begin{pmatrix} y_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_{\alpha\beta}^s = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ g_i e^{i\vartheta} + g'_i e^{-i\vartheta} & 0 \end{pmatrix}, \quad \Gamma_{\alpha\beta}^\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ i(g_i e^{i\vartheta} - g'_i e^{-i\vartheta}) & 0 \end{pmatrix},$$

$$\Gamma_{\alpha\beta}'^a = U_{L\alpha\gamma}^\dagger \Gamma_{\gamma\delta}^a U_{R\delta\beta}, \quad \tilde{\Gamma}_{\alpha\beta}'^a = \Gamma_{\alpha\beta}'^{a(\text{H})} + \gamma_5 \Gamma_{\alpha\beta}'^{a(\text{AH})},$$

where  $X^{(\text{H})} \equiv (X + X^\dagger)/2$  and  $X^{(\text{AH})} \equiv (X - X^\dagger)/2$  for a general matrix  $X$

$$I_{A,\alpha,\beta}(\not{p}) = \int \frac{d^4 k}{(2\pi)^4} \frac{i(\not{p} - \not{k} + m_\beta)}{(p-k)^2 - m_\beta^2} \frac{i}{k^2 - m_A^2}.$$

Changing the variable of integration  $k \rightarrow K$ , and using

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2},$$

along with

$$K^\mu = k^\mu - x p^\mu \quad \text{and} \quad M^2 = (1-x)m_A^2 + x m_\beta^2 - x(1-x)p^2,$$

we obtain

$$I_{A,\alpha,\beta}(\not{p}) = - \int \frac{d^4 K}{(2\pi)^4} \int_0^1 dx \frac{((1-x)\not{p} - \not{K} + m_\beta)}{(K^2 - M^2)^2}.$$

Wick rotating into Euclidean space, such that  $K^0 = iK_E^0$  and  $K^i = K_E^i$ , with  $i = 1, 2, 3$ , we get that

$$I_{A,\alpha,\beta}(\not{p}) = -i \int \frac{d^4 K_E}{(2\pi)^4} \int_0^1 dx \frac{((1-x)\not{p} - \not{K}_E + m_\beta)}{(K_E^2 + M^2)^2}.$$

The term in the numerator involving  $p$  does not contribute to  $\text{Im}(\delta m_\alpha)$ , so it vanishes and we get

$$I_{A,\alpha,\beta}(\not{p}) = -2i\pi^2 \int \frac{dK_E}{(2\pi)^4} \int_0^1 dx \frac{K_E^3 ((1-x)\not{p} - \not{K}_E + m_\beta)}{(K_E^2 + M^2)^2} \approx \frac{i}{16\pi^2} m_\beta \int_0^1 dx \left( \log\left(\frac{M^2}{\Lambda^2}\right) + 1 \right).$$

Using  $m_A \gg m_\alpha, m_\beta$ , we can approximate  $I_{A,\alpha,\beta}(\not{p})$  as

$$I_{A,\alpha,\beta}(\not{p}) \approx \frac{i m_\beta}{16\pi^2} \log\left(\frac{m_A^2}{\Lambda^2}\right).$$

This means that

$$\sum_\alpha \frac{\Sigma_\alpha}{m_\alpha} = \sum_{A,\alpha,\beta} \frac{1}{16\pi^2} R_A^a R_A^b m_\alpha^{-1} \tilde{\Gamma}_{\alpha\beta}'^a m_\beta \tilde{\Gamma}_{\beta\alpha}'^b J_{A,\alpha,\beta},$$

where we've defined

$$J_{A,\alpha,\beta} = \log\left(\frac{m_A^2}{\Lambda^2}\right).$$

Dropping terms that aren't proportional to  $\gamma_5$ , we have

$$\bar{\vartheta} = \sum_{A,\alpha,\beta} \frac{1}{16\pi^2} R_A^a R_A^b m_\alpha^{-1} \text{Im} \left[ \Gamma_{\alpha\beta}'^{a(\text{H})} m_\beta \Gamma_{\beta\alpha}'^{b(\text{AH})} + \Gamma_{\alpha\beta}'^{a(\text{AH})} m_\beta \Gamma_{\beta\alpha}'^{b(\text{H})} \right] J_{A,\alpha,\beta}.$$

