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Some Exceptional 2-Adic Buildings

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1. INTRODUCTION

Tits [19] has initiated the study of geometries having properties strongly resembling those of buildings (compare [6]). His main result is that, in general, such a geometry is the image of a building under a suitable type of morphism. These geometries that are almost buildings were called GABs in [10], and finite examples were described in [2, 8, 10, 15, 16] that are not buildings but have highly transitive groups. In this paper we will proceed in the opposite direction. An unexpected type of description is given for the affine buildings associated to some low-dimensional 2-adic orthogonal groups. This description makes it easy to produce infinitely many finite GABs having large groups.

Specifically, we construct subgroups of $\Omega^+(8, \mathbb{Q}_2)$, $\Omega(7, \mathbb{Q}_2)$, $\Omega^-(6, \mathbb{Q}_2)$ and $G_2(\mathbb{Q}_2)$ that are flag-transitive on the corresponding affine buildings and can be written using matrices with entries in the subring $\mathbb{Z}[\frac{1}{2}]$ of the rationals. (These flag-transitive groups are just $\Omega(\mathbb{Z}[\frac{1}{2}], f_k)$ and the automorphism group of the non-split Cayley algebra over $\mathbb{Z}[\frac{1}{2}]$, where $k = 8$, 7 or 6 and f_k is the quadratic form $\sum_1^k x_i^2$.) If m is any odd integer >1 and these matrices are viewed modulo m then a finite GAB is obtained having a flag-transitive group and one of the following diagrams.

Each connected rank 2 subdiagram corresponds to $PSL(3, 2)$, $Sp(4, 2)$ or $G_2(2)$.

This exceptional behavior of low-dimensional orthogonal groups is a reflection of the exceptional Weyl groups. Sections 3–5 are concerned with

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Weyl groups and root systems of type E_8 and GABs arising from them. These constructions are elementary, but some care is needed since infinite groups are involved. The next sections require additional background concerning 2-adic lattices and buildings (Section 6). This is used in Section 7 in order to prove the only theorem in this paper: the GABs constructed over $\mathbb{Z}[\frac{1}{2}]$ are buildings. However, the more complicated case of $G_2(\mathbb{Q}_2)$ is postponed to Section 8. Further GABs are indicated in Section 9 and at the end of Section 10.

2. PRELIMINARIES

In this section we will present a very special case of the situation studied by Tits [19] and Ronan [13]. Since all of our examples will be group-related, the general case will not be needed.

Let I be a finite index set, and let $\{P_i \mid i \in I\}$ be a family of subgroups of a group G. For $J \subseteq I$ set $P_J = \bigcap \{P_j \mid j \in J\}$, where $P_\varnothing = G$. We will require that

$$P_J = \langle P_{I-\{i\}} \mid i \in I - J \rangle \qquad \text{for all } J \subseteq I. \tag{2.1}$$

However, this condition will always be trivial to check: if $J \neq \varnothing$, some quotient group of P_J will be a Chevalley group whose minimal parabolic subgroups are the projections of the groups $P_{I-\{i\}}$, $i \in I - J$.

Let G/P_J be the set of right cosets of P_J, and let $\Delta(P_i \mid i \in I)$ be the simplicial complex whose set of simplexes is the disjoint union of copies of all the sets G/P_J, $\varnothing \neq J \subseteq I$, ordered by reverse inclusion [18, p. 5]. The *residue* (or link) of a simplex X is obtained by taking all the simplexes $Y \supset X$ and deleting X and all its faces from each such Y. We require that

If $|I - J| = 2$ then the residue of P_J is a generalized polygon (i.e., a rank 2 spherical building). (2.2)

The *diagram* of $\Delta(P_i \mid i \in I)$ is then defined in the obvious manner: its nodes are the elements of I, and distinct nodes i and j are joined by the number of bonds required by the generalized polygon for $P_{I-\{i,j\}}$. (N.B.—The residue of $P_{I-\{i,j\}}$ is a generalized 2-gon if $P_{I-\{i,j\}} = P_{I-\{i\}} P_{I-\{j\}}$.) We will occasionally draw the diagram with some nodes labeled by the members of I to which they correspond.

Two vertices $P_i g$ and $P_j h$ are called incident if $i \neq j$ and $P_i g \cap P_j h \neq \varnothing$. A *flag* is a family of pairwise incident vertices. A *chamber* is a maximal simplex $P_I g$, and clearly determines a maximal flag. Our final requirement is

$$G \text{ is transitive on maximal flags.} \tag{2.3}$$

Thus, each maximal flag will arise from a chamber (cf. [18, p. 5; 19; 1]).

DEFINITION. $\Delta(P_i \mid i \in I)$ is a flag-transitive *GAB* if (2.1)–(2.3) hold. Its *rank* is $|I|$.

A *cover* (or 2-cover [19, 13]) of $\Delta(P_i \mid i \in I)$ can be described as a GAB $\Delta(\tilde{P}_i \mid i \in I)$ with group $\tilde{G} = \langle \tilde{P}_i \mid i \in I \rangle$ together with a homomorphism $\tilde{G} \to G$ inducing an isomorphism $\tilde{P}_J \to P_J$ whenever $|I - J| = 2$ (compare [13]). Universal covers are studied in [19] and [13]. A GAB having no proper cover is called *simply connected.* For example, buildings are simply connected. In our situation the main results of [19] assert that universal covers exist and are buildings if each residue with a rank 3 spherical diagram corresponds to a building. While these results are not actually needed here, they motivate much of what follows. Moreover, in most of our examples covers are actually topological covers: $\tilde{P}_i \to P_i$ is an isomorphism for each i.

We will use the following notation when indicating the structure of a group: nG stands for a group having a normal subgroup of order n and quotient group $\cong G$, while $G \cdot n$ stands for a group having normal subgroup G and quotient group of order n. As usual, $O_2(G)$ is the largest normal 2-subgroup and G' is the derived group of G. Also, if G acts on a set X and $Y \subseteq X$, we will say that G fixes Y if $Y^G = Y$; if $x \in X$ then G_x denotes its stabilizer.

3. FOUR WEYL GROUPS

Let $f(x, y) = (x, y)$ be the usual dot product on $\mathbb{Q}^8$, and let $\mathscr{B} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the standard orthonormal basis. Let $GO(\mathbb{Q}, f)$ be the group of all $g \in GL(8, \mathbb{Q})$ preserving f projectively, so that $(u^g, v^g) = c_g(u, v)$ for some $c_g \in \mathbb{Q}^*$ and all $u, v \in \mathbb{Q}^8$. Finally, set $\Omega = GO(\mathbb{Q}, f)'$ and $\bar{\Omega} = \Omega/\langle -1 \rangle$. In general, if D is any set of linear transformations then $\bar{D}$ will denote $D\langle -1 \rangle / \langle -1 \rangle$.

