Formal development of fermionic path integrals

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Overview

Feynman Propagator

Our goal is to uncover a mathematically consistent Feynman Path Integral of the fermions. The desired result:

$$\langle 0| T\Psi(x_1)\overline{\Psi}(x_2)...\Psi(x_n)\overline{\Psi}(x_n-1)|0\rangle = \boxed{?}\, e^{\eta^\dagger A^{-1}\eta}$$

Richard Feynman did face significant challenges in incorporating fermions into the path integral formulation

The solution is very simple, but it introduced new mathematics: Grassmann Variables.

Grassmann Variables - Basics

Grassmann Variables are anti-commuting with themselves

- Given two distinct Grassmann Variables η and θ

$$\{\theta, \eta\} = 0$$

- Thus

$$\theta\eta = -\eta\theta, \quad \theta^2 = 0, \quad \eta^2 = 0$$

- We often consider exponential function:

$$f(\theta) = e^{\theta} = 1 + \theta + \frac{1}{2!}\theta^2 + \dots$$
$$= 1 + \theta$$

- In general form:

$$\boxed{\text{1Var: } f(\theta) = f_0 + f_1\theta}$$

$$\boxed{\text{2Var: } f(\theta_1, \theta_2) = f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2}$$

Grassmann Variables - Basics

Let's quickly build some results for these Grassmann functions.

Differentiation:

LHS: $\overrightarrow{\frac{d}{d\theta}} f(\theta_1, \theta_2) = f_1 + f_{12}\theta_2$

RHS: $\overleftarrow{\frac{d}{d\theta}} f(\theta_1, \theta_2) = f_1 - f_{12}\theta_2$

Integration Rules:

- (i) Linearity

$$\boxed{\int d\theta (f(\theta)a + g(\theta)b) = \left(\int f(\theta)\right) a + \left(\int g(\theta)\right) b}$$

- (ii) Integral Derivative

$$\boxed{\int d\theta \frac{d}{d\theta} f(\theta) = 0}$$

Grassmann Variables - Basics

- (iii) Integral of f_1

$$\boxed{\int d\theta f_1 = f \int d\theta = 0}$$

- (iv) Fix $\int d\theta\theta = 1$

$$\boxed{\int d\theta f(\theta) = f_1 = \frac{d}{d\theta} f(\theta)}$$

Let us now consider two Grassmann Integrals.

$$\int d\theta_1 d\theta_2 f(\theta_1, \theta_2) = \int d\theta_1 d\theta_2 (f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2)$$

$$= \boxed{\text{-}f_{12}}$$

Grassmann Variables - Basics

For n number of integrations across n number of grassman variables (θ_n)

$$\int d\theta_1...d\theta_n f(\theta_1, ..., \theta_n) = \int d\theta_1...d\theta_n(f_0 + ... + f_{1...n}\theta_1...\theta_n)$$

$$= \boxed{\text{-f}_{1...n}}$$

Gaussian Integrals

Let us now consider the gaussian integral instead of $f(\theta) = e^{\theta}$

- For two Grassmann Variables, the functional form:

  $f(\theta) = e^{-\theta_1 a_{12} \theta_2}$

$$\int d\theta_1 d\theta_2 f(\theta_1, \theta_2) = \int d\theta_1 d\theta_2 e^{-\theta_1 a_{12} \theta_2}$$

$$= \boxed{\mathrm{a}_{12}}$$

- For a more general form (i, j = (1,..,n)):

  $f(\theta) = e^{-\theta_i a_{ij} \theta_j}$

$$\int d\theta_1 ... d\theta_{2n} f(\theta) = \int d\theta_1 ... d\theta_{2n} e^{-\theta_i a_{ij} \theta_j}$$

$$= \boxed{\sqrt{\det(a)}}$$

Complex Gaussian Integrals

We're almost there. Let's quickly define some aspects of the complex Grassmann numbers.

