A $p$-adic Integral by Combinatorics  

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INTRODUCTION

For an integer $N \geq 2$, an algebraic number field $K$, and a prime ideal $\mathfrak{p}$ in the ring of integers $\mathcal{O}_K$, let $\zeta_K$ be the derivative function for the nonarchimedean absolute value defined by $|r|_K = \zeta_K(r)$. Let $X$ be the completion of $K$ with respect to $|r|_K$. Denote the unit balls in $X$ by $B(x) = \{r \in X : |r|_K < \delta(x)\}$, and let $\delta(x)$ be the additive Haar measure on $X$ which satisfies $\delta(x) = 1$. Writing $\mu_K$ for the $N$-fold Cartesian product of $X$ and $\mathbb{R}$ for Lobachevskii integration against the $N$-fold product measure, we have

$$\int_X \mu_K \, d\delta(x) = \delta(0).$$

As an $\mathfrak{a}_0$-isogenuous zeta function

Given a compactly supported locally constant function $\mathfrak{a}_0 : X \to \mathbb{C}$, a continuous homogeneous $\mathfrak{a}_N : X^N \to \mathbb{C}$, and a nonzero polynomial $\mathfrak{a}_s : X \to \mathbb{C}$, we associate the (gauge) zeta function defined by

$$Z_{\mathfrak{a}_0}(s; \mathfrak{a},\mathfrak{a}_N,\mathfrak{a}_s) = \int_X \mathfrak{a}_0 \prod_{i=1}^N |r_i|_K^{s-1} \mu_K \, d\delta(x).$$

Here $\mathfrak{a}_0$ is not to be confused with $\mathfrak{a}$ which is the group of fractional ideals of $K$. Each solution $s \in \mathfrak{a}$ is a polynomial and for every $s \in \mathfrak{a}$ we have an isomorphism of abelian groups $\mathfrak{a}_N \to \mathfrak{a}_N$ by

$$(\mathfrak{a}_N)(x_1,\ldots,x_N) \mapsto (s(x_1),\ldots,s(x_N)).$$

Every ball in $X$ is open, compact, and of the form $B(x) = \{r \in X : |r-x|_K < \delta(x)\}$. The measure of each ball is given by $\mu_K \, d\delta(x) = e^{-\delta(x)}$. The measure of each semicircle $r \to |r-x|_K$ is $2\pi$. If $N = 1$, the measure of each half-circle $r \to |r-x|_K$ is $\pi$. If $N = 2$, the measure of each half-circle $r \to |r-x_1, r-x_2|_K$ is $2\pi$. If $N = 3$, the measure of each half-circle $r \to |r-x_1, r-x_2, r-x_3|_K$ is $3\pi$. If $N = 4$, the measure of each half-circle $r \to |r-x_1, r-x_2, r-x_3, r-x_4|_K$ is $4\pi$.

What is a splitting sequence?

Definition 2. A splitting sequence $\eta$ of $N$ is a tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ of compositions $\eta_i : \mathfrak{a}_i \to \mathfrak{a}_i$ such that $\tau_{\eta_i} \preceq \tau_{\eta_i+1}$ for all $\eta_i \in \mathfrak{a}_i$. Writing $\mathfrak{a}_i(\mathfrak{d})$ for the integrand of $\zeta_K$, we may express the sum over all of $\mathfrak{a}_i$ of the “alphabet” $\mathfrak{d}$. Note that each $\mathfrak{d} \in \mathfrak{a}_i$ can be visualized as a path down an ordered elementary tree, in which case $\mathfrak{d} \notin \mathfrak{d}$ and only if the path is $\eta_{\mathfrak{d}}$ (eventually) to the left of the path for $\eta_{\mathfrak{d}}$.

The theorem follows by summing over all $\mathcal{C}$ and $\mathfrak{d}$ and returning to $\eta_{\mathfrak{d}}$.

THE MAIN RESULT

Theorem 4 (Main Theorem). Define the open sets

$$\Omega_{\eta} = \{ (x_1, x_2) \in \mathbb{R}^2 : \mathfrak{a}_0(\mathfrak{d}) \geq 0 \}$$

and

$$\Omega_{\eta}^c = \{ (x_1, x_2) \in \mathbb{R}^2 : \mathfrak{a}_0(\mathfrak{d}) < 0 \}$$

for all $\mathfrak{d} \in \mathcal{C}$.

RECURSIVE CONSTRUCTION OF SPLITTING SEQUENCES

For a particular $N \geq 2$, we need all $\mathfrak{a}_i \in \mathfrak{a}$ explicitly in order to compute $\zeta_K$. For Theorem 4, they can be constructed recursively as follows. Given $\mathfrak{c}_0, \mathfrak{c}_1$, let $\mathfrak{c}_n = \mathfrak{c}_0 \cup \mathfrak{c}_1$ and construct a family of splitting sequences $\eta_{\mathfrak{n}} (N \geq 1)$ with two types of modifications:

1. [Add a row] A composition $\mathfrak{c} = \{\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_N\}$ having $N$ parts must be composed of $N-1$ parts and a single $\mathfrak{d}_N$.

2. [Add a node] The last composition in $\mathfrak{c}$ has the form $\mathfrak{c}_N = \{\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_N\}$ and may be constructed from $\mathfrak{c}_{N-1}$,

   - by increasing one of the parts $\mathfrak{d}_i$ by $\mathfrak{d}_{N-1}$, which yields a splitting sequence $\mathfrak{c}_{N-1} = \{\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_N\}$,

   - or by replacing one of the parts $\mathfrak{d}_i$ by $\mathfrak{d}_{N-1}$.

Theorem 4 implies that $\mathfrak{c}_N$ is a $N$-adic smooth cover of $\mathfrak{c}_N^c$ in the sense of $\mathfrak{c}_N$.

A PROOF OUTLINE

Here fix a total order $\prec$ on $\mathfrak{d}$ such that $0$ is the last element. By identifying each $\mathfrak{d}$ with its coefficient word $(\tau_{\mathfrak{d}}, \mathfrak{d})$, we define a total (lexicographic) order $\prec$ on all of $\mathfrak{a}_i$ using the “alphabet” $\mathfrak{d}$. Note that each $\mathfrak{d} \in \mathfrak{a}_i$ can be visualized as a path down an ordered elementary tree, in which case $\mathfrak{d} \notin \mathfrak{d}$ and only if the path for $\mathfrak{d}$ (eventually) to the left of the path for $\mathfrak{d}$.

EXAMPLE: $N = 3$, $\mathfrak{d}_1 = 2$, $\mathfrak{d}_2 = 5$, $\mathfrak{d}_3 = 3$. Recall $\mathfrak{d} = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3\}$ and $\mathfrak{a}_i$ is prime. Given $\mathfrak{c}_0, \mathfrak{c}_1$, Definition 3 implies $\mathfrak{a}_N(\mathfrak{d}) = \mathfrak{a}_N(\mathfrak{d}_0) = \mathfrak{a}_N(\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3)$. Moreover, since $\mathfrak{c}_0 \cup \mathfrak{c}_1 = \mathfrak{c}_N$, the class $\mathfrak{c}_N(\mathfrak{d})$ is nonempty.

REFERENCES