If $0 \neq u \in \mathbb{Q}^8$ let r_u be the reflection $x \mapsto x - 2(x, u)\, u/(u, u)$. If $u = \Sigma a_i u_i$ with $a_i \in \{0, 1, -1\}$ it will be convenient to abbreviate r_u as $r_{\Sigma a_i i}$.

We will also use a second orthogonal basis

$$\begin{aligned} \mathscr{B}' &= \{u_1 + u_2, u_1 - u_2, u_3 + u_4, u_3 - u_4, u_5 + u_6, \\ &\qquad u_5 - u_6, u_7 + u_8, u_7 - u_8\} \\ &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, \end{aligned}$$

where $v_{2i-1} = u_{2i-1} + u_{2i}$ and $v_{2i} = u_{2i-1} - u_{2i}$ for $1 \leqslant i \leqslant 4$. Consider the following four root systems of type E_8.

$$\Phi^+ = \{\pm u_i \pm u_j | i \neq j\} \cup \left\{ \tfrac{1}{2} \sum_1^8 \varepsilon_i u_i | \varepsilon_i = \pm 1, \prod_1^8 \varepsilon_i = 1 \right\}$$

$$\Phi^- = \{\pm u_i \pm u_j | i \neq j\} \cup \left\{ \tfrac{1}{2} \sum_1^8 \varepsilon_i u_i | \varepsilon_i = \pm 1, \prod_1^8 \varepsilon_i = -1 \right\}$$

$$\Phi^\sharp = \{\pm v_i \pm v_j | i \neq j\} \cup \left\{ \tfrac{1}{2} \sum_1^8 \varepsilon_i v_i | \varepsilon_i = \pm 1, \prod_1^8 \varepsilon_i = 1 \right\}$$

$$\Phi^\flat = \{\pm v_i \pm v_j | i \neq j\} \cup \left\{ \tfrac{1}{2} \sum_1^8 \varepsilon_i v_i | \varepsilon_i = \pm 1, \prod_1^8 \varepsilon_i = -1 \right\}.$$

Set $A = \{+, -, \sharp, \flat\}$. For $\alpha \in A$ let W^α be the commutator subgroup of the corresponding Weyl group. Then $W^\alpha \cong 2\Omega^+(8, 2)$.

If each v_i is written in terms of $\mathscr{B}$, then $\Phi^\sharp$ is seen to consist of all the vectors $\pm 2u_i$, $\pm u_{2i-1} \pm u_{2i} \pm u_{2j-1} \pm u_{2j}$ $(i \neq j)$ and $\pm u_i \pm u_j \pm u_k \pm u_l$ with i, j, k, l in different blocks of the partition 12/34/56/78 and an even number of them even. Similarly, $\Phi^\flat$ contains those vectors $\pm u_i \pm u_j \pm u_k \pm u_l$ with i, j, k, l as above except that an odd number of them are even.

From this description, it is clear that r_8 induces the transposition $(+, -)$ on A, while r_{7-8} induces $(\sharp, \flat)$. Define $\theta \in GL(8, \mathbb{Q})$ by

$$\theta: u_i \mapsto v_i \qquad \text{for } 1 \leqslant i \leqslant 8.$$

Then $(u^\theta, v^\theta) = 2(u, v)$ and $u^{\theta^2} = 2u$ for all $u, v \in \mathbb{Q}^8$. It follows that $\theta \in GO(\mathbb{Q}, f)$, while θ induces $(+, \sharp)(-, \flat)$ on A. Thus, $\langle r_8, \theta \rangle$ induces a dihedral group of order 8 permuting the groups W^α by conjugation.

Clearly, $\Phi^+ \cap \Phi^-$ and $\Phi^\sharp \cap \Phi^\flat$ are root systems of type D_8, whose Weyl groups have subgroups of the form $2^7 A_8$ lying in Ω. Modulo $\langle -1 \rangle$ the latter are maximal parabolic subgroups of the appropriate groups $\Omega^+(8, 2)$. Thus, $W^+ \cap W^-$ and $W^\sharp \cap W^\flat$ have the form $2^7 A_8$.

All four groups W^α share a Sylow 2-subgroup S of order $2^7 \cdot 2^3 \cdot 2^2 \cdot 2$. Namely, with respect to either $\mathscr{B}$ or $\mathscr{B}'$ we define S to consist of all monomial transformations in Ω simultaneously preserving the partitions 12/34/56/78 and 1234/5678. We will use additional monomial transformations with respect to $\mathscr{B}$ or $\mathscr{B}'$ in the remainder of this section.

The group W^α is just the set stabilizer of Φ^α in Ω. Thus, by a straightforward calculation, $W^+ \cap W^- \cap W^\sharp \geqslant \langle S, (u_1, u_5, u_2, u_3, u_4, u_6, u_8) \rangle$, $W^+ \cap W^\sharp \cap W^\flat \geqslant \langle S, (v_1, v_5, v_2, v_3, v_4, v_6, v_8) \rangle$. Since $\bar{W}^+ \cap \bar{W}^\sharp$ is a parabolic subgroup of $\bar{W}^+$ it follows that it must have the form $2^6 A_8$. (In fact, $O_2(W^+ \cap W^\sharp)$ is extraspecial of order 2^7.) Applying r_{7-8} we find that $\bar{W}^+ \cap \bar{W}^\sharp$, $\bar{W}^+ \cap \bar{W}^\flat$ and $\bar{W}^+ \cap \bar{W}^-$ are three of the four maximal parabolic subgroups of $\bar{W}^+$ containing $\bar{S}$.

A further subgroup C of Ω is needed in order to describe the missing parabolic subgroup. Set $z = r_1 r_2 r_3 r_4$ and

$$E = \langle r_1 r_2, r_1 r_3, z, r_{1-2} r_{3-4}, r_{1-3} r_{2-4} \rangle$$
$$F = \langle r_5 r_6, r_5 r_7, -z, r_{5-6} r_{7-8}, r_{5-7} r_{6-8} \rangle$$
$$C = N_\Omega(E \times F).$$

Then E and F are extraspecial of order 2^5 and $\bar{E}\bar{F}$ is extraspecial of order 2^9 with center $\langle \bar{z} \rangle$. (Note that $r_i r_j = r_{i+j} r_{i-j}$ and $r_{i-j} = r_{j-i}$ for $i \neq j$.)

