Automorphism groups of some generalized quadrangles

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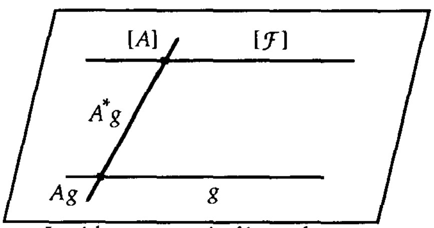
Dedicated to James Hirschfeld on the Occasion of his 50th Birthday.

The following construction was first given in Kantor (1980). Let Q be a finite group and $\mathcal{F}$ a family of subgroups; with each $A \in \mathcal{F}$ is associated another subgroup A^*. These subgroups satisfy the following conditions for some integers $s, t > 1$ and all distinct A, B, $C \in \mathcal{F}$:

$$|Q| = st^2,\ |\mathcal{F}| = s+1,\ |A| = t,\ |A^*| = st,$$
$$A < A^*,\ Q = A^*B,\ A^* \cap B = 1,\ AB \cap C = 1.$$

A generalized quadrangle $Q(\mathcal{F})$ is constructed by using cosets together with symbols $[A]$ and $[\mathcal{F}]$ as in the following picture (for all $A \in \mathcal{F}$ and $g \in Q$):



Incidence as indicated

Points:	*Lines:*
$[A]$	$[\mathcal{F}]$
cosets Ag	cosets A^*g
	elements g

"Most" of the known families of finite generalized quadrangles arise by this procedure (see Kantor (1980); Payne (1990)). It seems likely that Q must be a p-group for some prime p. While this remains open, there has been significant progress (Frohardt (1988); Chen and Frohardt (submitted)).

Proposition. Let $G = \mathrm{Aut}Q(\mathcal{F})$. Then one of the following holds:

(i) G fixes a line, namely $[\mathcal{F}]$.

(ii) G fixes a point ∞ but no line. Then ∞ is on $[\mathcal{F}]$, G is transitive on the points collinear with ∞ as well as those not, and G is 2-transitive on the lines on ∞.

(iii) G is transitive on both points and the lines of $Q(\mathcal{F})$, having rank 3 on each of these sets. G is flag-transitive. The stabilizer of a line is 2-transitive on the points of the line and dually.

Proof. **(i)** Q moves each line $\neq [\mathcal{F}]$.
(ii) Q moves each point not on $[\mathcal{F}]$, so that the fixed point ∞ must be on $[\mathcal{F}]$, say $\infty = [A]$. Then G must have an element g moving $[\mathcal{F}]$ to some other line on $[A]$. Write $H = \langle Q, Q^g \rangle$.

Note that Q fixes $[A]$ and is transitive on the lines $\neq [\mathcal{F}]$ on $[A]$, so that H is 2-transitive on the lines through ∞. Since A^* is transitive on the set of points $\neq [A]$ of the line A^*1, it follows that G is transitive on the set of points collinear with ∞.

Since Q is transitive on the lines missing $[\mathcal{F}]$, H is transitive on the lines not on ∞. It remains to consider the action of the points not collinear with ∞. Recall that B^* is transitive on the set of points $\neq [B]$ of the line B^*1; by transitivity on the lines not on ∞, the same is true for each such line. Thus, if x is any point not collinear with ∞, then x^H contains every point collinear with x but not ∞. Moreover, every such line must contain at least s points of x^H, so there can be only one such point-orbit x^H.
(iii) Now without loss of generality G moves every point and line. Then $[\mathcal{F}]^G$ contains two lines with no common points, and hence contains every line since Q is transitive on the lines missing $[\mathcal{F}]$. If $g \in G$ moves $[\mathcal{F}]$ to a line meeting $[\mathcal{F}]$ at a point x, then $\langle Q, Q^g \rangle$ fixes x and is transitive on the points collinear with x as well as those not collinear with x. Thus, G has rank 3 on points. Moreover, $\langle Q, Q^g \rangle$ is 2-transitive on the lines through x. In particular, G is flag-transitive.

Moreover, it follows that the stabilizer of $[\mathcal{F}]$ is transitive on the lines meeting $[\mathcal{F}]$. Thus, G has rank 3 on lines. □

Since all rank 3 groups are essentially known (using the classification of finite simple groups), it is clear that one can determine all the possibilities in (iii); and this does, indeed, lead to a characterization of the classical generalized quadrangles. However, this seems to be an uninteresting and uninformative approach. Too much information is available and too much is ignored in the above argument (especially the *regularity* of Q on the set of lines missing $[\mathcal{F}]$). In other words, a more geometric approach is needed.