Now, using  $m_\alpha = U_L^\dagger \mathcal{M}_0 U_R = U_R^\dagger \mathcal{M}_0^\dagger U_L$ , we can get

$$\bar{\vartheta} = \frac{1}{32\pi^2} \sum_A R_A^a R_A^b \text{Im} [\text{Tr}[\mathcal{M}^{-1} \Gamma^a \mathcal{M}^\dagger \Gamma^b] - \text{Tr}[(\mathcal{M}^\dagger)^{-1} (\Gamma^a)^\dagger \mathcal{M} (\Gamma^b)^\dagger]] \log\left(\frac{m_A^2}{\Lambda^2}\right).$$

Explicitly evaluating this, we find that only the combinations  $(a, b) = (h, \sigma), (\sigma, h)$  contribute and we get

$$\bar{\vartheta} = \frac{1}{32\pi^2} \frac{v_S}{v_H} \sum_{i,A} (g_i^2 - g_i'^2) R_A^h R_A^\sigma \log\left(\frac{m_A^2}{\Lambda^2}\right).$$

The quantity  $R_A^h R_A^\sigma$  is the roughly equal to the mixing angle between  $h$  and  $\sigma$ , which is roughly equal to  $c_1 \sin(2\alpha) v_H / v_S$ . Therefore  $\bar{\vartheta}$  is then

$$\bar{\vartheta} \sim \frac{1}{32\pi^2} c_1 \sin(2\alpha) \sum_i (g_i^2 - g_i'^2) \log\left(\frac{m_h^2}{m_\sigma^2}\right)$$

## A Appendix A

Starting with

$$A = \Gamma_{\alpha\beta}^{\prime a(H)} m_\beta \Gamma_{\beta\alpha}^{\prime b(AH)} + \Gamma_{\alpha\beta}^{\prime a(AH)} m_\beta \Gamma_{\beta\alpha}^{\prime b(H)},$$

we can expand in  $\Gamma_{\alpha\beta}^{\prime a,b}$  using Eq. 1.0.1 and get

$$\begin{aligned} A &= \left( \frac{\Gamma_{\alpha\beta}^{\prime a} + (\Gamma_{\alpha\beta}^{\prime a})^\dagger}{2} \right) m_\beta \left( \frac{\Gamma_{\beta\alpha}^{\prime b} - (\Gamma_{\beta\alpha}^{\prime b})^\dagger}{2} \right) + \left( \frac{\Gamma_{\alpha\beta}^{\prime a} - (\Gamma_{\alpha\beta}^{\prime a})^\dagger}{2} \right) m_\beta \left( \frac{\Gamma_{\beta\alpha}^{\prime b} + (\Gamma_{\beta\alpha}^{\prime b})^\dagger}{2} \right), \\ &= \frac{1}{4} [\Gamma_{\alpha\beta}^{\prime a} m_\beta \Gamma_{\beta\alpha}^{\prime b} - \Gamma_{\alpha\beta}^{\prime a} m_\beta (\Gamma_{\beta\alpha}^{\prime b})^\dagger + (\Gamma_{\alpha\beta}^{\prime a})^\dagger m_\beta \Gamma_{\beta\alpha}^{\prime b} - (\Gamma_{\alpha\beta}^{\prime a})^\dagger m_\beta (\Gamma_{\beta\alpha}^{\prime b})^\dagger \\ &\quad + \Gamma_{\alpha\beta}^{\prime a} m_\beta \Gamma_{\beta\alpha}^{\prime b} + \Gamma_{\alpha\beta}^{\prime a} m_\beta (\Gamma_{\beta\alpha}^{\prime b})^\dagger - (\Gamma_{\alpha\beta}^{\prime a})^\dagger m_\beta \Gamma_{\beta\alpha}^{\prime b} - (\Gamma_{\alpha\beta}^{\prime a})^\dagger m_\beta (\Gamma_{\beta\alpha}^{\prime b})^\dagger], \\ &= \frac{1}{2} [\Gamma_{\alpha\beta}^{\prime a} m_\beta \Gamma_{\beta\alpha}^{\prime b} - (\Gamma_{\alpha\beta}^{\prime a})^\dagger m_\beta (\Gamma_{\beta\alpha}^{\prime b})^\dagger]. \end{aligned}$$