Since $Z(E \times F) = \langle z, -z \rangle$, C fixes $\{\langle u_1, u_2, u_3, u_4 \rangle, \langle u_5, u_6, u_7, u_8 \rangle\}$. It follows that

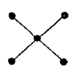
$$C = \langle S, (u_1, u_2, u_3), (u_5, u_6, u_7), (v_1, v_2, v_3), (v_5, v_6, v_7) \rangle$$

(which could have been used as the definition of C) and that $C = (E \times F) \cdot 3^4 \cdot 2^3$, where this 2^3 refers to a group generated by $r_{3-4} r_{7-8}$, $r_4 r_8$ and $r_{1-5} r_{2-6} r_{3-7} r_{4-8}$ (mod EF). Moreover, $\langle r_8, \theta \rangle$ normalizes EF and C, and

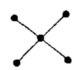
$$C \cap W^+ = \langle S, (u_1, u_2, u_3), (u_5, u_6, u_7), (v_1, v_2, v_3)(v_5, v_6, v_7) \rangle.$$

Thus, $\bar{C} \cap \bar{W}^\alpha = C_{\bar{W}^\alpha}(\bar{z})$ for all $\alpha \in A$. This proves

LEMMA 3.1. *If $\alpha \in A$ then the four groups $\bar{C} \cap \bar{W}^\alpha$ and $\bar{W}^\alpha \cap \bar{W}^\beta$, $\beta \in A - \{\alpha\}$, are the maximal parabolic subgroups of $\bar{W}^\alpha$ containing $\bar{S}$.*

4. THE GABs 

Set $G_8 = \langle C, W^\alpha \mid \alpha \in A \rangle$.

The five groups $\bar{C}$ and $\bar{W}^\alpha$ produce a GAB Δ_8 with diagram  *and group $\bar{G}_8$*. For, (2.1) and (2.2) follow immediately from (3.1), while (2.3) can be easily checked but also follows as in [1, (3.10)].

The group $\langle r_8, \theta \rangle$ clearly acts on Δ_8 while inducing a group of order 8 on A, and hence on the diagram. Thus, it is natural to regard this group as a group of graph automorphisms. We will obtain an S_4 of graph automorphisms in Section 8.

Set $\Lambda = (\mathbb{Z}[\frac{1}{2}])^8 < \mathbb{Q}^8$. Then G_8 leaves Λ invariant. If $m > 1$ is any odd integer then there is a natural homomorphism $G_8 \to G_8^{(m)}$ of G_8 onto a group of automorphisms of $\Lambda/m\Lambda$, obtained by reducing matrices (with respect to $\mathscr{B}$) modulo m. If $D \subseteq GL(8, \mathbb{Z}[\frac{1}{2}])$ let $D^{(m)}$ be its image in $\mathrm{Aut}(\Lambda/m\Lambda)$.

Our discussion of the structure of W^α and C applies equally well to $(W^\alpha)^{(m)}$ and $C^{(m)}$: simply interpret all transformations as elements of $\mathrm{Aut}(\Lambda/m\Lambda)$. In particular, the groups $(W^\alpha)^{(m)}$, $\alpha \in A$, are all distinct. Thus, $(\bar{W}^\alpha)^{(m)} \cong \bar{W}^\alpha$ and $\bar{C}^{(m)} \cong \bar{C}$, and the homomorphism $\bar{G}_8 \to \bar{G}_8^{(m)}$ preserves intersections among the groups $\bar{W}^\alpha$ and $\bar{C}$. Consequently, we obtain a *GAB* $\Delta_8^{(m)}$ *with diagram*  *and group* $\bar{G}_8^{(m)}$. (Flag-transitivity (2.3) is proved using [1, (3.8)].)

If p is an odd prime then $G_8^{(p)} = \Omega^+(8, p)$. For, $G_8^{(p)}$ has index 2 in $(G_8\langle r_8\rangle)^{(p)}$, and r_8 induces a reflection on $\Lambda/p\Lambda$. Thus, [20] applies.

When $p = 3$ we obtain one of the examples in [2].

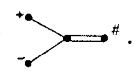
Remarks. 1. In (7.4) we will show that $G_8 = \Omega(\mathbb{Z}[\frac{1}{2}], f)$. From this one can immediately determine $G_8^{(m)}$ for any odd $m \geqslant 1$ (as well as the groups $G_7^{(m)}$ and $G_6^{(m)}$ appearing in the next section).

2. If $(\frac{2}{p}) = 1$ then $r_{7-8}^{(p)}$ and $r_8^{(p)}$ are conjugate by an element of $\Omega^+(8, p)$. Thus, in this case $(r_{7-8} r_8)^{(p)} \in \Omega^+(8, p)$. Also, $2^{-1/2}\theta^{(p)} \in \Omega^+(8, p)$. (For, $2^{-1/2}\theta^{(p)}$ preserves the form on $\Lambda/p\Lambda$; and $((1,3)(2,4)(5,7)(6,8))^{(p)}$ centralizes $2^{-1/2}\theta^{(p)}$ while interchanging its invariant subspaces $\langle u_{2i-1} + p\Lambda, u_{2i} + p\Lambda\rangle$ for $i = 1, 2$ and $i = 3, 4$.) That is, our dihedral group of graph automorphisms has four of its elements inside $\Omega^+(8, p)$, inducing inner automorphisms instead of outer ones. Thus, in this case all four $(W^\alpha)^{(p)}$ are conjugate in $G^{(p)} = \Omega^+(8, p)$.

5. Groups in 6 and 7 Dimensions

Set $H = \Omega_{u_7 - u_8}$. This is just the usual 7-dimensional orthogonal group on $(u_7 - u_8)^\perp$.

Set $G_7 = \langle W^+ \cap H, W^- \cap H\rangle$. Clearly $G_7 \leqslant G_8 \cap H$.

The complex $\Delta_7 = \Delta(C \cap H, W^\alpha \cap H \mid \alpha \in \{+, -, \sharp\})$ *is a GAB with group* G_7 *and diagram* . For, r_{7-8} normalizes $W^\pm$, C, S and $W^\sharp \cap W^\flat$, while $W^\pm \cap H \cong W(E_7)' \cong \Omega(7, 2)$ and $W^\sharp \cap H = W^\flat \cap H \cong 2^6 A_7$. Moreover, $\bar{W}^+ \cap \bar{W}^-$, $\bar{W}^+ \cap \bar{C}$ and $\bar{W}^+ \cap \bar{W}^\sharp \cap \bar{W}^\flat$ are the largest r_{7-8}-invariant parabolic subgroups of $\bar{W}^+$ containing $\bar{S}$. Intersecting with $\bar{H}$, we obtain the three maximal parabolic subgroups of $\bar{W}^+ \cap \bar{H}$ containing $\bar{S} \cap \bar{H}$. This takes care of the two left-hand nodes of the above diagram. In particular, $W^\pm \cap W^\sharp \cap W^\flat \cap H \cong 2^6 PSL(3, 2)$, and we obtain the desired behavior of the right-hand node as well. This produces the indicated diagram, and flag-transitivity is proved as before. Note that r_8 induces a graph automorphism, and that the diagram for Δ_7 arises from that of Δ_8 by a familiar "folding" process ( folds to •⟸•). Also note that $\Delta_7 = \Delta(C \cap G_7, W^\alpha \cap G_7 | \alpha \in \{+, -, \sharp\})$.

