Grand Unification

Josh Iascau

University of Oregon

joshuai@uoregon.edu

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Overview

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4. SU(5) $\supset$ $SU(3)_c \times SU(2)_L \times U(1)_Y$
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6. $SU(5)$ reps.
7. Gauge bosons of $SU(5)$
8. $p \rightarrow e^+\pi^0$

SU(?) - A motiviation

The gauge group of the full standard model is given by:

$$\boxed{\mathrm{SU}(3)_c \times SU(2)_L \times U(1)_Y}$$

$SU(3)_c$ respects color symmetry and contains the strong interaction.

$SU(2)_L \times U(1)_Y$ respects Isospin symmetry and contains the Electro-weak interaction.

Gauge groups are lie groups. Lie groups can be embedded in other lie groups. Consider:

$$SU(?) \supset SU(3)_c \times SU(2)_L \times U(1)_Y$$

The direct sum $\oplus$

The operations $\oplus$ and $\otimes$ are used frequently in representation theory. How do they work?

Consider two vectors $\vec{v}$ and $\vec{w}$

$$\vec{v} = v_x \vec{e_x} + v_y \vec{e_y} + v_z \vec{e_z}$$

$$\vec{w} = w_a \vec{e_a} + w_b \vec{e_b}$$

- $\vec{v} \oplus \vec{w}$

$$\vec{v} \oplus \vec{w} = v_x \vec{e_x} + v_y \vec{e_y} + v_z \vec{e_z} + w_a \vec{e_a} + w_b \vec{e_b}$$

$$= \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \oplus \begin{bmatrix} w_a \\ w_b \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ w_a \\ w_b \end{bmatrix}$$

The tensor product $\otimes$

The $\otimes$ operation is a bit more elusive. As a quick example, consider the Kronecker product.

- $\vec{v} \otimes \vec{w}$

$$\vec{v} \otimes \vec{w} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \otimes \begin{bmatrix} w^a \\ w^b \end{bmatrix} = \begin{bmatrix} v_x \begin{bmatrix} w^a \\ w^b \end{bmatrix} \\ v_y \begin{bmatrix} w^a \\ w^b \end{bmatrix} \\ v_z \begin{bmatrix} w^a \\ w^b \end{bmatrix} \end{bmatrix} = \begin{bmatrix} v_x w^a \\ v_x w^b \\ v_y w^a \\ v_y w^b \\ v_z w^a \\ v_z w^b \end{bmatrix}$$

Representation of $SU(3)_c \times SU(2)_L \times U(1)_Y$

Recall the gauge group for the full Standard Model:

$$\boxed{\mathrm{SU}(3)_c \times SU(2)_L \times U(1)_Y}$$

- Specify the representations of the gauge group using a rep. $(\boldsymbol{p}, \boldsymbol{q})_n$
  - $\boldsymbol{p}$ is the dim. of the $SU(3)_c$ rep.
  - $\boldsymbol{q}$ is the dim of the $SU(2)_L$ rep.
  - n is the hyper charge.

Representation of $SU(3)_c \times SU(2)_L \times U(1)_Y$

The standard model gauge group is based on Weyl fields in the representation of a scalar $(1,2)_{-1/2}$ and:

Particle	**Name**	**Representation**
Spin 1		
B	Z boson	$(1,1)_0$
W	W boson	$(1,3)_0$
G	gluon	$(8,1)_0$
Spin 1/2		
q_L	left-handed quark	$(3,2)_{+1/6}$
u_L^c	left-handed antiquark (up)	$(\overline{3},1)_{-2/3}$
d_L^c	left-handed antiquark (down)	$(\overline{3},1)_{1/3}$
ℓ_L	left-handed lepton	$(1,2)_{-1/2}$
ℓ_L^c	left-handed antilepton	$(1,1)_1$
Spin 0		
H	Higgs boson	$(1,2)_{-1/2}$

$$(1,2)_{-1/2} \oplus (1,1)_{+1} \oplus (3,2)_{+1/6} \oplus (\overline{3},1)_{-2/3} \oplus (\overline{3},1)_{1/3}$$

SU(5)

Now consider $SU(3)_c \times SU(2)_L \times U(1)_Y$ as a subgroup of $SU(5)$.

$$\boxed{\text{SU}(5) \supset SU(3)_c \times SU(2)_L \times U(1)_Y}$$

This is a new lie group! We have taken the standard model and embedded it into an SU(5).

