log-Coulomb gas in a nonarchimedean local field

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The known story for $\mathbb{R}$

- A one-dimensional **log-Coulomb gas** (or **log-gas**) is a system of $N \geq 2$ point charges $q_1, q_2, \ldots, q_N > 0$ constrained to a line and subject to a “repulsive log-Coulomb potential.”

- Let $x_i \in \mathbb{R}$ be the location of $q_i$ and call $\vec{x} = (x_1, x_2, \ldots, x_N)$ a **microstate**. Let $T > 0$ be a fixed temperature.

- Define the potential energy of a microstate $\vec{x} \in \mathbb{R}^N$ by

$$V(\vec{x}) := \frac{1}{2} T \|\vec{x}\|^2 - \sum_{i<j} q_i q_j \log |x_i - x_j| .$$

- **Rough interpretation:** A microstate has high energy if charges are far from the origin (the quadratic term) or close together (the logarithmic term).
• Call $\beta = \frac{1}{T}$ the “coldness” of the system and set $\beta_{ij} := q_i q_j \beta$ for all $i < j$.

• Define the **canonical partition function** $Z_N$ by

$$Z_N(\beta) := \int_{\mathbb{R}^N} e^{-\beta V(\vec{x})} \, d\vec{x} = \int_{\mathbb{R}^N} e^{-\frac{1}{2}||\vec{x}||^2} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}.$$ 

• **Fundamental idea of Boltzmann statistics:** The microstates $\vec{x} \in \mathbb{R}^N$ have probability density $\frac{1}{Z_N(\beta)} e^{-\beta V(\vec{x})}$.

• **Rough interpretation:** High energy microstates are less probable. This effect is more severe if the system is cold $(\beta \gg 0)$ and less severe if the system is hot $(\beta \approx 0)$.
The known story for $\mathbb{R}$ (continued)

- It is hard to compute $Z_N(\beta)$ for general $q_i$. In the special case $q_i = 1$ for all $i$, we have $\beta_{ij} = \beta$ for all $ij$ and $Z_N(\beta)$ becomes the value known as **Mehta’s integral**:

\[
\int_{\mathbb{R}^N} e^{-\frac{1}{2} \|x\|^2} \prod_{i < j} |x_i - x_j|^\beta \, d\vec{x} = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)} .
\]

- **Early 1960’s**: Mehta and Dyson proved the above formula only for $\beta = 1, 2, 4$, while developing random matrix theory.

- **Late 1970’s**: A lucky encounter with Selberg’s integral led Bombieri to a clever proof. It is valid for all complex $\beta$ at which the integral converges.
Motivating questions

• How do things change if we replace \( \mathbb{R} \) with a non-archimedean local field, such as \( \mathbb{Q}_p \)? What becomes easier/harder?

• Are there interesting properties common to \( p \)-adic log-gases and real log-gas? Is there a unified way to handle them all?

• What do these “local log-gases” together imply about adèle rings and idèle groups of number fields?

• Today’s goal: Discuss the first question and some examples.
• Let $K$ be a non-archimedean local field with discrete valuation $\nu : K \to \mathbb{Z} \cup \{\infty\}$. Recall $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ with $\nu(0) = \infty$, and $\nu$ is a homomorphism of $(K^\times, \cdot)$ onto $(\mathbb{Z}, +)$.

• $\mathfrak{o} := \{x \in K : \nu(x) \geq 0\}$ is a DVR, $\mathfrak{m} := \{x \in K : \nu(x) > 0\}$ is its maximal ideal, and local compactness of $K$ provides a unique integer $q = p^f$ satisfying $\mathfrak{o}/\mathfrak{m} \cong \mathbb{F}_q$.

• Note $K$ is one of the following:
  • a finite extension of $\mathbb{Q}_p$ with $\nu$ a rescaled extension of ord$_p$
  • the field $\mathbb{F}_q((t))$ of formal Laurent series with $\nu = \text{ord}_t$

• Pick the absolute value $|\cdot|$ on $K$ satisfying $|x| = q^{-\nu(x)}$, note

$$\mathfrak{o} = \{x \in K : |x| \leq 1\} \quad \text{and} \quad \mathfrak{m} = \{x \in K : |x| < 1\}.$$  