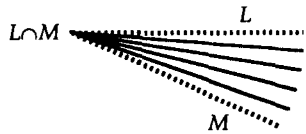
As a side remark, it is also clear that all generalized quadrangles whose automorphism groups have rank 3 on the set of points can be determined. Again, this is a straightforward question. (Note, however, that there is one nonclassical example: the quadrangle with

$s = 3$, $t = 5$.) A more interesting question is the determination of all flag-transitive quadrangles — using the aforementioned classification! Besides the classical ones, presumably there are only two others, with $s = 3$, $t = 5$ and $s = 15$, $t = 17$.

In the only known nonclassical examples for which (ii) holds, Q is elementary abelian of order q^4 (Kantor (1986); Payne (1989)).

In order to make further progress in the context of the Proposition, it seems necessary to have a way to recover Q from $Q(\mathcal{F})$. In general this is entirely open. However, under "reasonable" assumptions this can be accomplished:

For intersecting lines L and M, let $U_{L,M}$ denote the group of all automorphisms fixing every point of L, every point of M and every line on $L \cap M$.



Then $U_{L,M}$ is semiregular on the t lines $\neq L$ through each point $\neq L \cap M$ of L, so that $|U_{L,M}| \leq t$ with inequality if and only if $U_{L,M}$ is regular on those t lines. The groups $U_{L,M}$, and their duals, are exactly those involved in the Moufang condition for generalized quadrangles (Tits (1976); Payne and Thas (1984)).

It is clear that the collection of groups $U_{L,M}$ is canonically determined by a quadrangle. Thus, if Q is generated by some such subgroups then it can be recovered from the quadrangle.

Remark. $U_{[\mathcal{F}],A^{\cdot 1}} = A \Leftrightarrow (A \triangleleft A^*$ and $g^{-1}Ag \leq A^*, \forall g \in Q)$. Hence, if this condition holds for (at least) three members of $\mathcal{F}$ then $Q = \langle U_{[\mathcal{F}],M} \mid M \text{ meets } [\mathcal{F}]\rangle$.

This is straightforward to check. The stated condition holds for all known examples of the construction given at the beginning of this note.

Let q be a prime power.

Example 1. Let Q be the set $GF(q)^2 \times GF(q) \times GF(q)^2$ equipped with the multiplication $(u, c, v)(u', c', v') = (u + u', c + c' + v \cdot u', v + v')$, where $v \cdot u'$ is the usual dot-product. Then Q is a group, whose center is $Z(Q) = 0 \times GF(q) \times 0$. Moreover, $Q/Z(Q)$ can be viewed as a vector space over $GF(q)$. Let

$$A(\infty) = 0 \times 0 \times GF(q)^2,\ A(r) = \{(u, uB_r u^t, uM_r) \mid u \in GF(q)^2\},$$
$$\text{and } A^* = AZ(Q) \text{ for } r \in GF(q)^2 \text{ and each } A = A(\infty) \text{ or } A(r),$$

where B_r and M_r are 2×2 matrices satisfying suitable conditions in order to produce a quadrangle with $s = q$ and $t = q^2$ (Payne (1980, 1989, 1990); Kantor (1986)).

In this situation, *if* $Q(\mathcal{F})$ *is* ***not*** *the* $O(5, q)$ *quadrangle then* $Q \trianglelefteq \mathrm{Aut}Q(\mathcal{F})$. For, in Payne (1989) it is shown that the line $[\mathcal{F}]$ is the *only* line L with the following property: If x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 are two sets of pairwise noncollinear points such that x_1 and y_1 are on L and x_i and y_j are collinear for all i, j except perhaps for $i = j = 4$, then also x_4 and y_4 are collinear. Thus, $\mathrm{Aut}Q(\mathcal{F})$ fixes $[\mathcal{F}]$, and then the preceding Remark implies that Q is normal in $\mathrm{Aut}Q(\mathcal{F})$.

Example 2. Let $s = q$ and $t = q^2$, and assume that $Q/Z(Q)$ is elementary abelian of order q^4 and $A^* = AZ(Q)$ for all $A \in Z$. Note that quadrangles behaving in this manner, but not as in Example 1, have been constructed in Payne (1989).

Once again, *if* $Q(\mathcal{F})$ *is* ***not*** *the* $O(5, q)$ *quadrangle then* $Q \trianglelefteq \mathrm{Aut}Q(\mathcal{F})$. This time no purely geometric argument presently is available.

By the Remark, Q can be recovered as the group generated by those $U_{[\mathcal{F}],M}$ with M meeting $[\mathcal{F}]$. Also, $Z(Q)$ is the group $U_{[\mathcal{F}]}$ of all automorphisms of $Q(\mathcal{F})$ fixing every line meeting $[\mathcal{F}]$, and is regular on the set of points not on $[\mathcal{F}]$ of each such line.

Now consider possibilities (ii) and (iii) for $\mathrm{Aut}Q(\mathcal{F})$ in the Proposition. If (iii) holds, then $Q(\mathcal{F})$ is Moufang, and the main theorem in Fong and Seitz (1973) can be applied. However, this will not be needed: possibilities (ii) and (iii) will be handled simultaneously.