Similarly, if $K = \Omega_{u_7, u_8}$ then $\Delta_6 = \Delta(W^+ \cap K, C \cap K, W^\sharp \cap K)$ *is a GAB with diagram* +•⟸•—• # *and group* $G_6 = \langle W^+ \cap K, \quad W^\sharp \cap K \rangle$. Here, $W^+ \cap K = W^- \cap K \cong 2^5 \Omega(5, 2)' \cong W^\sharp \cap K = W^\flat \cap K$. These groups can be increased slightly by adjoining r_1 and r_{1-2}, since both of these normalize the three groups defining Δ_6. This produces an isomorphic GAB, but now we have two groups $2^6 \Omega(5, 2)$ interchanged by θ; that is, θ induces a graph automorphism. (N.B.—We have used G_6, but we could just as well have used $G_6 / \langle -r_7 r_8 \rangle$, since $-r_7 r_8$ induces the identity on Δ_6. A similar remark applies to the use of G_8 instead of $\bar{G}_8$ in Section 4.)

As in the preceding section, *for each odd* $m > 1$ *we obtain GABs in* $G_7^{(m)}$ *and* $G_6^{(m)}$ *covered by* Δ_7 *and* Δ_6, *respectively*. In particular, the GAB for $G_6^{(3)}$ is in [2, 10, 15], while the one for $G_7^{(3)}$ is in [2].

Note that $G_7^{(p)} = \Omega(7, p)$ and $G_6^{(p)} = \Omega^\pm(6, p)$ for each odd prime p (where the sign is determined by the congruence $p \equiv \pm 1 \pmod 4$, since the form inherited by $GF(p)^6$ is $\Sigma_1^6 x_i^2$).

6. Lattices

The present section summarizes some known properties of 2-adic groups needed later. Let $\mathbb{Q}_2$ denote the field of 2-adic numbers and $\mathbb{Z}_2$ its ring of integers.

Let V be an 8-dimensional vector space over $\mathbb{Q}_2$, equipped with a nonsingular symmetric bilinear form (,) of maximal index. Let $GO^+(8, \mathbb{Q}_2)$ be the group of all linear transformations preserving the form projectively, and set $G = \Omega^+(8, \mathbb{Q}_2) = GO^+(8, \mathbb{Q}_2)'$.

If $\Sigma \subset V$ then $\langle \Sigma \rangle$ will denote the $\mathbb{Z}_2$-submodule of V generated by Σ. A *lattice in* V is a submodule generated by a basis of V.

Let $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ be a basis of V such that all inner products are 0 except that $(e_i, f_i) = (f_i, e_i) = 1$ for all i.

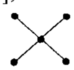
Lemma 6.1. *If* Φ *is the* E_8 *root lattice (over* $\mathbb{Z}$*) then* $\Phi \otimes_{\mathbb{Z}} \mathbb{Z}_2$ *is* $\mathbb{Z}_2$-*isometric to* $\langle e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4 \rangle$.

Proof. See [3, (3.3) and (4.3)]. Alternatively, this is stated almost explicitly in [12, pp. 324–325], and is also an immediate consequence of [7, p. 119].

Next, consider the following five lattices:

$$L_0 = \langle e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4 \rangle$$
$$L_1 = \langle \tfrac{1}{2}e_1, e_2, e_3, e_4, 2f_1, f_2, f_3, f_4 \rangle$$
$$L_2 = \langle \tfrac{1}{2}e_1, \tfrac{1}{2}e_2, e_3, e_4, f_1, f_2, f_3, f_4 \rangle$$
$$L_4 = \langle \tfrac{1}{2}e_1, \tfrac{1}{2}e_2, \tfrac{1}{2}e_3, \tfrac{1}{2}e_4, f_1, f_2, f_3, f_4 \rangle$$
$$L_{4'} = \langle \tfrac{1}{2}e_1, \tfrac{1}{2}e_2, \tfrac{1}{2}e_3, \tfrac{1}{2}f_4, f_1, f_2, f_3, e_4 \rangle.$$

Set $I = \{0, 1, 2, 4, 4'\}$. If $i \in I$ let P_i be the stabilizer of L_i in G, and let U_i be the centralizer of $L_i/2L_i$ in P_i. Note that if $i \in I - \{2\}$ then $P_i \cong \Omega^+(8, \mathbb{Z}_2)$ and $P_i/U_i \cong \Omega^+(8, 2)$. In fact, the reflections $r_{(1/2)e_1+f_1}$ and $r_{e_4+f_4}$ induce the transpositions $(0, 1)$ and $(4, 4')$ on $\{L_i | i \in I\}$, while the transformation $t: e_j \mapsto 2f_{5-j} \mapsto 2e_j$, $1 \leqslant j \leqslant 4$, belongs to $GO^+(8, \mathbb{Q}_2)$ and induces $(0, 4)(1, 4')$ on $\{P_i | i \in I\}$. (Note that $L_0^t = 2L_4$, $L_1^t = 2L_{4'}$ and $L_2^t = \{v \in V | (v, L_2) \subseteq \mathbb{Z}_2\}$.) Thus, the four groups P_i, $i \neq 2$, are conjugate in $GO^+(8, \mathbb{Q}_2)$.

By [5], the complex $\Delta = \Delta(P_i | i \in I)$ is just the affine building for G, with diagram . The groups P_i are the maximal parahoric subgroups containing P_I, where P_I is the "B" of a BN-pair for G [5].

7. Return of the Weyl Groups

We can now determine the GABs Δ_8, Δ_7 and Δ_6.

THEOREM 7.1. *Δ_8 is the affine building for $P\Omega^+(8, \mathbb{Q}_2)$.*

Proof. Let Δ be the affine building for $G = \Omega^+(8, \mathbb{Q}_2)$, described in the preceding section. We will show that G_8 *is flag-transitive on* Δ, from which the theorem will then follow easily.

By (6.1), V has an orthonormal basis $u_1, \ldots, u_8$. Use this basis in order to identify V with $(\mathbb{Q}_2)^8$, and define $\Phi^\alpha \subset \mathbb{Q}^8 \subset V$ and $W^\alpha < G$ as in Section 3. Set $\Lambda^\alpha = \mathbb{Z}\Phi^\alpha$. By (6.1) we may assume that $L_0 = \langle \Lambda^+ \rangle$. Then $L_0/2L_0$ and $\Lambda^+/2\Lambda^+$ are $\Omega^+(8, 2)$-spaces which can be identified. (The quadratic form Q on each takes the coset containing v to $(v, v)/2$ (mod 2).) We can use P_0 in order to assume that the Sylow 2-subgroup S of Section 3 fixes each member of the family $(L_0 \cap L_1/2L_0)^\perp$, $2L_2/2L_0$, $2L_4/2L_0$, $2L_{4'}/2L_0$ of totally singular subspaces of $L_0/2L_0$ (of respective dimensions 1, 2, 4, 4) fixed by P_I. Note that P_I is the stabilizer in G of L_0 and these four subspaces.