Is this compatible with the representation of the standard model?

$SU(5) \to SU(3) \times SU(2)_L \times U(1)_Y$

Consider the adjoint representation of SU(5): $\mathbf{5} \otimes \bar{\mathbf{5}} = \mathbf{24} \oplus \mathbf{1}$

For the one-dimensional rep, we have a scalar $\Phi^a T^a$. Hence, we assume that the scalar potential results in a VEV:

$$\langle 0|\Phi|0\rangle = V \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

This VEV spontaneously breaks the gauge symmetry down to $SU(3) \times SU(2)_L \times U(1)_Y$. Hence,

$$\boxed{\text{SU}(5) \to SU(3) \times SU(2)_L \times U(1)_Y}$$

SU(5) reps.

SU(5)

The fundamental and anti-fundemental rep. of $SU(5)$ is **5** and $\bar{\mathbf{5}}$. They transform as:

$$\boxed{\mathbf{5} \to (3,1)_{-1/3} \oplus (1,2)_{+1/2}}$$

$$\boxed{\bar{\mathbf{5}} \to (\bar{3},1)_{-1/3} \oplus (1,2)_{+1/2}}$$

Let us now consider $\mathbf{5} \otimes \mathbf{5}$ of the subgroup.

- $\mathbf{5} \otimes \mathbf{5}$

$$\boxed{\mathbf{5} \otimes \mathbf{5} = 15_S + 10_A}$$

SU(5) - $\mathbf{5} \otimes \mathbf{5}$

Next we find the product of $\mathbf{5} \otimes \mathbf{5}$.

$$\mathbf{5} \otimes \mathbf{5} \rightarrow [(3,1)_{-1/3} \oplus (1,2)_{+1/2}] \otimes [(3,1)_{-1/3} \oplus (1,2)_{+1/2}]$$

By distributivity of tensor products:

$\mathbf{5} \otimes \mathbf{5} \rightarrow [(3 \otimes 3, 1)_{(-2/3)} \oplus (3,2)_{+1/6}] \oplus [(3,2)_{+1/6} \oplus (1, 2 \otimes 2)_{+1}]$

$= [(6_S \oplus \overline{3}_A, 1)_{-2/3} \oplus (3,2)_{+1/6}] \oplus [(3,2)_{+1/6} \oplus (1, 3_S \oplus 1_A)_{+1}]$

$= [(6,1)_{-2/3} \oplus (3,2)_{+1/6} \oplus (3,2)_{1/6}]_S \oplus [(\overline{3},1)_{-2/3} \oplus (3,2)_{+1/6} \oplus (1,1)_{+1}]_A$

Recall that

$$\boxed{\mathbf{5} \otimes \mathbf{5} = 15_S + 10_A}$$

So, our 10 rep. transforms as:

$$\boxed{\mathbf{10} \rightarrow (\overline{3},1)_{-2/3} \oplus (3,2)_{+1/6} \oplus (1,1)_{+1}}$$

SU(5) - $\overline{\mathbf{5}} \oplus \mathbf{10}$

Recall the $\overline{\mathbf{5}}$ fundamental

$$\boxed{\overline{\mathbf{5}} \rightarrow (3,1)_{-1/3} \oplus (1,2)_{+1/2}}$$

Consider the direct sum of $\overline{\mathbf{5}} \oplus \mathbf{10}$

$$\boxed{\overline{\mathbf{5}} \oplus \mathbf{10} = (3,1)_{-1/3} \oplus (1,2)_{+1/2} \oplus (\overline{3},1)_{-2/3} \oplus (3,2)_{+1/6} \oplus (1,1)_{+1}}$$

Particle	**Name**	**Representation**
Spin 1/2		
q_L	left-handed quark	$(3,2)_{+1/6}$
u_L^c	left-handed antiquark (up)	$(\overline{3},1)_{-2/3}$
d_L^c	left-handed antiquark (down)	$(\overline{3},1)_{+1/3}$
ℓ_L	left-handed lepton	$(1,2)_{-1/2}$
ℓ_L^c	left-handed antilepton	$(1,1)_{+1}$

This is exactly the rep. corresponding to the matter content of the standard model (of 1-gen)!

SU(5) - $\mathbf{5} \otimes \overline{\mathbf{5}}$

What about the Gauge bosons? Gauge bosons live in the adjoint representation of a gauge group. The adjoint rep. of $SU(5)$ is $\mathbf{5} \otimes \overline{\mathbf{5}}$.