• Pick the additive Haar measure $\mu$ satisfying $\mu(\mathfrak{o}) = 1$. 
Suppose \( q_1, q_2, \ldots, q_N > 0, \beta > 0, \) and \( \beta_{ij} = q_i q_j \beta \) as before. Now suppose \( q_i \) is located at \( x_i \in K \) and define the potential energy of a microstate \( \vec{x} \in K^N \) by

\[
V(\vec{x}) := \begin{cases} 
- \sum_{i<j} q_i q_j \log |x_i - x_j| & \text{if all } x_i \in o, \\
\infty & \text{otherwise}.
\end{cases}
\]

Set \( d\vec{x} := d\mu^N(\vec{x}) \) and define the canonical partition function:

\[
Z_N(\beta) := \int_{K^N} e^{-\beta V(\vec{x})} \, d\vec{x} = \int_{o^N} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}.
\]

The microstates \( \vec{x} \in K^N \) have probability density

\[
\frac{1}{Z_N(\beta)} e^{-\beta V(\vec{x})} = \frac{1}{Z_N(\beta)} 1_{o^N}(\vec{x}) \prod_{i<j} |x_i - x_j|^{\beta_{ij}}.
\]
A preview of $Z_N(\beta)$ values

- $Z_2(\beta) = \frac{(q-1)q^{\beta_{12}}}{q^{\beta_{12}+1} - 1}$
- $Z_3(\beta) = \frac{(q-1)q^{\beta_{12}+\beta_{13}+\beta_{23}}}{q^{\beta_{12}+\beta_{13}+\beta_{23}+2} - 1} \cdot \left[ (q - 2) + \frac{q-1}{(q^{\beta_{12}+1} - 1)} + \frac{q-1}{(q^{\beta_{13}+1} - 1)} + \frac{q-1}{(q^{\beta_{23}+1} - 1)} \right]$

$$Z_4(\beta) = \frac{(q-1)q^{\beta_{12}+\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}+\beta_{34}}}{q^{\beta_{12}+\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}+\beta_{34}+3} - 1} \cdot \left\{ (q - 2)(q - 3) + (q - 2) \left[ \frac{q-1}{q^{\beta_{12}+1} - 1} + \frac{q-1}{q^{\beta_{23}+1} - 1} + \frac{q-1}{q^{\beta_{13}+1} - 1} + \frac{q-1}{q^{\beta_{14}+1} - 1} + \frac{q-1}{q^{\beta_{24}+1} - 1} + \frac{q-1}{q^{\beta_{34}+1} - 1} \right] \right. + \left. \frac{q-1}{q^{\beta_{12}+\beta_{23}+\beta_{13}+2} - 1} \right. + \left. \frac{q-1}{q^{\beta_{12}+\beta_{14}+\beta_{24}+2} - 1} \right. + \left. \frac{q-1}{q^{\beta_{13}+\beta_{14}+\beta_{34}+2} - 1} \right. + \left. \frac{q-1}{q^{\beta_{23}+\beta_{34}+\beta_{24}+2} - 1} \right. + \left. \frac{q-1}{q^{\beta_{12}+1} - 1} \cdot \frac{q-1}{q^{\beta_{34}+1} - 1} + \frac{q-1}{q^{\beta_{13}+1} - 1} \cdot \frac{q-1}{q^{\beta_{24}+1} - 1} + \frac{q-1}{q^{\beta_{14}+1} - 1} \cdot \frac{q-1}{q^{\beta_{23}+1} - 1} \right\}$$
Average values

- The **expectation** (or **average value**) of a Borel measurable function $f : o^N \to \mathbb{C}$ is

$$
\mathbb{E}[f(\vec{X})] := \frac{1}{Z_N(\beta)} \int_{o^N} f(\vec{x}) \prod_{i<j} |x_i - x_j|^{\beta_{ij}} d\vec{x}.
$$

- Some meaningful examples of $f(\vec{x})$:
  - $\prod_i x_i^{s_i}$ = an unramified quasicharacter of $(K^\times)^N$ ($s_i \in \mathbb{C}$)
  - $\min_{i<j} v(x_i - x_j) = \min\{n : x_i \not\equiv x_j \mod m^{n+1} \text{ for some } i < j\}$
  - $\max_{i<j} v(x_i - x_j) = \min\{n : x_i \not\equiv x_j \mod m^{n+1} \text{ for all } i < j\}$
  - $V(\vec{x}) = \text{the total potential energy of the system}$
  - $\min_{i<j} |x_i - x_j|$ = the minimum distance between charges
  - $\max_{i<j} |x_i - x_j|$ = the diameter of the gas
Example: eight unit charges in $\mathbb{Z}_3$

A microstate $\vec{x} \in \mathbb{Z}_3^8$ with...