We will relate the $\mathbb{Z}$-lattices Λ^α and the $\mathbb{Z}_2$-lattices L_i by comparing the actions of S on $L_0/2L_0$ and $\Lambda^+/2\Lambda^+$. Let $v \mapsto [v]$ denote the natural projection from Λ^+ onto $\Lambda^+/2\Lambda^+$. Then S fixes the subspaces $[\Lambda^+ \cap \Lambda^-]$, $[\Lambda^\sharp]$ and $[\Lambda^\flat]$ of $[\Lambda^+]$. (Note that $\Lambda^+ \supseteq \Lambda^\sharp, \Lambda^\flat$ since $v_i \in \Lambda^+$ and $\frac{1}{2}\sum_1^8 v_i = u_1 + u_3 + u_5 + u_7 \in \Lambda^+$.) We will relate these three subspaces to the totally singular subspaces of $L_0/2L_0$ mentioned in the preceding paragraph.

Since $\Lambda^+ = (\Lambda^+ \cap \Lambda^-) + \frac{1}{2}\mathbb{Z}\sum_1^8 u_i$ and $([2u_1], [u_i \pm u_j]) \equiv (2u_1, u_i) \pm (2u_1, u_j) \equiv 0 \pmod 2$ for all i,j, $[\Lambda^+ \cap \Lambda^-]$ is a hyperplane of $[\Lambda^+]$ perpendicular to $[2u_1]$. Thus, $[\Lambda^+ \cap \Lambda^-]^\perp = \langle [2u_1] \rangle$, where $Q([2u_1]) \equiv (2u_1, 2u_1)/2 \equiv 0 \pmod 2$.

Also, $\Lambda^\sharp$ is generated by $v_1 + v_2$, $v_i - v_{i+1}$ for $1 \leqslant i \leqslant 6$, and $\frac{1}{2}\sum_1^8 v_i = u_1 + u_3 + u_5 + u_7$. Since $[v_{2j-1} - v_{2j}] = [2u_{2j}] = [2u_1] = [v_1 + v_2]$ and $v_2 - v_3 + v_6 - v_7 = u_1 - u_2 - u_3 - u_4 + u_5 - u_6 - u_7 - u_8 \in 2\Lambda^+$, it follows that $[\Lambda^\sharp] = \langle [2u_1], [v_2 - v_3], [v_4 - v_5], [u_1 + u_3 + u_5 + u_7] \rangle$; this is a totally singular 4-space. Similarly, $[\Lambda^\flat] = \langle [2u_1], [v_2 - v_3]\ [v_4 - v_5], [u_1 + u_3 + u_5 + u_8] \rangle$. Moreover, $\Lambda^\sharp \supseteq 2\Lambda^+$ and $\Lambda^\flat \supseteq 2\Lambda^+$ (since $\Lambda^\sharp = \Lambda^{+\theta} \supseteq \Lambda^{\sharp\theta} = \Lambda^{+\theta^2} = 2\Lambda^+$), while $\Lambda^+ \cap \Lambda^- \supseteq 2\Lambda^+$ (since $\sum_1^8 u_i \in \Lambda^+ \cap \Lambda^-$).

Consequently, we can choose the notation L_4 and $L_{4'}$ so that $(L_0 \cap L_1/2L_0)^\perp = (\langle \Lambda^+ \cap \Lambda^- \rangle/2L_0)^\perp$, $2L_4/2L_0 = \langle \Lambda^\sharp \rangle/2L_0$ and $2L_{4'}/2L_0 = \langle \Lambda^\flat \rangle/2L_0$. Then $L_0 \cap L_1 = \langle \Lambda^+ \cap \Lambda^- \rangle$, $2L_4 = \langle \Lambda^\sharp \rangle$ and $2L_{4'} = \langle \Lambda^\flat \rangle$. Also, r_8 interchanges Φ^+ and Φ^-, and hence normalizes the stabilizer $P_0 \cap P_1$ of $\langle \Lambda^+ \cap \Lambda^- \rangle$. Then $(P_0)^{r_8} = P_0$ or P_1. Since $W^- < (P_0)^{r_8}$, it follows that $(P_0)^{r_8} = P_1$.

This proves that $W^+ < P_0$, $W^- < P_1$, $W^\sharp < P_4$ and $W^\flat < P_{4'}$.

Next, $C < P_2$. For, C leaves invariant the $\mathbb{Z}$-lattice Λ_2 spanned by the vectors $\pm v_i \pm v_j$ with $i \neq j$ and $i, j \leqslant 4$ or $i, j \geqslant 5$. (This is a root lattice of type $D_4 + D_4$.) Clearly, $\Lambda_2 \subseteq \Lambda^+$. Also, $2\Lambda^+ \subseteq \Lambda_2$ (since $2u_i$ and $\sum_1^8 u_i$ are in Λ_2). Finally, $[\Lambda_2] = \langle [2u_1], [v_1 + v_3] \rangle$ (since $[v_1 + v_2] = [v_1 - v_2] = [2u_1] = [v_3 + v_4]$ and $[v_1 + v_3] = [v_5 + v_7]$). Thus, $\langle \Lambda_2 \rangle/2L_0$ and $2L_2/2L_0$ are totally singular 2-spaces fixed by S. This proves that $2L_2 = \langle \Lambda_2 \rangle$, and hence that $C < P_2$.

Thus, each quotient P_i/U_i, $i \neq 2$, is covered by some W^α, while C acts flag-transitively within P_2/U_2. Then G_8 induces a flag-transitive group on the residue of each P_i. Consequently, G_8 is flag-transitive on Δ (by the connectedness of Δ).

It follows that $\Delta \cong \Delta(G_8 \cap P_i \mid i \in I)$. There is a natural projection $\pi: \Delta_8 \to \Delta(G_8 \cap P_i \mid i \in I)$ sending $W^+ g \mapsto (G_8 \cap P_0) g$, $g \in G_8$, and so on. It remains to show that π is an isomorphism. This follows from the (topological) simple connectivity of Δ (cf. [19]), or can be proved directly as follows. First, $G_8 \cap P_0 = W^+$. (This will be proved below in Lemma 7.2.) Now let $i \neq 0$. Then $G_8 \cap U_i \leqslant U_i < P_0$, and $\bar{G}_8 \cap \bar{U}_i$ acts faithfully on $L_0/2L_0$, agreeing with a subgroup of W^+ in its action there. Since $\bar{G}_8 \cap \bar{U}_0 = 1$ it follows that $\bar{G}_8 \cap \bar{U}_i = 1$. This proves that π is an isomorphism, and completes the proof of the theorem.

LEMMA 7.2. $\Omega(\mathbb{Z}[\frac{1}{2}], f) \cap P_0 = W^+$.

Proof. Consider a fundamental system $\mathscr{F}$ of roots of Φ^+ (such as $-\frac{1}{2}\sum_1^8 u_i$, $u_2 \pm u_1$, $u_3 - u_2$, $u_4 - u_3$, $u_5 - u_4$, $u_6 - u_5$, $u_7 - u_6$). If T is the transition matrix from $\mathscr{B}$ to $\mathscr{F}$ then all entries of both T and T^{-1} lie in $\frac{1}{2}\mathbb{Z}$.