- $\mathbf{5} \otimes \overline{\mathbf{5}}$

$$\boxed{\mathbf{5} \otimes \overline{\mathbf{5}} = 24 \oplus 1}$$

And hence, consider.

$$\mathbf{5} \otimes \overline{\mathbf{5}} \rightarrow [(3,1)_{-1/3} \oplus (1,2)_{+1/2}] \otimes [(\overline{3},1)_{-1/3} \oplus (1,2)_{+1/2}]$$

If we expand this out, and compare to the 24 of $\mathbf{5} \otimes \overline{\mathbf{5}}$.

$$\boxed{\mathbf{24} \rightarrow (8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0 \oplus (3,2)_{-5/6} \oplus (\overline{3},2)_{5/6}}$$

SU(5) - Gauge Bosons

Again, $\mathbf{5} \otimes \overline{\mathbf{5}}$ gives us the gauge bosons as well as the **24** rep (excluding scalar).

$$\boxed{\mathbf{24} \to (8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0 \oplus (3,2)_{-5/6} \oplus (\overline{3},2)_{5/6}}$$

The gauge bosons of the standard model are as so.

Particle	**Name**	**Representation**
Spin 1 (Gauge Bosons)		
B	Z boson (hypercharge)	$(1,1)_0$
W	W boson (weak isospin)	$(1,3)_0$
G	gluon (color)	$(8,1)_0$
Spin 0 (Scalar)		
H	Higgs boson	$(1,2)_{-1/2}$

Our reps. are present! But wait... what about $(3,2)_{-5/6} \oplus (\overline{3},2)_{5/6}$?

SU(5) - 12 new Gauge Bosons

In $SU(5)$, since $\mathbf{5} \otimes \ \overline{\mathbf{5}} = 24 \oplus 1$, we will have 24 massive Gauge bosons. That is 12 more than $SU(3)_c \times SU(2)_L \times U(1)_Y$.

Consider the Weyl field Ψ^i in $\overline{\mathbf{5}}$ and a Weyl field χ_{ij} in the $\mathbf{10}$ rep. i,j = (1,2,3,4,5).

Writing out the gauge field in the generator basis we find:

$$\begin{bmatrix} G_r^r - \frac{1}{3}cB & G_r^b & G_r^g & \frac{1}{\sqrt{2}}X_1^r & \frac{1}{\sqrt{2}}X_2^r \\ G_b^r & G_b^b - \frac{1}{3}cB & G_b^g & \frac{1}{\sqrt{2}}X_1^b & \frac{1}{\sqrt{2}}X_2^b \\ G_g^r & G_g^b & G_g^g - \frac{1}{3}cB & \frac{1}{\sqrt{2}}X_1^g & \frac{1}{\sqrt{2}}X_2^g \\ \frac{1}{\sqrt{2}}X_1^{\dagger r} & \frac{1}{\sqrt{2}}X_1^{\dagger b} & \frac{1}{\sqrt{2}}X_1^{\dagger g} & \frac{1}{2}W^3 + \frac{1}{2}cB & \frac{1}{\sqrt{2}}W^+ \\ \frac{1}{\sqrt{2}}X_2^{\dagger r} & \frac{1}{\sqrt{2}}X_2^{\dagger b} & \frac{1}{\sqrt{2}}X_2^{\dagger g} & \frac{1}{\sqrt{2}}W^- & -\frac{1}{2}W^3 + \frac{1}{2}cB \end{bmatrix}$$

SU(5) - Results

We can finally conclude that:

$$\boxed{\text{SU}(5) \supset SU(3)_c \times SU(2)_L \times U(1)_Y}$$

Recall the following:

- $\overline{\mathbf{5}} \oplus \mathbf{10}$ will give you the matter content of the SM
- $\mathbf{5} \otimes \overline{\mathbf{5}}$ rep. gives you Gauge bosons of the SM + 12 new Gauge bosons X and $X^\dagger$

$X_\mu \sim (3,2)_{-5/6}$ and $X_\mu^\dagger \sim (\bar{3},2)_{+5/6}$ carry *both* colour and weak charge. A single X couples a quark to a lepton. So, a proton (uud) can convert to $e^+\pi^0$.

$$\boxed{\text{p} \to e^+\pi^0}$$