Using Eq. 1.0.1 once more, we can expand in the weak basis and get

$$A = \frac{1}{2} [V_{\alpha\gamma}^\dagger \Gamma_{\gamma\delta}^a U_{\delta\beta} m_\beta V_{\beta\rho}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - U_{\alpha\gamma}^\dagger (\Gamma_{\gamma\delta}^a)^\dagger V_{\delta\beta} m_\beta U_{\beta\rho}^\dagger (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha}],$$

where we used  $V = U_L$  and  $U = U_R$ . Using  $m_\beta = V_{\beta\pi}^\dagger \mathcal{M}_{\pi\tau} U_{\tau\beta} = U_{\beta\pi}^\dagger \mathcal{M}_{\pi\tau}^\dagger V_{\tau\beta}$ , we get

$$\begin{aligned} A &= \frac{1}{2} [V_{\alpha\gamma}^\dagger \Gamma_{\gamma\delta}^a U_{\delta\beta} U_{\beta\pi}^\dagger \mathcal{M}_{\pi\tau}^\dagger V_{\tau\beta} V_{\beta\rho}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - U_{\alpha\gamma}^\dagger (\Gamma_{\gamma\delta}^a)^\dagger V_{\delta\beta} V_{\beta\pi}^\dagger \mathcal{M}_{\pi\tau} U_{\tau\beta} U_{\beta\rho}^\dagger (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha}], \\ &= \frac{1}{2} \delta_{\delta\pi} \delta_{\tau\rho} [V_{\alpha\gamma}^\dagger \Gamma_{\gamma\delta}^a \mathcal{M}_{\pi\tau}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - U_{\alpha\gamma}^\dagger (\Gamma_{\gamma\delta}^a)^\dagger \mathcal{M}_{\pi\tau} (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha}], \\ &= \frac{1}{2} [V_{\alpha\gamma}^\dagger \Gamma_{\gamma\delta}^a \mathcal{M}_{\delta\rho}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - U_{\alpha\gamma}^\dagger (\Gamma_{\gamma\delta}^a)^\dagger \mathcal{M}_{\delta\rho} (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha}], \end{aligned}$$

Now we come to the point of this ride where we must multiply by  $m_\alpha^{-1}$ , take the imaginary part, and then take the trace. Using  $m_\alpha^{-1} = U_{\alpha\pi}^\dagger \mathcal{M}_{\pi\tau}^{-1} V_{\tau\alpha} = V_{\alpha\pi}^\dagger (\mathcal{M}_{\pi\tau}^\dagger)^{-1} U_{\tau\alpha}$ , we find that