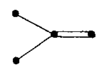
Let $g \in \Omega(\mathbb{Z}[\frac{1}{2}], f) \cap P_0$. Let F and B be the matrices of g with respect to $\mathscr{F}$ and $\mathscr{B}$, respectively. Then $F = T^{-1}BT \in GL(8, \mathbb{Z}[\frac{1}{2}])$, while $F \in GL(8, \mathbb{Z}_2)$ since $g \in P_0$. Thus, $g \in GL(8, \mathbb{Z})$. It follows that g preserves the $\mathbb{Z}$-lattice $\mathbb{Z}\mathscr{F} = \mathbb{Z}\Phi^+$, and hence also preserves its set Φ^+ of minimal vectors. Thus, g lies in the stabilizer W^+ of Φ^+ in $\Omega(\mathbb{Q}, f)$. This proves the lemma.

COROLLARY 7.3. $G_8 = \Omega(\mathbb{Z}[\frac{1}{2}], f)$.

Proof. By (6.1), $J = \Omega(\mathbb{Z}[\frac{1}{2}], f_8)$ acts on Δ. Since $G_8 \leqslant J$, both groups act flag-transitively on Δ by (7.1). Then $J = G_8(J \cap P_0) = G_8(J \cap U_0)$. By (7.2), $J \cap U_0 = \langle -1 \rangle$. Thus, $J = G_8$.

Let f_k denote the quadratic form $\sum_1^k x_i^2$ for $k = 6, 7, 8$, so that $f_8 = f$.

The stabilizer of v_8 in $\Omega^+(8, \mathbb{Q}_2)$ is the group $\Omega(7, \mathbb{Q}_2)$ of the orthogonal space $v_8^\perp$ of index 3. Moreover, using the basis $v_1, \ldots, v_7$ of $v_8^\perp$ we see that $\Omega(7, \mathbb{Q}_2) = \Omega(\mathbb{Q}_2, f_7)$. Since $(v_8, v_8) = (e_4 + f_4, e_4 + f_4)$, v_8 and $e_4 + f_4$ are in the same orbit of $\Omega^+(8, \mathbb{Q}_2)$. Thus, the following description of the affine building for $(e_4 + f_4)^\perp$ has an analogue for $v_8^\perp$.

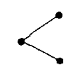
The building for $(e_4 + f_4)^\perp$ has the diagram  , where the four nodes correspond to the stabilizers in $\Omega(7, \mathbb{Q}_2)$ of the following lattices in $(e_4 + f_4)^\perp$.

$$\langle e_1, e_2, e_3, f_1, f_2, f_3, e_4 - f_4 \rangle$$

$$\langle \tfrac{1}{2}e_1, e_2, e_3, 2f_1, f_2, f_3, e_4 - f_4 \rangle$$

$$\langle \tfrac{1}{2}e_1, \tfrac{1}{2}e_2, e_3, f_1, f_2, f_3, e_4 - f_4 \rangle$$

$$\langle \tfrac{1}{2}e_1, \tfrac{1}{2}e_2, \tfrac{1}{2}e_3, f_1, f_2, f_3, e_4 - f_4 \rangle$$

Thus, the chambers of this building are obtained from the set of chambers of Δ fixed by $r_{e_4+f_4}$ by identifying a pair of vertices in each fixed chamber (the "folding" process taking  to). With this in mind, we can now prove

COROLLARY 7.4. (i) *Δ_7 is the affine building for $\Omega(7, \mathbb{Q}_2) = \Omega(\mathbb{Q}_2, f_7)$.*

(ii) $G_7 \cong \Omega(\mathbb{Z}[\frac{1}{2}, f_7)$.

Proof. Clearly, $G_7 \leqslant C_{G_8}(r_{7-8})$, and G_7 acts on the set of all chambers of Δ_8 fixed by $r_{7-8} = r_{v_8}$. One such chamber is S, and G_7 induces $C \cap G_7$ and

$W^{\alpha} \cap G_7$ on the appropriate residues of the vertices of S. Since the affine building for $v_8^{\perp}$ is connected, it follows that $\langle C \cap G_7,\ W^{\alpha} \cap G_7 | \alpha \in \{+, -, \#\}\rangle$ is flag-transitive on that building. The preceding group is contained in G_7, while the chambers of Δ_7 are among the chambers fixed by r_{7-8}. Thus, Δ_7 is (isomorphic to) the affine building for $v_8^{\perp}$. This proves (i), and (ii) follows from the fact that $\Omega(\mathbb{Z}[\frac{1}{2}], f_7) = \Omega(\mathbb{Z}[\frac{1}{2}], f_8)_{v_8}$.

The same argument also proves

COROLLARY 7.5. (i) *Δ_6 is the affine building for $\Omega^-(6, \mathbb{Q}_2) = \Omega(\mathbb{Q}_2, f_6)$.*

(ii) $G_6 = \Omega(\mathbb{Z}[\frac{1}{2}], f_6)$.

8. CAYLEY ALGEBRAS

The last affine building considered here arises from $G_2(\mathbb{Q}_2)$. We will use Cayley algebras and results from [3] and [9] in order to construct a subgroup of G_8 acting flag-transitively on this building. It is interesting to note that these references contain hints of parts of Sections 3 and 6: the bases $\mathscr{B}$ and $\mathscr{B}'$ appear in [9] (see below), while (6.1) is proved and used in [3].

PROPOSITION 8.1. (i) *There is a subgroup $G_* = \langle Q_0, Q_2, Q_{4'}\rangle$ of G_8 such that $\Delta_*^{(m)} = \Delta(Q_0^{(m)}, Q_2^{(m)}, Q_{4'}^{(m)})$ is a GAB for each odd integer $m \geqslant 1$. Its diagram is $\Lleftarrow\!\!\!\Rrightarrow\!\!-\!\!\bullet$ and its rank 2 residues corresponding to $Q_0^{(m)}$ and $Q_{4'}^{(m)}$ are $PG(2, 2)$ and the generalized hexagon for $G_2(2)$.*

(ii) *$\Delta_*^{(1)}$ is isomorphic to the affine building for $G_2(\mathbb{Q}_2)$.*

(iii) *G_* is the automorphism group of the non-split Cayley algebra over $\mathbb{Z}[\frac{1}{2}]$.*

(iv) *If p is an odd prime then $G_*^{(p)} \cong G_2(p)$.*

Proof. (i) Let $1, i, j, k$ be the usual basis of the rational quaternions. The non-split rational Cayley algebra $\mathbf{A}$ has basis $1, i, j, k, e, ie, je, ke$ (in the notaton of [9]) and multiplication given by $(p + qe)(r + se) = pr - \bar{s}q + (sp + q\bar{r})\, e$, where p, q, r and s are quaternions and $\bar{s}$ is the usual quaternion conjugate of s. This basis will be identified with $\mathscr{B}$ (cf. Section 3), as follows:

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) = (1, e, i, ie, j, je, k, ke).$$

Define v_i as in Section 3. Then $v_i = 2l_i$ in the notation of [9], and $\frac{1}{2}\mathbb{Z}\Phi^{\flat}$ is precisely the ring of integral Cayley numbers studied in [9]. By [3], the automorphism group $Q_{4'}$ of this ring is $G_2(2)$. Clearly, $Q_{4'} < W^{\flat}$. (This somewhat awkward notation is caused by the desirability of relating our notation to that of [9].) Note that $\mathbb{Z}[\frac{1}{2}]\,\Phi^\flat$ is the algebra occurring in (iii).