Consider the group H generated by Q and Q^g for some $g \in G$ such that $[\mathcal{F}]^g$ meets $[\mathcal{F}]$ at a point $[A]$. Without loss of generality $[\mathcal{F}]^g = A^*1$. The stabilizer $H_{[\mathcal{F}]}$ normalizes Q (since Q is canonically determined by $[\mathcal{F}]$), so that the stabilizer $H_{[\mathcal{F}],1}$ lies in $\mathrm{Out}(Q)$.

Let K be the kernel of the 2-transitive action of H on the set of $t + 1$ lines on $[A]$. Then $K \cap Q = A^*$, while A is just the pointwise stabilizer of A^*1 in Q Moreover, $Q/A^* \cong QK/K$ induces a normal subgroup of $H_{[\mathcal{F}]}/K$ regular on the t lines $\neq [\mathcal{F}]$ through $[A]$.

Without loss of generality g interchanges $[\mathcal{F}]$ and A^*1, and hence normalizes $U_{[\mathcal{F}],A^*1} = A$.

Note that $Z(Q)^g$ is transitive on the points $[B] \neq [A]$ of $[\mathcal{F}]$, while A^* is transitive on the points not on $[\mathcal{F}]$ collinear with such a $[B]$. Thus, $\langle A^*, A^{*g}\rangle$ is a subgroup of K transitive on the set of points not collinear with $[A]$. Let E denote the group generated by all the conjugates of A^* under the action of H. Then E is a p-group (as each such conjugate is a p-group normal in K). *Claim:* $E = \langle A^*, A^{*g}\rangle$. For, it suffices to show that *E is regular on the set of points not collinear with* $[A]$. Let $e \in E$ fix a point x not collinear with $[A]$. Each line on $[A]$ has a unique point collinear with x, and hence has a point $\neq [A]$ fixed by e; and since e is a p-element it must fix yet another point on that line. It follows that the set of fixed points and lines of e is a subquadrangle with $s' \leq s = q$ and $t' = t = q^2$. By Higman's inequality (Payne and Thas (1984, p.4)) $t \leq s'^2$, so that $s' = s$ and hence $e = 1$.

Thus, $E = \langle A^*, A^{*g}\rangle$ has order q^4. It contains a set Ω of $t + 1 = q^2 + 1$ subgroups conjugate under H to $Z(Q)$ and permuted 2-transitively by H. It follows that E must be elementary abelian, and that H acts irreducibly on it.

At this point, I do not know how to show that E is a $GF(q)[H]$-module without invoking some (preclassification!) group theory. The 2-transitive group $\overline{H}$ induced by H on Ω has the property that the stabilizer of $Z(Q)$ has a normal elementary abelian subgroup of order q^2 regular on the remaining members of Ω. It follows from Shult (1972) and Hering, Kantor and Seitz (1972) that $\overline{H} \trianglerighteq PSL(2, q^2)$. This group $PSL(2, q^2)$ acts (projectively and) irreducibly on the $GF(p)$-space E. However, up to twisting by field automorphisms, $PSL(2, q^2)$ has exactly two projective irreducible modules of size q^4: the natural one (over $GF(q^2)$) and the $\Omega^-(4, q)$-module (Fong and Seitz (1973 4B, C)). Since there is an orbit Ω of $q^2 + 1$ subgroups of size q, the only possibility is that E can be viewed as the $\Omega^-(4, q)$-module with Ω the associated ovoid of singular points.

Now it is easy to recover the generalized quadrangle from the group E. For, the stabilizer in E of a point $[B]$ of $[\mathcal{F}]$ is known — namely, A^*, which fixes every such point; as is that of a line B^* on $[B]$ but not on $[A]$ — namely, $Z(Q)$, which fixes every such line. Since E is regular on the points not collinear with $[A]$, while H is transitive on the lines on $[A]$, the lines not on $[A]$, and the points collinear with $[A]$, it follows that $Q(\mathcal{F})$ can be described in terms of the $\Omega^-(4, q)$-ovoid Ω consisting of $q^2 + 1$ subgroups of E, together with the tangent planes (such as A^*) to that ovoid. Consequently, $Q(\mathcal{F}) \cong Q(\Omega)$ and $Q(\mathcal{F})$ is the $O(5, q)$-quadrangle. □

Remark. Payne and Thas (1984) came very close to obtaining a geometric proof of the classification of finite Moufang quadrangles. Their obstacle was the same one appearing in Example 2! Thus, the above argument can be inserted into the appropriate part of Payne and

Thas (op. cit., Ch. 9) in order to complete their approach to that classification theorem.

Note, however, that the amount of group theory employed was fairly small, certainly miniscule compared to Fong and Seitz (1973): information was required concerning a relatively restricted type of 2-transitive permutation group, together with a fact about very small degree representations of such a group.

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