- $\prod_i |x_i|^{s_i} = 3^{-(s_1+s_2)}$
- $\min_{i<j} v(x_i - x_j) = 0$
- $\max_{i<j} v(x_i - x_j) = 5$
- $V(\vec{x}) = 27 \log(3)$
- $\min_{i<j} |x_i - x_j| = 3^{-5}$
- $\max_{i<j} |x_i - x_j| = 1$
Expectation of quasicharacters and energy

- **Unramified quasi-characters**: Given \( \vec{\beta} = (\beta_{ij}) \) as before and \((s_1, s_2, \ldots, s_N) \in \mathbb{C}^N \) with \( \Re(s_i) > -1 \), set \( \beta_{i(N+1)} := s_i \) and

\[
Z_{N+1}^*(\beta) := \int_{0}^{N+1} \prod_{i<N} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}.
\]

A simple change of variables gives

\[
Z_{N+1}^*(\beta) = \int_{0}^{N} \prod_{i} |x_i|^{s_i} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} \, d\vec{x}
\]

and hence

\[
\mathbb{E}[\prod_i |X_i|^{s_i}] = \frac{Z_{N+1}^*(\beta)}{Z_N(\beta)}.
\]

- **Potential energy**: \( \frac{d}{d\beta} \) can pass through the integral for \( Z_N \), so

\[
\mathbb{E}[V(\vec{X})] = -\frac{d}{d\beta} \log Z_N(\beta).
\]
Expectation of max’s and min’s

• For $\alpha_1, \alpha_2 \geq 0$ and $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, define

$$f(\vec{\alpha}, \vec{n}, \vec{x}) := (\max_i v(x_i - x_j))^{n_1} (\min_i v(x_i - x_j))^{n_2} (\min_i |x_i - x_j|)^{\alpha_1} (\max_i |x_i - x_j|)^{\alpha_2},$$

• An important auxiliary function: Define

$$F_N(\vec{\alpha}, \vec{\beta}) := \int_{(m)^N} (\min_i |x_i - x_j|)^{\alpha_1} (\max_i |x_i - x_j|)^{\alpha_2} \prod_{i<j} |x_i - x_j|^{\beta_{ij}} d\vec{x}$$

for all suitable $(\vec{\alpha}, \vec{\beta}) \in \mathbb{C}^2 \times \mathbb{C}^{(N)}_2$.

• If $\beta > 0$ and $\beta_{ij} = q_i q_j \beta$ as before, note

$$\mathbb{E}[f(\vec{\alpha}, \vec{n}, \vec{X})] = \left( \frac{-1}{\log(q)} \frac{\partial}{\partial \alpha_1} \right)^{n_1} \left( \frac{-1}{\log(q)} \frac{\partial}{\partial \alpha_2} \right)^{n_2} \left[ q^{\alpha_1 + \alpha_2} \cdot \frac{F_N(\vec{\alpha}, \vec{\beta})}{F_N(0, \vec{\beta})} \right].$$
• If $\vec{\beta} = (\beta_{ij})_{1 \leq i < j \leq N}$ with $\beta_{ij} = q_i q_j \beta$ and $\vec{\beta}' = (\beta_{ij})_{1 \leq i < j \leq N+1}$ with $\beta_{i(N+1)} = s_i$ (where $\beta > 0$ and $\Re(s_i) > -1$), then

$$Z_N(\beta) = q^{N+\sum_{1 \leq i < j \leq N} \beta_{ij}} \cdot F_N(0, \vec{\beta}) ,$$
$$Z^*_N(\beta) = q^{N+1+\sum_{1 \leq i < j \leq N+1} \beta_{ij}} \cdot F_N(0, \vec{\beta}') ,$$

and hence $\mathbb{E}[\prod_i |X_i|^{s_i}] = q^{1+\sum_i s_i} \cdot \frac{F_N(0, \vec{\beta}')} {F_N(0, \vec{\beta})} .$

• Big idea: The canonical partition function $Z_N(\beta)$ and the expectations $\mathbb{E}[\prod_i |X_i|^{s_i}]$, $\mathbb{E}[V(\vec{X})]$, and $\mathbb{E}[f(\vec{\alpha}, \vec{n}, \vec{X})]$ can all be expressed in terms of $F_N$.