$$\begin{aligned} \text{Im}[\text{Tr}(m_\alpha^{-1} A)] &= \text{Im}[\text{Tr} \left[ \frac{1}{2} m_\alpha^{-1} \left( V_{\alpha\gamma}^\dagger \Gamma_{\gamma\delta}^a \mathcal{M}_{\delta\rho}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - U_{\alpha\gamma}^\dagger (\Gamma_{\gamma\delta}^a)^\dagger \mathcal{M}_{\delta\rho} (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha} \right) \right]], \\ &= \frac{1}{2} \text{Im}[\text{Tr} \left[ U_{\alpha\pi}^\dagger \mathcal{M}_{\pi\tau}^{-1} V_{\tau\alpha} V_{\alpha\gamma}^\dagger \Gamma_{\gamma\delta}^a \mathcal{M}_{\delta\rho}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - V_{\alpha\pi}^\dagger (\mathcal{M}_{\pi\tau}^\dagger)^{-1} U_{\tau\alpha} U_{\alpha\gamma}^\dagger (\Gamma_{\gamma\delta}^a)^\dagger \mathcal{M}_{\delta\rho} (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha} \right]], \\ &= \frac{1}{2} \text{Im}[\text{Tr} \left[ \delta_{\tau\gamma} \left( U_{\alpha\pi}^\dagger \mathcal{M}_{\pi\tau}^{-1} \Gamma_{\gamma\delta}^a \mathcal{M}_{\delta\rho}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - V_{\alpha\pi}^\dagger (\mathcal{M}_{\pi\tau}^\dagger)^{-1} (\Gamma_{\gamma\delta}^a)^\dagger (\mathcal{M}_{\delta\rho})^{-1} (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha} \right) \right]], \\ &= \frac{1}{2} \text{Im}[\text{Tr} \left[ U_{\alpha\pi}^\dagger \mathcal{M}_{\pi\gamma}^{-1} \Gamma_{\gamma\delta}^a \mathcal{M}_{\delta\rho}^\dagger \Gamma_{\rho\sigma}^b U_{\sigma\alpha} - V_{\alpha\pi}^\dagger (\mathcal{M}_{\pi\gamma}^\dagger)^{-1} (\Gamma_{\gamma\delta}^a)^\dagger \mathcal{M}_{\delta\rho} (\Gamma_{\rho\sigma}^b)^\dagger V_{\sigma\alpha} \right]], \\ &= \frac{1}{2} \text{Im}[\text{Tr} [U^\dagger \mathcal{M}^{-1} \Gamma^a \mathcal{M}^\dagger \Gamma^b U] - \text{Tr} [V^\dagger (\mathcal{M}^\dagger)^{-1} (\Gamma^a)^\dagger \mathcal{M} (\Gamma^b)^\dagger V]], \\ &= \frac{1}{2} \text{Im}[\text{Tr} [\mathcal{M}^{-1} \Gamma^a \mathcal{M}^\dagger \Gamma^b] - \text{Tr} [(\mathcal{M}^\dagger)^{-1} (\Gamma^a)^\dagger \mathcal{M} (\Gamma^b)^\dagger]]. \end{aligned}$$

It turns out that by explicitly evaluating the above trace and taking its imaginary part, one can see that the combinations  $(a, b) = (h, \sigma), (\sigma, h)$  contribute and sum together such that

$$\text{Im}[\text{Tr}(m_\alpha^{-1} A)] = \sum_i \frac{1}{2} \frac{v_S}{v_H} (g_i^2 - g_i'^2).$$

Next, we look at the Higgs mass matrix, which is formulated from the potential. Assuming that the mass of the Higgs is  $m_h^2 \sim v_H^2$  and that  $m_s^2, m_\sigma^2 \sim v_S^2$ , we can approximate the mass matrix  $M_\Phi^2 = RDR^\dagger$  as

$$M_\Phi^2 \approx \begin{pmatrix} \sim v_H^2 & v_S v_H (c_2 + 2c_1 \cos 2\alpha) & -2c_1 v_s v_H \sin 2\alpha \\ v_S v_H (c_2 + 2c_1 \cos 2\alpha) & \sim v_S^2 & \sim v_S^2 \\ -2c_1 v_s v_H \sin 2\alpha & \sim v_S^2 & \sim v_S^2 \end{pmatrix},$$

where  $R$  are the rotation matrices and  $D = \text{diag}(m_h^2, m_s^2, m_\sigma^2)$  is the mass matrix in its diagonal basis. The upper right hand corner of the matrix  $RDR^\top$  is roughly proportional to  $v_S^2$ , meaning that the mixing angle between  $h$  and  $\sigma$ , which is approximately equal to  $R_A^h R_A^\sigma$ , is roughly equal to

$$R_A^h R_A^\sigma \sim R_\sigma^h R_\sigma^\sigma \sim c_1 \frac{v_H}{v_S} \sin 2\alpha.$$

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