Next, Q_0 is defined to be the group of all monomial transformations (with respect to $\mathscr{B}$) preserving the multiplication table of $\mathscr{B}$. Then $Q_0 \cong 2^3PSL(3, 2)$ (non-split extension) by [9]. There are exactly 7 subsets of $\mathscr{B}$ each of which spans a quaternion subalgebra. Each involution in $O_2(Q_0)$ is 1 on one of these subalgebras and -1 on its orthogonal complement. One of these involutions is $-z = r_5r_6r_7r_8$.

Note that $Q_0 \cap Q_{4'} \cap S = \langle O_2(Q_0), (u_3, u_4, -u_3, -u_4)(u_7, u_8, -u_7, -u_8), (u_5, u_7, -u_5, -u_7)(u_6, u_8, -u_6, -u_8)\rangle$ is a Sylow 2-subgroup of both Q_0 and $Q_{4'}$. Then $Q_0 \cap Q_{4'} = \langle Q_0 \cap Q_{4'} \cap S, (u_3, u_5, u_7)(u_4, u_6, u_8)\rangle$ is a maximal parabolic subgroup of $Q_{4'}$, and $Q_0 = \langle Q_0 \cap Q_{4'}, (u_2, u_3, u_4)(u_6, -u_7, -u_8)\rangle < W^+ \cap W^- \cap W^\sharp$.

Set $G_* = \langle Q_0, Q_{4'}\rangle$ and $Q_2 = \langle C_{Q_0}(z), C_{Q_{4'}}(z)\rangle < \langle C_{W^+}(z), C_{W^\flat}(z)\rangle = C$. Then $Q_0 \cap Q_{4'}$ and $Q_2 \cap Q_{4'}$ are maximal parbolic subgroups of $Q_{4'}$, while $Q_0 \cap Q_2$ and $Q_0 \cap Q_{4'}$ project onto maximal parabolic subgroups of $Q_0/O_2(Q_0)$. This produces the diagram required in (i), and $\Delta_*^{(m)}$ is a GAB by [1].

(ii)–(iv) Clearly, $G_* \leqslant \operatorname{Aut} \mathbf{A} < J = G \cap \operatorname{Aut}(\mathbf{A} \otimes_{\mathbb{Q}} \mathbb{Q}_2)$. There is a unique triality automorphism τ of $\bar{G}$ that centralizes $\bar{J} \cong G_2(\mathbb{Q}_2)$. Moreover, τ normalizes $\bar{\Omega}$, $\bar{G}_8 = P\Omega(\mathbb{Z}[\frac{1}{2}], f_8)$, and the kernel of $\bar{G}_8 \to \bar{G}_8^{(m)}$ for each odd $m \geqslant 1$. (This follows, for example, from the description of triality in [4].)

There is a natural action of τ on the building Δ of (7.1). The building Δ_* referred to in (ii) can be described as follows: its chambers are the τ-invariant chambers of Δ; its vertices are the $\langle\tau\rangle$-orbits of vertices of these fixed chambers (compare Section 5). Then Δ_* has diagram $\Lleftarrow\!\!\!\!\Rrightarrow\!\!-\!\!\bullet$, and $G_2(2)$, a group of order $3^2 \cdot 2^2$, and $PSL(3, 2)$ are induced (within $\bar{J}$) on the appropriate residues. (Note that the first and last of these are precisely the same groups induced by $Q_{4'}$ and Q_0 on the corresponding residues of $\Delta_*^{(1)}$.)

We claim that τ normalizes $\bar{W}^\flat$. For, $G_2(2) \cong \operatorname{Aut}(\frac{1}{2}\mathbb{Z}\Phi^\flat) \leqslant \bar{W}^\flat \cap J$, where $\operatorname{Aut}(\frac{1}{2}\mathbb{Z}\Phi^\flat)$ is contained in a unique $\bar{G}_8$-conjugate of $\bar{W}^\flat$. (Since $\bar{W}^\flat$ has a single conjugacy class of subgroups isomorphic to $G_2(2)$, each such subgroup is transitive on the $\bar{G}_8$-conjugates of $\bar{W}^\flat$ containing it. Thus, there is only one such conjugate.)

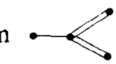
Now $\tau \in \operatorname{Aut} \bar{W}^\flat$ and τ centralizes $\bar{S} \cap \bar{J}$. But the latter group is contained in a unique Sylow 2-subgroup of $\bar{W}^\flat$ (since it fixes a unique maximal flag of the usual $\Omega^+(8, 2)$-space). Thus, τ normalizes $\bar{S}$.

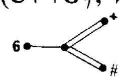
In particular, P_I is a simplex in Δ_*. Then so are $P_{4'}$, P_2 and $P_0 \cap P_1 \cap P_4$, and $Q_{4'}$, Q_2 and Q_0 act flag-transitively on their respective residues (in Δ_*). Thus, G_* is flag-transitive on Δ_*.

Now we can argue exactly as in Section 7 in order to deduce that $\Delta_*^{(1)} \cong \Delta_*$ and that $G_* = \operatorname{Aut}(\mathbb{Z}[\frac{1}{2}]\,\Phi^\flat)$. This proves (ii) and (iii), and then (iv) follows easily.

Remark. A Cayley algebra also appears in the construction of $\Delta_*^{(3)}$ given in [8]. Its role there is different than here. Namely, the Cayley algebra over $GF(3)$ is used there in order to give a particularly nice geometric realization of $\Delta_*^{(3)}$. The cosets of $Q_{4'}^{(3)}$ are identified with an orbit of nonsingular 1-spaces, and the cosets of $Q_2^{(3)}$ and $Q_0^{(3)}$ are identified with suitable triples (e.g., $\{i, j, k\}$) and septuples (e.g., $\mathscr{B}$-$\{1\}$) of these 1-spaces. That description depends upon the fact that $Q_{4'}^{(p)}$ acts reducibly on the 7-dimensional module for $G_2(p)$ when $p = 3$.

9. More GABs

In this section we will describe GABs with diagram  related to a remarkable GAB for A_7 not arising from a building. The later GAB is $\Delta(X_1, X_2, X_3)$ with $X_1 = A_6$, $X_2 = (A_4 \times A_3) \cdot 2$ and $X_3 = PSL(3, 2)$ [11, 2, 16]; its universal cover cannot be a building (and, in fact, the GAB is simply connected [14]). Thus, the GABs we will construct cannot be covered by buildings.