• We will show that $F_N$ can be computed via combinatorics.
Definition

Let $L$ be a positive integer. A **splitting sequence of** $N$ is a tuple $\vec{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{L-1})$ of compositions (i.e., ordered partitions) $\lambda_\ell = [\lambda_\ell^{(1)}, \lambda_\ell^{(2)}, \ldots, \lambda_\ell^{(N_\ell)}] \vdash N_{\ell+1}$ such that

$$1 = N_0 < N_1 < N_2 < \cdots < N_L = N.$$ 

Call $L(\vec{\lambda}) := L$ the **length** of $\vec{\lambda}$ and denote the set of all splitting sequences of $N$ by $\nabla(N)$.

Example

The tuple $\vec{\lambda} = ([3], [1, 1, 4], [2, 1, 1, 1, 2, 1], [1, 2, 1, 1, 1, 1, 1, 1])$ is a length 4 splitting sequence of 9.
Other ways to think about a splitting sequence

Our definition of splitting sequences has been chosen for brevity. There are two equivalent and useful alternatives.

Example

\[ \vec{\lambda} = ([3], [1, 1, 4], [2, 1, 1, 2, 1], [1, 2, 1, 1, 1, 1, 1, 1]) \]

\[ \Lambda_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \]
\[ \Lambda_1 = \{1, 2, 3\}{4}{5, 6, 7, 8, 9} \]
\[ \Lambda_2 = \{1, 2, 3\}{4}{5}{6}{7, 8}\{9\} \]
\[ \Lambda_3 = \{1\}{2, 3\}{4}{5}{6}{7}{8\}{9} \]
Special symbols associated to a splitting sequence

- If $\vec{\lambda} \in \mathfrak{r}(N)$ and $\Lambda_0, \Lambda_1, \ldots, \Lambda_{L(\vec{\lambda})-1}$ are the corresponding partitions of $\{1, 2, \ldots, N\}$, denote the $m$th part of $\Lambda_\ell$ by $\Lambda^{(m)}_\ell$.

**Definition**
For each $\vec{\lambda} \in \mathfrak{r}(N)$ and $\ell \in \{0, 1, \ldots, L(\vec{\lambda}) - 1\}$, define the $\ell$th multiplicity and $\ell$th exponent respectively by

$$M_\ell(\vec{\lambda}, n) := \prod_{m=1}^{N_\ell} \frac{1}{n} \binom{n}{\lambda^{(m)}_\ell}$$

and

$$E_\ell(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) := \alpha_1 + \delta_{0\ell} \alpha_2 + \sum_{m=1}^{N_\ell} \left( |\Lambda^{(m)}_\ell| - 1 + \sum_{i < j \atop i, j \in \Lambda^{(m)}_\ell} \beta_{ij} \right).$$
\[ \vec{\lambda} = ([3], [1, 1, 4], [2, 1, 1, 1, 2, 1], [1, 2, 1, 1, 1, 1, 1, 1, 1]) \]

- \( \Lambda_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \)

\[ M_0(\vec{\lambda}, n) = \frac{1}{n} \binom{n}{3}, \quad E_0(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + \alpha_2 + 8 + \sum_{1 \leq i < j \leq 9} \beta_{ij} \]

- \( \Lambda_1 = \{1, 2, 3\} \{4\} \{5, 6, 7, 8, 9\} \)

\[ M_1(\vec{\lambda}, n) = \frac{1}{n} \binom{n}{4}, \quad E_1(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + 6 + \sum_{1 \leq i < j \leq 3 \atop 5 \leq i < j \leq 9} \beta_{ij} \]

- \( \Lambda_2 = \{1, 2, 3\} \{4\} \{5\} \{6\} \{7, 8\} \{9\} \)

\[ M_2(\vec{\lambda}, n) = \left( \frac{1}{n} \binom{n}{2} \right)^2, \quad E_2(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + 3 + \beta_{12} + \beta_{13} + \beta_{23} + \beta_{78} \]

- \( \Lambda_3 = \{1\} \{2, 3\} \{4\} \{5\} \{6\} \{7\} \{8\} \{9\} \)

\[ M_3(\vec{\lambda}, n) = \frac{1}{n} \binom{n}{2}, \quad E_3(\vec{\alpha}, \vec{\beta}, \vec{\lambda}) = \alpha_1 + 1 + \beta_{23} \]
An explicit formula for $F_N$

- Define $\sigma(\vec{\beta}) := (\beta_{\sigma^{-1}(i)\sigma^{-1}(j)})$ if $\sigma \in S_N$ and define open sets
  \[
  \Omega^+ := \{ (\vec{\alpha}, \vec{\beta}) : \Re(E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda})) > 0 \text{ for all } \sigma, \vec{\lambda}, \ell \} ,
  \]
  \[
  \Omega := \{ (\vec{\alpha}, \vec{\beta}) : E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda}) \notin \frac{2\pi i \mathbb{Z}}{\log(q)} \text{ for all } \sigma, \vec{\lambda}, \ell \} .
  \]

Theorem (W.)

