

Automorphism Groups of Hadamard Matrices*

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ABSTRACT

Automorphism groups of Hadamard matrices are related to automorphism groups of designs, and the automorphism groups of the Paley-Hadamard matrices are determined.

According to Hall [3], an automorphism of a Hadamard matrix H of size n is a pair (P, Q) of $n \times n$ monomial matrices such that $PHQ = H$. The automorphisms of H form a group Γ . $1 = (I, I)$ and $\sigma = (-I, -I)$ are in the center of Γ . $\Gamma' = \Gamma / \langle \sigma \rangle$ acts faithfully as a permutation group on the union of the sets of rows and columns of H .

If \mathcal{D} is a Hadamard design and M a $(-1, 1)$ incidence matrix of \mathcal{D} , then $H = M^+$ is the Hadamard matrix obtained from M by adjoining a column c and a row r of 1's.

\mathcal{D} determines a design \mathcal{D}^+ as follows. The points of \mathcal{D}^+ are the points of \mathcal{D} together with a new point r ; the blocks of \mathcal{D}^+ are (i) the blocks of \mathcal{D} with r adjoined, and (ii) the complements $\mathcal{C}B$ of the blocks B of \mathcal{D} . $(B \cup \{r\}, \mathcal{C}B)$ is called a parallel class of blocks. A $(-1, 1)$ incidence matrix of \mathcal{D}^+ may be obtained from the $n \times 2n$ matrix $(M^+, -M^+)$ by removing the columns c and $-c$. This implies the following

THEOREM 1. *The automorphism group of \mathcal{D}^+ is isomorphic to Γ_c .*

If M is symmetric, and $\gamma \in \Gamma_c$ moves r , then there is an element $\gamma' \in \Gamma_c$ moving c . Then $\gamma\gamma'$ moves r and c :

THEOREM 2. *If \mathcal{D} admits a polarity, and Γ_c has an element moving r , then Γ has an element moving both r and c .*

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THEOREM 3. *If Γ is 2-transitive on rows but not faithful on columns, then \mathcal{D} is a projective space.*

PROOF: The subset Σ of elements of Γ fixing all columns is a normal subgroup of Γ , and $\Sigma = \Sigma/\langle\sigma\rangle$ acts regularly on rows. It follows from the 2-transitivity of Γ that Σ is row-transitive and elementary Abelian. Σ may be regarded as an automorphism group of \mathcal{D}^+ transitive on points and fixing each parallel class. Then the stabilizer of a block in Σ has index ≤ 2 , so that the blocks correspond to the cosets of subgroups of index 2, and \mathcal{D}^+ is an affine space.

If $\gamma \in \Gamma$ fixes all rows of H and $\delta \in \Gamma$ fixes all columns of H , then $\gamma^{-1}\delta^{-1}\gamma\delta$ fixes all rows and columns and thus $= 1$ or σ . This implies the following:

THEOREM 4. *If $\mathcal{D} = PG(d, 2)$ then Γ is a semidirect product of $PSL(d, 2)$ with an elementary Abelian group of order 2^{2d} .*

Let $q > 3$ be a prime power $\equiv 3 \pmod{4}$. The Paley design $\mathcal{P}(q)$ is the Hadamard design defined by the difference set of squares in $GF(q)$. Let $\mathcal{D} = \mathcal{P}(q)$, so that $H = M^+$ is the Paley-Hadamard matrix [6]. Hall [3] has shown that Γ has a subgroup Π containing σ such that $\Pi = \Pi/\langle\sigma\rangle$ acts faithfully on both the rows and columns of H as the group of all permutations of $GF(q) \cup \{\infty\}$ of the form $x \rightarrow (ax^\theta + b)/(cx^\theta + d)$, $a, b, c, d \in GF(q)$, $ad - bc = 1$, and $\theta \in \text{Aut } GF(q)$; moreover $\Pi_c = \Pi_r$.

THEOREM 5 (Hall [3]). *If $\mathcal{D} = \mathcal{P}(11)$, then Γ acts on both rows and columns as the Mathieu group M_{12} .*

PROOF: By Hughes [4] and Todd [7], the full automorphism group of \mathcal{D}^+ is M_{11} . By Theorem 1, Γ_c is M_{11} . Since Γ_c is transitive on the columns $\neq c$, M_{11} is thus represented as a group transitive on these 11 columns. By Theorems 2 and 3, Γ acts faithfully on columns as a 2-transitive group of degree 12 such that the stabilizer of a column is isomorphic to M_{11} . It is now easy to see that Γ is M_{12} .

THEOREM 6. *If $\mathcal{D} = \mathcal{P}(q)$, $q > 11$, then $\Gamma = \Pi$.*

Special cases of this result are found in [1].

PROOF: Assume that $\Gamma > \Pi$. Γ_{cr} acts as an automorphism group of $\mathcal{P}(q)$, and thus $\Gamma_{cr} = \Pi_{cr}$ by [5, Theorem 12.1]. Then $\Gamma_c > \Pi_c = \Pi_{cr}$ implies that Γ_c moves r and thus is 2-transitive on rows. By Theorem 3, Γ_c acts faithfully on rows as a 2-transitive permutation group such that the stabilizer of a row r acts on the remaining rows as Π_{cr} . It is then not

difficult to show that $\bar{\Gamma}_c$ is isomorphic to $\bar{\Pi}$ (cf. Zassenhaus [8]; Bender [0]). Since $\bar{\Gamma}_c$ has a faithful transitive representation of degree q on the columns $\neq c$, this readily contradicts a classical result of Galois and Dickson [2, p. 286].

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