Define $C \cap G_7$, $W^+ \cap G_7 \cong \Omega(7, 2)$, $W^\sharp \cap G_7 \cong 2^6 A_7$, $G_6^* = \Omega(\mathbb{Z}[\frac{1}{2}], f_8)_{\langle u_7+u_8\rangle, u_7-u_8} = G_6\langle r_{1-2} r_{7+8}\rangle$ and $G_7 = \Omega(\mathbb{Z}[\frac{1}{2}], f_8)_{u_7-u_8}$ as in Section 5 (compare (7.3) and (7.4)). Then $\Delta_{7'} = \Delta(C \cap G_7, W^+ \cap G_7, W^\sharp \cap G_7, G_6^*)$ *is a GAB with group* G_7 *and diagram* . For, $G_6^* \cap W^+ \cap G_7 \cong 2^5\Omega(5, 2)$, $G_6^* \cap W^\sharp \cap G_7 \cong 2^6\Omega(5, 2)'$ and $G_6^* \cap C \cap G_7 = G_6^* \cap C$, which takes care of the G_6 node of the diagram. The remaining checks are also straightforward, and produce a GAB as before.

Once again *we obtain a GAB from* $G_7^{(m)}$ *for each odd* $m > 1$. When $m = 3$ this is one of the GABs in [2].

Conjecture. $\Delta_{7'}$ is simply connected.

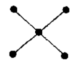
Note that passage modulo m is *not* a topological cover: just consider residues of vertices of type 6.

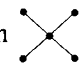
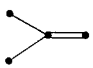
10. Remarks

1. Lemma 6.1 can be proved using the affine building Δ for $P\Omega^+(8, \mathbb{Q}_2)$ as follows. Since W^+ is a bounded subgroup it has a fixed point on Δ [5, pp. 64–65]. Then W^+ must fix some lattice L_i^g, $g \in \Omega^+(8, \mathbb{Q}_2)$, and the lemma follows easily.

The building can also be used to avoid some calculations in Section 3. Namely, once it is known that $W^+ < P_0$ and $W^- < P_1$, it follows that

$W^{\pm} \cap P_i$ $(i \neq 0, 1)$ projects onto a maximal parabolic subgroup of P_i/U_i. Then the groups C, $W^{\sharp}$ and $W^{\flat}$ can be defined by $C = \langle W^+ \cap P_2, W^- \cap P_2 \rangle$, $W^{\sharp} = \langle W^+ \cap P_4, W^- \cap P_4 \rangle$ and $W^{\flat} = \langle W^+ \cap P_{4'}, W^- \cap P_{4'} \rangle$, and will act flag-transitively on the residues of the appropriate P_i. That they act faithfully on these residues is proved at the end of the proof of (7.1). The approach we used in Sections 3–7 was intended to be more concrete than the one just outlined, and much more elementary in Sections 3–5.

2. Theorem 7.1 can be proved very indirectly, as follows. Starting with Δ_8, form a universal cover $\tilde{\Delta}_8$. By Tits [19], $\tilde{\Delta}_8$ is a building. By an unpublished result of Tits, any affine building with diagram  arises from $P\Omega^+(8, F)$ for some complete local field F. The various groups W^{α} and C lift to subgroups of $\Omega^+(8, F)$. Some calculations yield that $F = \mathbb{Q}_2$, and that G_8 lifts to itself (the proof resembles parts of Section 7). This implies the theorem.

3. The argument indicated in the preceding remark can also be used to prove the following *Theorem*: Let $\Gamma = \Delta(F_i \mid i \in I)$ be a finite flag-transitive GAB with group F and diagram . Then Δ_8 is a universal cover of Γ, and F lifts to (a conjugate of) G_8. (More generally, Γ only needs to be a chamber system [19] instead of a GAB. There is also an analogue of the theorem for the case , assuming that the universal cover is an $\Omega(7, F)$ affine building.)

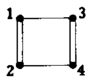
4. Those wishing to avoid Cayley algebras in Section 8 can use [2] instead. In that paper, a GAB $\Delta_*^{(3)}$ is constructed using $G_2(3)$. Let σ' be a triality automorphism of $P\Omega^+(8, 3)$ centralizing this $G_2(3)$ and normalizing $\bar{S}^{(3)}$ and $(\bar{W}^{\flat})^{(3)}$ (cf. [2]). Since $\Delta_8 \to \Delta_8^{(3)}$ is a universal cover, σ' lifts to an automorphism σ of Δ_8 of order 3 that normalizes $\bar{S}$ and $\bar{W}^{\flat}$. One then shows that $C_{\bar{G}}(\sigma) = G_2(\mathbb{Q}_2)$, the affine building of σ-invariant simplexes or triples of simplexes is a cover of $\Delta_*^{(3)}$, and σ normalizes the kernel of $\bar{G}_8 \to \bar{G}_8^{(m)}$ for each odd m. (Of course, $\sigma^{\pm 1}$ is just the triality automorphism τ appearing in Section 8.)

5. The construction in Section 4 can be carried out in part using other lattices. Conceivably, interesting objects can arise, though probably not diagram geometries. The most tempting candidate is, of course, the Leech lattice Λ. (For example, if Φ is an E_8 sublattice then there is an element in the orthogonal group that normalizes the stabilizer of Φ in Aut Λ but moves Λ.)

6. The properties of GABs discussed in [10]—such as the intersection property [6] or diameter—remain open for the GABs constructed here.

7. Corollary 7.5 directly contradicts a remark at the end of [10]. That remark is therefore incorrect. The proof in [10, p. 241] that apartments do not exist is also incorrect, and it is not clear whether or not that assertion is correct.

The following errors were introduced into [10] after the galley proofs had been corrected. On p. 240, *ll.* 15 and −10, read $\rtimes$ for $\times$; on p. 241, between lines 18 and 19, insert "Apartments do not exist." (a deletion for which the printer should perhaps be commended); and insert "generate N/T and satisfy" in the blank space at the beginning of *l.* −1 of p. 246. Finally, on p. 244, *l.* −12, read $\{x, y, a, b\}$ for $\{a, y, a, b\}$.

8. Another example of a GAB involving the A_7 GAB is constructed as follows (and was also discovered, earlier, by A. Neumaier [11]). Set $G = PSU(3, 5)$. Let X and Y be two of its three classes of subgroups $\cong A_7$. Then $P_1, P_2 \in X$ and $P_3, P_4 \in Y$ can be chosen so that $P_1 \cap P_2 \cong P_3 \cap P_4 \cong A_6$, $P_1 \cap P_3 \cong P_2 \cap P_4 \cong PSL(3, 2)$ and $P_1 \cap P_4 \cong P_2 \cap P_3 \cong (A_3 \times A_4) \cdot 2$. Moreover, *$\Delta(P_1, P_2, P_3, P_4)$ is a GAB with diagram* 

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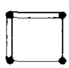
Ronan and Stroth [16] have found still another GAB involving the A_7 GAB. Theirs has diagram  and group M^c.

9. There are 3-adic analogues of parts of Sections 3–8.

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Note added in proof. The conjecture in Section 9 has been proved by Li Hui-Ling (Two pairs of simply connected geometries, to appear).

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