The function $F_N$ defined by

\[
F_N(\vec{\alpha}, \vec{\beta}) := \int_{m)^N} \left( \min_{i<j} |x_i - x_j| \right)^{\alpha_1} \left( \max_{i<j} |x_i - x_j| \right)^{\alpha_2} \prod_{i<j} |x_i - x_j|^{|\beta_{ij}} \, d\vec{x}
\]

is analytic on $\Omega^+$ and extends to all of $\Omega$ via

\[
F_N(\vec{\alpha}, \vec{\beta}) = \frac{1}{q^N} \sum_{\sigma \in S_N} \sum_{\vec{\lambda} \in \mathfrak{H}(N)} \prod_{\ell=0}^{L(\vec{\lambda})-1} \frac{M_\ell(\vec{\lambda}, q)}{q^{E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda}) - 1}} .
\]
A fun corollary

• For all $\sigma \in S_N$ and all $\vec{\lambda} \in \mathfrak{R}(N)$ we have

$$E_0(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda}) = \alpha_1 + \alpha_2 + N - 1 + \sum_{i<j} \beta_{ij}$$

so $(q^{E_0(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda})} - 1)^{-1}$ is independent of $\sigma$ and $\vec{\lambda}$ and factors out of the whole sum.

• If $\ell > 0$, $E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda})$ is independent of $\alpha_2$.

Corollary

If $\alpha \geq 0$, $\beta > 0$, and $\beta_{ij} = q_i q_j \beta$, then

$$\mathbb{E}[(\max_{i<j}|X_i - X_j|)^\alpha] = \frac{q^{N-1+\sum_{i<j} \beta_{ij}} - 1}{q^{N-1+\sum_{i<j} \beta_{ij}} - q^{-\alpha}}.$$
Another fun corollary

- If $\alpha_1 = \alpha_2 = 0$ and all $q_i = 1$, then $\beta_{ij} = \beta$ for all $ij$, then

$$
E_\ell(\vec{\alpha}, \sigma(\vec{\beta}), \vec{\lambda}) = E_\ell(0, \vec{\beta}, \vec{\lambda}) = \sum_{m=1}^{N_\ell} \left( |\Lambda_{\ell}^{(m)}| - 1 + \left( \frac{|\Lambda_{\ell}^{(m)}|}{2} \right) \beta \right)
$$

for all $\sigma \in S_N$ and we get an analogue of Mehta’s integral:

**Corollary**

$$
\int_{K^N} \mathbf{1}_{o_N}(\vec{x}) \prod_{i<j} |x_i - x_j|^\beta \, d\vec{x} = N! q^{(N/2)} \beta \sum_{\vec{\lambda} \in \Phi(N)} \prod_{\ell=0}^{L(\vec{\lambda})-1} \frac{M_\ell(\vec{\lambda}, q)}{q^{E_\ell(0,\vec{\beta}, \vec{\lambda})} - 1}.
$$
Recursive construction of $\mathfrak{r}(N)$

All splitting sequences in $\mathfrak{r}(N)$ can be constructed by adding nodes and edges to those in $\mathfrak{r}(N - 1)$. An inductive argument gives $|\mathfrak{r}(N)| \leq (2N - 3)!!$ with equality only if $N = 2$ or $N = 3$. 
Everything simplifies considerably when all $q_i = 1$:

- $Z_2(\beta) = \frac{(q-1)q^\beta}{q^{\beta+1}-1}$
- $Z_3(\beta) = \frac{(q-1)q^{3\beta}}{q^{3\beta+2}-1} \cdot \left[(q-2) + \frac{3(q-1)}{q^{\beta+1}-1}\right]$
- $Z_4(\beta) = \frac{(q-1)q^{6\beta}}{q^{6\beta+3}-1} \cdot \left\{ (q-2)(q-3) + \frac{6(q-1)(q-2)}{q^{\beta+1}-1} \right. $
  \quad \quad \quad + \frac{4(q-1)(q-2)}{q^{3\beta+2}-1} \left. \right. $
  \quad \quad \quad + \frac{12(q-1)^2}{(q^{3\beta+2}-1)(q^{\beta+1}-1)} \right. $
  \quad \quad \quad + \frac{3(q-1)^2}{(q^{\beta+1}-1)^2} \right\}$
Thank you!