Automorphism Groups of Hadamard Matrices

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ABSTRACT

Automorphism groups of Hadamard matrices are related to automorphism groups of designs, and the automorphism groups of the Paley-Hadamard matrices are determined.

According to Hall [3], an automorphism of a Hadamard matrix $H$ of size $n$ is a pair $(P, Q)$ of $n \times n$ monomial matrices such that $PHQ = H$. The automorphisms of $H$ form a group $\Gamma$. $1 = (I, I)$ and $\sigma = (-I, -I)$ are in the center of $\Gamma$. $\Gamma / \langle \sigma \rangle$ acts faithfully as a permutation group on the union of the sets of rows and columns of $H$.

If $D$ is a Hadamard design and $M$ a $(-1, 1)$ incidence matrix of $D$, then $H = M^+$ is the Hadamard matrix obtained from $M$ by adjoining a column $c$ and a row $r$ of 1’s.

$D$ determines a design $D^+$ as follows. The points of $D^+$ are the points of $D$ together with a new point $r$; the blocks of $D^+$ are (i) the blocks of $D$ with $r$ adjoined, and (ii) the complements $B^c$ of the blocks $B$ of $D$. $(B \cup \{r\}, C^c B)$ is called a parallel class of blocks. A $(-1, 1)$ incidence matrix of $D^+$ may be obtained from the $n \times 2n$ matrix $(M^+, -M^+)$ by removing the columns $c$ and $-c$. This implies the following

THEOREM 1. The automorphism group of $D^+$ is isomorphic to $\Gamma_c$.

If $M$ is symmetric, and $\gamma \in \Gamma_c$ moves $r$, then there is an element $\gamma' \in \Gamma_r$ moving $c$. Then $\gamma \gamma'$ moves $r$ and $c$.

THEOREM 2. If $D$ admits a polarity, and $\Gamma_c$ has an element moving $r$, then $\Gamma$ has an element moving both $r$ and $c$.

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Theorem 3. If $\Gamma$ is 2-transitive on rows but not faithful on columns, then $D$ is a projective space.

Proof: The subset $\Sigma$ of elements of $\Gamma$ fixing all columns is a normal subgroup of $\Gamma$, and $\tilde{\Sigma} = \Sigma/\langle \sigma \rangle$ acts regularly on rows. It follows from the 2-transitivity of $\Gamma$ that $\tilde{\Sigma}$ is row-transitive and elementary Abelian. $\tilde{\Sigma}$ may be regarded as an automorphism group of $D^+$ transitive on points and fixing each parallel class. Then the stabilizer of a block in $\tilde{\Sigma}$ has index $\leqslant 2$, so that the blocks correspond to the cosets of subgroups of index 2, and $D^+$ is an affine space.

If $\gamma \in \Gamma$ fixes all rows of $H$ and $\delta \in \Gamma$ fixes all columns of $H$, then $\gamma^{-1}\delta^{-1}\gamma\delta$ fixes all rows and columns and thus $=1$ or $\sigma$. This implies the following:

Theorem 4. If $D = PG(d, 2)$ then $\Gamma$ is a semidirect product of $PSL(d, 2)$ with an elementary Abelian group of order $2^a$.

Let $q > 3$ be a prime power $\equiv 3 \pmod{4}$. The Paley design $P(q)$ is the Hadamard design defined by the difference set of squares in $GF(q)$. Let $D = P(q)$, so that $H = M^+$ is the Paley-Hadamard matrix [6]. Hall [3] has shown that $\Gamma$ has a subgroup $\Pi$ containing $\sigma$ such that $\Pi = \Pi/\langle \sigma \rangle$ acts faithfully on both the rows and columns of $H$ as the group of all permutations of $GF(q) \cup \{ \infty \}$ of the form $x \rightarrow (ax^b + b)/(cx^d + d)$, $a, b, c, d \in GF(q)$, $ad - bc = 1$, and $\theta \in \text{Aut} GF(q)$; moreover $\Pi_c = \Pi_r$.

Theorem 5 (Hall [3]). If $D = P(11)$, then $\Gamma$ acts on both rows and columns as the Mathieu group $M_{12}$.

Proof: By Hughes [4] and Todd [7], the full automorphism group of $D^+$ is $M_{11}$. By Theorem 1, $\Gamma_c$ is $M_{11}$. Since $\Gamma_c$ is transitive on the columns $\neq c$, $M_{11}$ is thus represented as a group transitive on these 11 columns. By Theorems 2 and 3, $\Gamma$ acts faithfully on columns as a 2-transitive group of degree 12 such that the stabilizer of a column is isomorphic to $M_{11}$. It is now easy to see that $\Gamma$ is $M_{12}$.

Theorem 6. If $D = P(q)$, $q > 11$, then $\Gamma = \Pi$.

Special cases of this result are found in [1].

Proof: Assume that $\Gamma > \Pi$. $\Gamma_c$ acts as an automorphism group of $P(q)$, and thus $\Gamma_c = \Pi_c$. By [5, Theorem 8.1]. Then $\Gamma_c > \Pi_c = \Pi_c$ implies that $\Gamma_c$ moves $r$ and thus is 2-transitive on rows. By Theorem 3, $\Gamma_c$ acts faithfully on rows as a 2-transitive permutation group such that the stabilizer of a row $r$ acts on the remaining rows as $\Pi_{cr}$. It is then not
difficult to show that $\bar{T}_c$ is isomorphic to $\bar{\Pi}$ (cf. Zassenhaus [8]; Bender [0]). Since $\bar{T}_c$ has a faithful transitive representation of degree $q$ on the columns $\neq c$, this readily contradicts a classical result of Galois and Dickson [2, p. 286].

REFERENCES

0. H. Bender, Endliche zweifach transitive Permutationsgruppen, deren Inversionen keine Fixpunkte haben. Math. Z. 104 (1968), 175–204.