- For a complex Grassmann, we can define:

  $\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2)$

  $\theta^* = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)$

Hence, we can define a complex Grassmann function of variables θ and θ^*.

$$\boxed{\text{2Var: } f(\theta, \theta^*) = c_0 + c_1\theta + \overline{c_1}\theta^* + c_{11}\theta\theta^*}$$

Note: $\int d\theta d\theta^* (\theta^*\theta) = 1$

Complex Gaussian Integral generalized

For n number of integrations across n number of grassman variables (θ_n)

$$\int \prod_{j=1}^{n} (d\theta_j^* d\theta_j) e^{-\theta_j^\dagger A_{ij} \theta_j} = \boxed{\text{det(A)}}$$

Complex Variables of a vector type

Now, we can consider column vectors of the Grassmann variables:

$$\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

As with the case of the Guassian integral, this is the exact same:

$$\int \prod_{j=1}^{n} (d\theta_j^* d\theta_j) e^{-\theta_j^\dagger A_{ij} \theta_j} = \boxed{\text{det(A)}}$$

Consider the case in which we just add a linear term:

$$I(\eta, \overline{\eta}) = \int \prod_{j=1}^{n} (d\theta_j^* d\theta_j) e^{-\theta_j^\dagger A_{ij} \theta_j + \eta^\dagger \theta + \theta^\dagger \eta} = \boxed{\text{det(A)}(\eta^\dagger A^{-1} \eta)}$$

Complex Variables of a vector type

Just as an exercise, what is $I(0,0)$?

Now, consider our last formula:

$$= \frac{1}{I(0,0)} \int \prod_{j=1}^{n} (d\theta_j^* \, d\theta_j) \, \theta_k \theta_l^* \, e^{-\theta_j^\dagger A_{ij} \theta_j}$$

$$= \frac{1}{I} \frac{\partial}{\partial \eta_k^*} \frac{\partial}{\partial \eta_l} \int \prod_{j=1}^{n} (d\theta_j^* \, d\theta_j) \, e^{-\theta_j^\dagger A_{ij} \theta_j + \eta^\dagger \theta + \theta^\dagger \eta}$$

$$= \frac{1}{I} \frac{\partial}{\partial \eta_k^*} \frac{\partial}{\partial \eta_l} I(\eta, \eta^*)$$

Fermion Path Integral

Now, with all the Grassmann technology I've built, we can easily generalize to the fermion path integral.

The Dirac spinor field is a field with 4-components

$$\Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_4 \end{pmatrix}$$

The Dirac object $\Psi_\alpha(x)$: takes values as Grassmann numbers. These anti-commute! Because fermions anti-commute. $\{\Psi_\alpha(x), \Psi_\beta(x)\} = 0$

$$\langle 0| T\Psi(x_1)\overline{\Psi}(x_2)...\Psi(x_n)\overline{\Psi}(x_n - 1)|0\rangle = \int D\overline{\Psi}D\Psi X e^{i(s(\Psi,\overline{\Psi}))}$$

Fermion Path Integral

$$= \int D\overline{\Psi} D\Psi X e^{is_0 + \int d^4x(\overline{\eta}\Psi + \overline{\Psi\eta})}$$

This is general. But, for Fermions, we consider the Dirac theory:

Dirac:

$$s_0 = -i \int d^4x d^4y \overline{\Psi}_\alpha(x) A_{\alpha\beta}(x-y) \Psi_\beta(y)$$

$$= \int D\overline{\Psi} D\Psi X e^{i(-i\int d^4x d^4y \overline{\Psi}_\alpha(x) A_{\alpha\beta}(x-y)\Psi_\beta(y)) + \int d^4x(\overline{\eta}\Psi + \overline{\Psi\eta})}$$

$$= \boxed{\det(\mathrm{A}) e^{\overline{\eta} A^{-1} \eta}}$$

$$\langle 0| T\Psi(x_1)\overline{\Psi}(x_2)...\Psi(x_n)\overline{\Psi}(x_n - 1)|0\rangle = \boxed{\det(\mathrm{A}) e^{\overline{\eta} A^{-1} \eta}}$$